<u>Review</u>

A : 2×2 行列, 実数値.

$$\boxed{\frac{d}{dt}\vec{x}(t) = A\,\vec{x}(t)}$$

D方系.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Phi_A(\lambda) = \det(\lambda I_2 - A)$$
$$= \lambda^2 - (a+d)\lambda + \det(A)$$

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① $\Phi_A(\lambda) = 0$ が 相異なる 2実数解 α, β

$$\alpha, \beta \in \mathbb{R} \quad \alpha \neq \beta.$$

$$A\vec{v_1} = \alpha\vec{v_1}, \quad A\vec{v_2} = \beta\vec{v_2} \quad (\vec{v_j} \neq \vec{0})$$

とする.

$P = (\vec{v_1}\ \vec{v_2})$ とする (P : 正則となる)

$$\rightsquigarrow \quad AP = P\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$ 対角化.

$$\vec{z}(t) = P^{-1}\vec{x}(t)$$

$$\frac{d}{dt}\vec{z}(t) = P^{-1}\frac{d}{dt}\vec{x}(t) = P^{-1}AP\cdot P^{-1}\vec{x}(t)$$
$$= \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\vec{z}(t) \iff \begin{cases} z_1'(t) = \alpha z_1(t) \\ z_2'(t) = \beta z_2(t) \end{cases}$$

$$\begin{cases} z_1(t) = z_1(0)e^{\alpha t} \\ z_2(t) = z_2(0)e^{\beta t} \end{cases}$$

$$\vec{z}(t) = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix}\begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}$$

$$P^{-1}\vec{x}(t) = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix} P^{-1}\vec{x}(0)$$

$$\rightsquigarrow \quad \boxed{\vec{x}(t) = P\begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix} P^{-1}\cdot\vec{x}(0)}$$

D: $\Phi_A(\lambda)=0$ の判別式.
(discriminant)

② $D=0$ 　$\Phi_A(\lambda)=0$ が重根を持つ.
$\overset{\shortparallel}{(\lambda-\alpha)^2}$.

$AP = P\begin{pmatrix}\alpha & 0\\ 1 & \alpha\end{pmatrix}$ をみたす P がある.

$\rightsquigarrow$ 次回 演習.
「ジョルダン標準形」

③ $D<0$

(#) $\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$
　　$a_j \in \mathbb{R}$ $(j=1,\cdots,n)$

$\lambda = \alpha$ が (#) の解 $\Rightarrow$ $\lambda = a - ib$ も (#) の解.
$\alpha = a+ib$
$\overline{a_j} = a_j$

$\overline{a+ib} = a-ib$. 複素共役.
complex conjugate

$\alpha, \beta \in \mathbb{C}$
$\overline{\alpha\cdot\beta} = \overline{\alpha}\cdot\overline{\beta}$
$\overline{\alpha\pm\beta} = \overline{\alpha}\pm\overline{\beta}$ etc.

証明
$\alpha^n + a_1\alpha^{n-1} + \cdots + a_{n-1}\alpha + a_n = 0$
両辺の複素共役をとる

$\overline{\alpha^n + \cdots + a_{n-1}\alpha + a_n} = \overline{0} = 0$
$\shortparallel$
$(\overline{\alpha})^n + a_1(\overline{\alpha})^{n-1} + \cdots + a_{n-1}\overline{\alpha} + a_n = 0$

$\rightsquigarrow$ $\lambda = \overline{\alpha}$ が (#) の解.

$$\lambda^2 + a_1\lambda + a_2 = 0 \qquad D = a_1^2 - 4a_2 < 0$$

$$\lambda = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$= \frac{-a_1 \pm \sqrt{-D}\cdot i}{2}$$

$$\alpha = \frac{-a_1 + \sqrt{-D}\, i}{2}, \quad \beta = \frac{-a_1 - \sqrt{-D}\, i}{2}$$

$\overline{\alpha} = \beta$ が言える.

$$\Phi_A(\lambda) = (\lambda - \alpha)(\lambda - \beta)$$

$\alpha = a + ib,\ \beta = a - ib \quad (a, b \in \mathbb{R})$

とおける.

(例1) $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$

$$\begin{vmatrix} \lambda - 1 & 1 \\ -2 & \lambda + 1 \end{vmatrix} = (\lambda^2 - 1) - (-2)$$
$$= \lambda^2 + 1.$$

$\lambda = \pm i$, Aの固有値である.

