直交行列　P: $\mathbb{R}$値の n次正方行列

$$(P\vec{x}, P\vec{y}) = (\vec{x}, \vec{y})$$

$$\iff \|P\vec{x}\|^2 = \|\vec{x}\|^2$$

$$\iff {}^tPP = P{}^tP = I_n \rightsquigarrow {}^tP = P^{-1}$$

$$\iff P = (\vec{p}_1 \cdots \vec{p}_n)$$

$$(\vec{p}_i, \vec{p}_j) = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

A: 実対称.　${}^tA = A$

$A = (a_{ij})$ とす　$a_{ij} = a_{ji}$

定理　A: 実対称. ある直交行列Pで $= P^{-1}$

$$A = P \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} {}^tP$$

と対角化できる　$\alpha_1, \cdots, \alpha_n \in \mathbb{R}$.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Phi_A(\lambda) = |\lambda I_3 - A| = \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda+1 & 0 \\ -1 & 0 & \lambda \end{vmatrix}$$

234の余因子展開.

$$= (-1)^{2+2}(\lambda+1)\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix}$$

$$= (\lambda+1)(\lambda^2-1) = (\lambda+1)^2(\lambda-1)$$

$\underline{\lambda = -1}$ $\quad V(-1) = \mathrm{Ker}(-I_3 - A)$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \longrightarrow \cdots \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V(-1) \Longleftrightarrow \underline{x + z = 0}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ y \\ z \end{pmatrix} = z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{b}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ \vec{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightsquigarrow \vec{a} \perp \vec{b}$$

$$\vec{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ \vec{p}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 1}$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V(1) \Longleftrightarrow x = z, \ y = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{p}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

定理 A:対称, $\alpha, \beta \in \mathbb{R}$ 固有値.

$$A\vec{v}_1 = \alpha\vec{v}_1,\quad A\vec{v}_2 = \beta\vec{v}_2$$

$$\alpha \neq \beta \Longrightarrow (\vec{v}_1, \vec{v}_2) = 0$$

$(\vec{p}_1, \vec{p}_3) = (\vec{p}_2, \vec{p}_3) = 0$ が従う.

$\leadsto P = (\vec{p}_1\ \vec{p}_2\ \vec{p}_3)$ は直交行列.

$$A\vec{p}_1 = -\vec{p}_1,\quad A\vec{p}_2 = -\vec{p}_2,\quad A\vec{p}_3 = \vec{p}_3$$

$$AP = (-\vec{p}_1\ -\vec{p}_2\ \vec{p}_3) = (\vec{p}_1\ \vec{p}_2\ \vec{p}_3)\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

$$= P\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \leadsto A = P\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}{}^tP$$

$${}^tPP = I_3$$
$${}^tP = P^{-1}$$

定理の証明

$$(\alpha\vec{v}_1, \vec{v}_2) = (A\vec{v}_1, \vec{v}_2) = (\vec{v}_1, \overset{A}{\overset{\shortparallel}{{}^tA}}\vec{v}_2)$$

$$\alpha(\vec{v}_1, \vec{v}_2) \qquad\qquad (\vec{v}_1, \beta\vec{v}_2) = \beta(\vec{v}_1, \vec{v}_2)$$

$$\overset{\neq 0}{(\alpha-\beta)}(\vec{v}_1, \vec{v}_2) = 0$$

$$\leadsto (\vec{v}_1, \vec{v}_2) = 0$$

x	y	z
x_1	y_1	z_1
x_2	y_2	z_2
$\vdots$	$\vdots$	$\vdots$
x_N	y_N	z_N

$$\vec{x} = \frac{1}{\sqrt{N}} \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_N - \bar{x} \end{pmatrix} \quad \vec{y},\ \vec{z} \ \cdots$$

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$\|\vec{x}\|^2 = V(x) = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2$$

$$(\vec{x}, \vec{z}) = \sigma_{xz} \quad \text{共分散}$$

$$= \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(z_i - \bar{z})$$

$$w = \alpha x + \beta y + \gamma z$$

$$\boxed{w_i = \alpha x_i + \beta y_i + \gamma z_i} \quad i = 1, \cdots, N.$$

$$\bar{w} = \alpha \bar{x} + \beta \bar{y} + \gamma \bar{z}$$

$$\frac{1}{\sqrt{N}} (w_i - \bar{w}) = \frac{\alpha}{\sqrt{N}} (x_i - \bar{x}) + \frac{\beta}{\sqrt{N}} (y_i - \bar{y}) + \frac{\gamma}{\sqrt{N}} (z_i - \bar{z})$$

$$\vec{w} = \alpha \vec{x} + \beta \vec{y} + \gamma \vec{z} = (\vec{x}\ \vec{y}\ \vec{z}) \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$= D \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$\vec{w}$ の第 i 成分

