6月14日 統計入門I.

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

の固有値. 固有ベクトルを計算

対称行列

x	y
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_N	y_N

$z_i = \alpha x_i + \beta y_i$

$(i = 1, 2, \cdots, N)$

$\boxed{z = \alpha x + \beta y}$

として新たな変数を作る.

zの分散.

共分散

$$V(z) = \alpha^2 V(x) + 2\alpha\beta C_{xy} + \beta^2 V(y)$$

$$= \left(C \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right)$$

Cが定める2次形式

$$C = \begin{pmatrix} V(x) & C_{xy} \\ C_{xy} & V(y) \end{pmatrix}$$

分散行列.

${}^tC = C$ より対称行列.

$\alpha^2 + \beta^2 = 1$ の下で $V(z)$ 最小化・最大化.

$A = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ ${}^tA = A$ より対称.

固有方程式 $|\lambda I_2 - A| = \begin{vmatrix}\lambda-2 & -1\\ -1 & \lambda-2\end{vmatrix}$

$= (\lambda-2)\times(\lambda-2) - (-1)\times(-1)$

$= (\lambda-2)^2 - 1 = 0$

$\lambda = 2 \pm 1$ 　 3, 1 が固有値.

固有ベクトル ① $\lambda = 3$. $\begin{pmatrix}\boxed{1 \ -1}\\ \boxed{-1 \ 1}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \vec{0}$

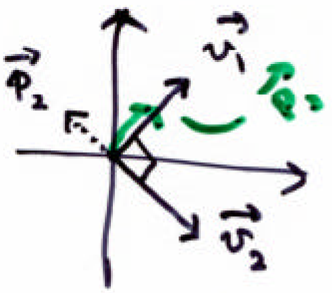
$\Longleftrightarrow \begin{cases} x - y = 0 \\ -x + y = 0 \end{cases}$

$\Longleftrightarrow x = y$

$$\boxed{\vec{v}_1 = \begin{pmatrix}x\\ x\end{pmatrix} = x\begin{pmatrix}1\\ 1\end{pmatrix}}$$

② $\lambda = 1$ のとき. $\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \vec{0} \Longleftrightarrow x + y = 0$

$$\boxed{\vec{v}_2 = \begin{pmatrix}x\\ -x\end{pmatrix} = x\begin{pmatrix}1\\ -1\end{pmatrix}}$$

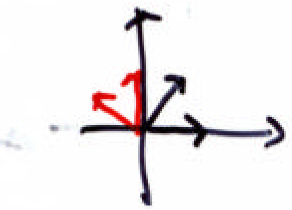


$\boxed{\vec{v}_1 \perp \vec{v}_2}$ ← 一般的に成立.

$\vec{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$, $\vec{p}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ +1\end{pmatrix}$ $P = \begin{pmatrix}\cos 45^\circ & -\sin 45^\circ\\ \sin 45^\circ & \cos 45^\circ\end{pmatrix}$

45°回転

$P = (\vec{p}_1\ \vec{p}_2)$ 45°回転



$P\begin{pmatrix}1\\0\end{pmatrix} = \vec{p}_1$

$P\begin{pmatrix}0\\1\end{pmatrix} = \vec{p}_2$

${}^tP = P^{-1}$ −45°回転

$$= \begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix} = \begin{pmatrix}\cos(-45^\circ) & -\sin(-45^\circ)\\ \sin(-45^\circ) & \cos(-45^\circ)\end{pmatrix}$$

一般論　P: 2次正方行列 $P = (\vec{p}_1\ \vec{p}_2)$

${}^tPP = P\,{}^tP = I_2$　直交行列.

$\Leftrightarrow$ $(P\vec{x}, P\vec{y}) = (\vec{x}, \vec{y})$ ←
(全ての $\vec{x}, \vec{y}$)

$\Leftrightarrow$ $\|P\vec{x}\| = \|\vec{x}\|$ (全ての $\vec{x}$)

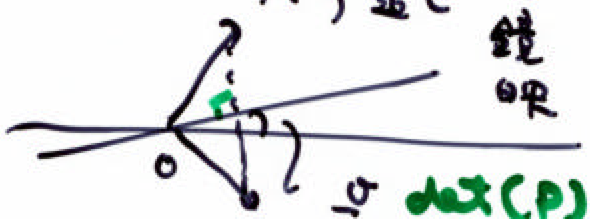
$\Leftrightarrow$ $\|\vec{p}_1\|^2 = \|\vec{p}_2\| = 1$, $(\vec{p}_1, \vec{p}_2) = 0$

$$P = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \text{ or } \begin{pmatrix}\cos\theta & +\sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}$$