$\lambda = i$ の固有ベクトル

$(i+1)(i-1 \quad 1)$
$\|$
$(-2 \quad i+1)$

$$\begin{pmatrix} \boxed{i-1 \quad 1} \\ -2 \quad i+1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \vec{0}$$

$\Leftrightarrow$

$z_2 = -(i-1)z_1$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -i+1 \end{pmatrix} \qquad \boxed{A\vec{v}_1 = i\vec{v}_1}$$

$\lambda = -i$ の固有ベクトル

$$A\vec{v}_1 = i\vec{v}_1$$

$$\bar{A} = A$$

A : 実数値

$$\boxed{A\vec{v}_2 = -i\vec{v}_2}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

A : m×n 行列

$$= (\alpha_{ij})$$

$$\bar{A} = (\bar{\alpha}_{ij})$$

$$\begin{cases} \vec{v}_1 = \vec{w}_1 + i\vec{w}_2 \\ \vec{v}_2 = \vec{w}_1 - i\vec{w}_2 \end{cases}$$

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{w}_1 = \frac{1}{2}(\vec{v}_1 + \vec{v}_2) \\ \vec{w}_2 = \frac{1}{2i}(\vec{v}_1 - \vec{v}_2) \end{cases}$$

$P = (\vec{w}_1\ \vec{w}_2)$ とおく。

$$A\vec{w}_1 = \frac{1}{2}(A\vec{v}_1 + A\vec{v}_2)$$

$$= \frac{1}{2}(i\vec{v}_1 - i\vec{v}_2)$$

$$= -\frac{1}{2i}(\vec{v}_1 - \vec{v}_2)$$

$$= -\vec{w}_2.$$

① $A\vec{w}_1 = -\vec{w}_2$

② $A\vec{w}_2 = \vec{w}_1$

$$A\vec{w}_2 = \frac{1}{2i}(A\vec{v}_1 - A\vec{v}_2)$$

$$= \frac{1}{2i}(i\vec{v}_1 + i\vec{v}_2)$$

$$= \frac{1}{2}(\vec{v}_1 + \vec{v}_2) = \vec{w}_1$$

$$AP$$

$$= A(\vec{w}_1\ \vec{w}_2)$$

$$= (A\vec{w}_1\ A\vec{w}_2)$$

$$A\vec{w}_1 = -\vec{w}_2, \quad A\vec{w}_2 = \vec{w}_1$$

$$AP = (A\vec{w}_1\ A\vec{w}_2) = (-\vec{w}_2\ \vec{w}_1)$$

$$= (\vec{w}_1\ \vec{w}_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= P\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

$\det(P) \neq 0.$

一般論でok.

$$P^{-1}AP = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$b \neq 0$

$\alpha = a + ib, \quad \beta = a - ib.$

正則

$$\vec{v}_1 = \vec{w}_1 + i\vec{w}_2, \quad \vec{v}_2 = \vec{w}_1 - \vec{w}_2$$

$$P = (\vec{w}_1\ \vec{w}_2)$$

$$AP = P\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \longleftarrow \text{実標準形.}$$

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

$$\vec{z}(t) = P^{-1}\vec{x}(t) \longrightarrow \boxed{\vec{z}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\vec{z}(t)}$$

同じ計算.

$$X(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$(\cos t)' = -\sin t$

$(\sin t)' = \cos t$

$$X'(t) = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$//$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$X'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}X(t) = X(t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\vec{z}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{z}(t)$$

$$= X(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$X(t) = \begin{pmatrix} \cos t & \overset{\sin}{\cancel{\cos}}\, t \\ -\sin t & \underset{\cos}{\cancel{\sin}}\, t \end{pmatrix} \qquad X'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(t)$$

$$\left(X(-t) \vec{z}(t) \right)'$$

$$= \underbrace{X(-t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times (-1)} \vec{z}(t) + X(-t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{z}(t)$$

$$= X(-t) \Bigg(\underbrace{-\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{= O_2} \Bigg) \vec{z}(t)$$

$$= \vec{0}.$$

$$X(-t) \vec{z}(t) = X(0) \vec{z}(0)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{z}(0) = \vec{z}(0)$$

$$X(s) X(t) = X(s+t). \rightsquigarrow X(-t) X(t) = X(0) = I_2$$

$$X(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \qquad X(-t) = (X(t))^{-1}$$

$-t$ 回転

$$\boxed{\vec{z}(t) = X(t) \vec{z}(0)}$$

$$P^{-1} \vec{x}(t) = \vec{z}(t) = X(t) \vec{z}(0)$$

$$= X(t) P^{-1} \vec{x}(0)$$

$$\rightsquigarrow \boxed{\vec{x}(t) = P X(t) P^{-1} \vec{x}(0)}$$

問題

$$x(t) = e^{at}\begin{pmatrix} \cos \overset{b}{t} & \sin \overset{b}{t} \\ -\sin \underset{b}{t} & \cos \underset{b}{t} \end{pmatrix}$$

$$x'(t) = x(t)\,\boxed{}$$

$$= \boxed{}\;x(t)$$

2×2 1つに
行列.