$D = (\vec{x}\ \vec{y}\ \vec{z})$ とおく

$$\|\vec{w}\|^2 = \left(D \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, D \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \right)$$

$$= \left({}^t D D \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \right).$$

$V(w)$ は ${}^t DD$ の 2次形式.

$D = (\vec{x}\ \vec{y}\ \vec{z})$

$${}^tDD = \begin{pmatrix} {}^t\vec{x} \\ {}^t\vec{y} \\ {}^t\vec{z} \end{pmatrix} (\vec{x}\ \vec{y}\ \vec{z})$$

$$= \begin{pmatrix} {}^t\vec{x}\vec{x} & {}^t\vec{x}\vec{y} & {}^t\vec{x}\vec{z} \\ {}^t\vec{y}\vec{x} & {}^t\vec{y}\vec{y} & {}^t\vec{y}\vec{z} \\ {}^t\vec{z}\vec{x} & {}^t\vec{z}\vec{y} & {}^t\vec{z}\vec{z} \end{pmatrix}$$

$$= \begin{pmatrix} V(x) & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & V(y) & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & V(z) \end{pmatrix}$$

$(\vec{a}, \vec{b}) = {}^t\vec{a}\vec{b}$

分散行列

$$({}^tDD)^t = {}^tD\,{}^t({}^tD)$$

$$= {}^tDD \quad \rightsquigarrow \quad {}^tDD：対称.$$

$$\begin{cases} {}^t(AB) = {}^tB\,{}^tA \\ {}^t({}^tA) = A \end{cases}$$

$\alpha^2 + \beta^2 + \gamma^2 = 1$ の下で $\|\vec{w}\|^2$ を最大化・最小化.

$\|\vec{w}\|^2 = V(w)$

主成分分析

$$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$(A\vec{v}, \vec{v}) = \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} z \\ -y \\ x \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = -y^2 + xz$$

A が定める2次形式.

P: 直交 $\longrightarrow$ ${}^tP$: 直交. $\quad {}^tPP = I_3$, $\quad P\,{}^tP = I_3$

$$(A\vec{v}, \vec{v}) = ({}^tPA\vec{v}, {}^tP\vec{v})$$

$P\,{}^tP$

$${}^tPAP = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

$$= \left(\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} {}^tP\vec{v}, {}^tP\vec{v} \right)$$

$${}^tP\vec{v} = \vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

$$\| \quad \left(\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \right)$$

$$= \begin{pmatrix} -\xi_1 \\ -\xi_2 \\ \xi_3 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

$$= \underline{-\xi_1^2 - \xi_2^2 + \xi_3^2}$$

$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\qquad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$\|\vec{v}\|^2 = 1$ のとき $(A\vec{v}, \vec{v})$ の最大値を求めよ

P：直交

$$(A\vec{v}, \vec{v}) = -\xi_1^2 - \xi_2^2 + \xi_3^2$$

最大値 $= 1$ $\qquad \vec{\xi} = {}^tP\vec{v}$

$$1 = \|\vec{v}\| \underset{\uparrow}{=} \|{}^tP\vec{v}\| = \|\vec{\xi}\|$$

${}^tP$ は直交

$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$ のとき $-\xi_1^2 - \xi_2^2 + \xi_3^2$ を最大化.

$$= -\xi_1^2 - \xi_2^2 + (1 - \xi_1^2 - \xi_2^2)$$

$$= 1 - 2\xi_1^2 - 2\xi_2^2 \leq 1$$

等号成立 $\xi_1 = \xi_2 = 0$

$\xi_3 = \pm 1$ $\qquad \vec{\xi} = \pm\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ で 最大値 1

$${}^tP\vec{v} = \vec{\xi} \Longleftrightarrow \vec{v} = P\vec{\xi}$$

$$\vec{v} = \pm P \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \pm\vec{p}_3$$

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 7 & -4 \\ 2 & -4 & 1 \end{pmatrix}$$

を直交行列で対角化する.