回転　　　　　　折り返し

$\det(P) = 1$

鏡映



$\det(P) = -1$

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 固有値 3, 1

$\vec{P_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad A\vec{P_1} = 3\vec{P_1}$

$\vec{P_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad A\vec{P_2} = \vec{P_2}$

$AP = (A\vec{P_1} \ A\vec{P_2}) = (3\vec{P_1} \ 1\cdot\vec{P_2})$

$= (\vec{P_1} \ \vec{P_2})\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = P\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

まとめ $AP = P\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

${}^tP\cdot$

$\boxed{{}^tPAP = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}$ 対角化可能

${}^tPP = P\,{}^tP = I_2 \quad (P^{-1} = {}^tP)$

${}^tP$も直交

${}^tP\cdot{}^t({}^tP) = {}^t({}^tP)\,{}^tP = I_2$

2次形式

$\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) = \begin{pmatrix} 2\alpha+\beta \\ \alpha+2\beta \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ 内積

$= (2\alpha+\beta)\alpha + (\alpha+2\beta)\beta$

$= 2\alpha^2 + \underset{2\cdot 1}{2\alpha\beta} + 2\beta^2$

演習 $A = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix} \quad \left(A\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) = \cdots$

$\left(A\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) \underset{\uparrow}{=} \left(\boxed{{}^tPAP}\,\underbrace{{}^tP}_{I_2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, {}^tP\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$

${}^tP$: 直交

${}^tPAP = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

P : 直交

$$\left(A\begin{pmatrix}\alpha\\\beta\end{pmatrix}, \begin{pmatrix}\alpha\\\beta\end{pmatrix}\right) = \Big(\underbrace{\boxed{{}^tPAP}}_{= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}} \overbrace{\cdot\, {}^tP}^{I_2} \underbrace{\begin{pmatrix}\alpha\\\beta\end{pmatrix}}_{= \begin{pmatrix}\xi\\\eta\end{pmatrix}}, \underbrace{{}^tP\begin{pmatrix}\alpha\\\beta\end{pmatrix}}_{= \begin{pmatrix}\xi\\\eta\end{pmatrix}}\Big)$$

$$= \left(\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix}, \begin{pmatrix}\xi\\\eta\end{pmatrix}\right) = \begin{pmatrix}3\xi\\\eta\end{pmatrix}\cdot\begin{pmatrix}\xi\\\eta\end{pmatrix}$$

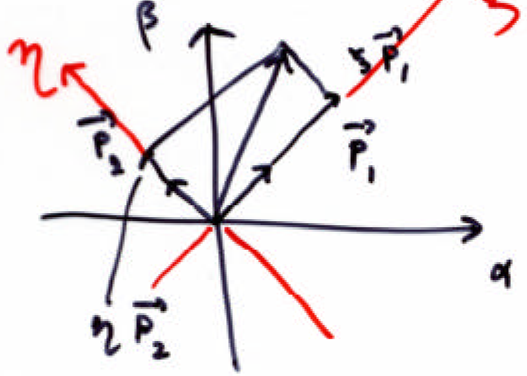
$$= 3\xi^2 + \eta^2$$

$$\underbrace{{}^tP}_{= P^{-1}}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}\xi\\\eta\end{pmatrix} \Rightarrow \begin{pmatrix}\alpha\\\beta\end{pmatrix} = P\begin{pmatrix}\xi\\\eta\end{pmatrix}$$

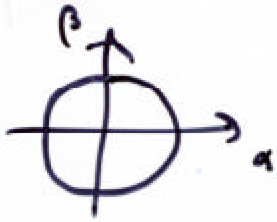
$P \cdot$

$$= (\vec{P_1}\ \vec{P_2})\begin{pmatrix}\xi\\\eta\end{pmatrix}$$

$$= \xi\vec{P_1} + \eta\vec{P_2}$$



$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$



$\alpha^2 + \beta^2 = 1$ の下で

$$\left(A \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) = 2(\alpha^2 + \alpha\beta + \beta^2)$$

の最大値・最小値. （制約条件付 極値問題）

$$f = \left(A \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) = 3\xi^2 + \eta^2$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = P \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightsquigarrow \left\| P \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\| = \left\| \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\|$$

P 直交 $\qquad \left\| P \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\| = \left\| \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\|$

$$\rightsquigarrow 1 = \alpha^2 + \beta^2 = \xi^2 + \eta^2$$

最大値 $\quad f = 3(1 - \eta^2) + \eta^2$

$$= 3 - 2\eta^2 \underset{\uparrow}{\leqq} 3$$

等号成立 $\eta = 0, \ \xi = \pm 1$

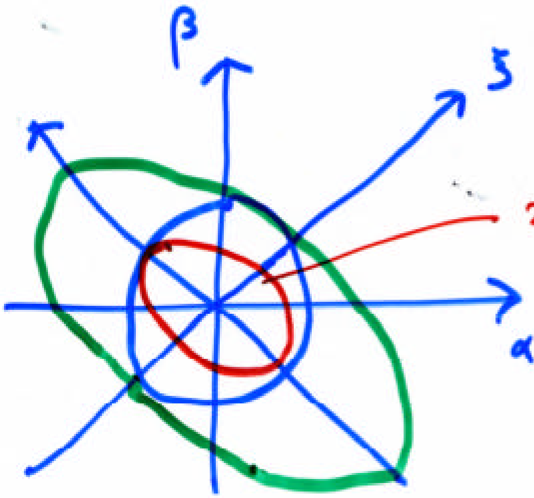
$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = P \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} = (\vec{p}_1\ \vec{p}_2) \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} = \pm \vec{p}_1$$

最小値 $\quad f = 3\xi^2 + (1 - \xi^2)$

$$= 1 + 2\xi^2 \underset{\uparrow}{\geqq} 1$$

等号成立 $\xi = 0, \ \eta = \pm 1$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = P \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} = (\vec{p}_1\ \vec{p}_2) \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} = \pm \vec{p}_2$$



$2(\alpha^2+\alpha\beta+\beta^2) = \text{一定} = 1$

だ円　$2(\alpha^2+\alpha\beta+\beta^2) = \text{一定} = 3$

→ Lagrangeの未定乗数法

Lagrangeの　未定乗数法

$A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ に対して A^n を2通りで計算

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

I

$$\boxed{\begin{array}{l} \text{ケーリー・ハミルトン} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ A^2 - (a+d)A + (ad-bc)I_2 = O_2 \end{array}}$$

$$A^2 - 4A + 3I_2 = O_2$$

$$A^2 - (3+1)A + 3\times 1\, I_2 = O_2$$

$$\begin{cases} A^2 - A = 3(A - I_2) \\ A^2 - 3A = A - 3I_2 \end{cases}$$

$$\begin{cases} A(A - I_2) = 3(A - I_2) \\ A(A - 3I_2) = A - 3I_2 \end{cases}$$

$$\begin{aligned} A^2(A - I_2) &= A \cdot A(A - I_2) \\ &= A \cdot 3(A - I_2) \\ &= 3A(A - I_2) \\ &= 3^2(A - I_2) \end{aligned}$$

$$\begin{aligned} A^{n+1} - A^n &= A^n(A - I_2) = 3^n(A - I_2) \\ -)\ A^{n+1} - 3A^n &= A^n(A - 3I_2) = 1^n(A - I_2) \\ \hline 2A^n &= 3^n(A - I_2) - (A - I_2) \end{aligned}$$

$$A^n = \frac{1}{2}3^n(A - I_2) - \frac{1}{2}(A - I_2)$$

対角化の方法

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{固有値 } 3,\ 1$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad P = (\vec{v}_1\ \vec{v}_2)$$

$$A\vec{v}_1 = 3\vec{v}_1,\ A\vec{v}_2 = 1 \cdot \vec{v}_2$$

$$AP = A(\vec{v}_1\ \vec{v}_2) = (A\vec{v}_1\ A\vec{v}_2)$$
$$= (3\vec{v}_1\ \vec{v}_2) = (\vec{v}_1\ \vec{v}_2)\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= P\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

$\det(P) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$ より P：正則

（逆行列あり）

$$AP = P\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \rightsquigarrow A = P\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}P^{-1}$$

$$A^2 = P\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\underbrace{P^{-1} \cdot P}_{I_2}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}P^{-1}$$

$$= P\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}^2 P^{-1} = P\begin{pmatrix} 3^2 & 0 \\ 0 & 1^2 \end{pmatrix}P^{-1}$$

より一般に

$$A^n = P\begin{pmatrix} 3^n & 0 \\ 0 & 1^n \end{pmatrix}P^{-1}$$