x	y	z
x_1	y_1	z_1
$\vdots$	$\vdots$	$\vdots$
x_N	y_N	z_N

$$\vec{x} = \frac{1}{\sqrt{N}} \begin{pmatrix} x_1 - \bar{x} \\ \vdots \\ x_N - \bar{x} \end{pmatrix}$$

$V(x) = \|\vec{x}\|^2$

$\vec{y}, \vec{z}$: 同様.

$$\vec{w} = \alpha x + \beta y + \gamma z$$

α, β, γ : 定数.

$$\vec{w} = \frac{1}{\sqrt{N}} \begin{pmatrix} w_1 - \bar{w} \\ \vdots \\ w_N - \bar{w} \end{pmatrix} = \alpha \vec{x} + \beta \vec{y} + \gamma \vec{z}$$

$$w_i = \alpha x_i + \beta y_i + \gamma z_i$$
$$-) \quad \bar{w} = \alpha \bar{x} + \beta \bar{y} + \gamma \bar{z}$$

$$w_i - \bar{w} = \alpha (x_i - \bar{x}) + \beta (y_i - \bar{y}) + \gamma (z_i - \bar{z})$$

$$\boxed{\vec{w} = \alpha \vec{x} + \beta \vec{y} + \gamma \vec{z}}$$

$$= (\vec{x} \ \vec{y} \ \vec{z}) \vec{a} \qquad \vec{a} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\boxed{= D \vec{a}}$$

$$V(w) = \|\vec{w}\|^2$$
$$= (D\vec{a}, D\vec{a})$$
$$= ({}^tDD\vec{a}, \vec{a})$$

$\boxed{(\vec{e}, \vec{e}) = \|\vec{e}\|^2}$

公式

$${}^tDD = \begin{pmatrix} \boxed{{}^t\vec{x}} \\ {}^t\vec{y} \\ {}^t\vec{z} \end{pmatrix} \begin{pmatrix} \vec{x} & \vec{y} & \vec{z} \end{pmatrix}$$

分共分行列

$$= \begin{pmatrix} V(x) & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & V(y) & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & V(z) \end{pmatrix}$$

${}^t\vec{x}\,\vec{x}$
$= \|\vec{x}\|^2$
$= V(x)$

${}^t\vec{y}\,\vec{x} = (\vec{y}, \vec{x})$
$= \sigma_{xy}$

共分散

$${}^t({}^tDD) = {}^tD \cdot {}^t({}^tD) = {}^tDD$$

$$\boxed{{}^t(AB) = {}^tB\,{}^tA}$$

$\rightsquigarrow$ ${}^tDD$: 対称.

${}^tDD$: Dのグラム行列.

$A=(a_{ij})$ $a_{ij}=a_{ji}$ 対称.

${}^tA = A$

$(A\vec{v}, \vec{v})$ Aの2次形式.

$\underline{n=2}$ $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ 実対称.

$\Phi_A(\lambda)$

$$= \det(\lambda I_2 - A) = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix}$$

$$= (\lambda-a)(\lambda-b) - c^2$$

$$= \lambda^2 - (a+b)\lambda + (ab - c^2)$$

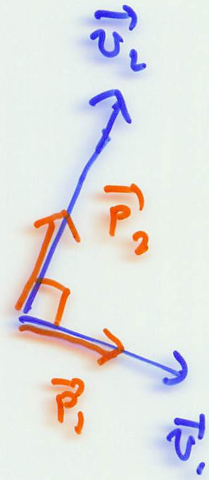
$$D = (a+b)^2 - 4(ab - c^2)$$
$$= (a-b)^2 + 4c^2 \geqq 0$$

- $\Phi_A(\lambda)$ の根は実根.
- $\Phi_A(\lambda)$ の根が重根 $\Longleftrightarrow D = 0$

$\Longleftrightarrow a = b,\ c = 0$

$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI_2$

$$\vec{p}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1, \quad \vec{p}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2$$



$$\|\vec{p}_1\| = \|\vec{p}_2\| = 1, \quad (\vec{p}_1, \vec{p}_2) = 0$$

$P = (\vec{p}_1\ \vec{p}_2)$ ：直交行列.

$$ {}^tPP = \begin{pmatrix} \boxed{{}^t\vec{p}_1} \\ \boxed{{}^t\vec{p}_2} \end{pmatrix} (\vec{p}_1\ \vec{p}_2) $$

$$ = \begin{pmatrix} \|\vec{p}_1\|^2 & (\vec{p}_1, \vec{p}_2) \\ (\vec{p}_1, \vec{p}_2) & \|\vec{p}_2\|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} $$

$\rightsquigarrow\ {}^tPP = I_2$

$$\det({}^tPP) = \det(I_2) = 1$$
$$\|$$
$$\det({}^tP)\det(P) = \det(P)^2$$

$$\boxed{\det({}^tC) = \det(C)}$$

$\rightsquigarrow\ \det(P) = \pm 1$
$\neq 0$

P: 正則.

${}^tPP = I_2$

$\cdot P^{-1} \searrow\ {}^tP = P^{-1}$

$P\,{}^tP = I_2.$

<u>直交行列</u> $\quad {}^tPP = P\,{}^tP = I_2.$

$D > 0$ とする。 $\longrightarrow$ $\Phi_A(\lambda) = (\lambda - \alpha)(\lambda - \beta)$

($A = aI_2$ は考えない) $\quad \alpha \neq \beta$.

α の固有ベクトル $\vec{v}_1 \neq \vec{0}$ $\quad A\vec{v}_1 = \alpha \vec{v}_1$

β ―― $\vec{v}_2 \neq \vec{0}$ $\quad A\vec{v}_2 = \beta \vec{v}_2$

$\vec{v}_1 \nparallel \vec{v}_2$

復習

A : 一般の行列

$\alpha_i \neq \alpha_j$ 固有値

$A\vec{v}_i = \alpha_i \vec{v}_i \quad (i = 1, \cdots, k)$

$$\left(\begin{array}{l} c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0} \\ \Longrightarrow c_1 = c_2 = \cdots = c_k = 0 \end{array} \right.$$

$\vec{v}_1, \cdots, \vec{v}_k$: 1次独立

定理 $(\vec{v}_1, \vec{v}_2) = 0$.

$$(A\vec{v}_1, \vec{v}_2) = (\vec{v}_1, {}^tA\vec{v}_2) = (\vec{v}_1, A\vec{v}_2)$$

$$(\alpha\vec{v}_1, \vec{v}_2) = \alpha(\vec{v}_1, \vec{v}_2) \qquad (\vec{v}_1, \beta\vec{v}_2) = \beta(\vec{v}_1, \vec{v}_2)$$

$$\leadsto \underbrace{(\alpha - \beta)}_{\neq 0}(\vec{v}_1, \vec{v}_2) = 0$$

$$\leadsto (\vec{v}_1, \vec{v}_2) = 0$$

$$A\vec{p_1} = \alpha \vec{p_1}, \quad A\vec{p_2} = \beta \vec{p_2}$$

$$A(\vec{p_1}\ \vec{p_2}) = (\vec{p_1}\ \vec{p_2})\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

$${}^tPAP = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad P^{-1} = {}^tP.$$

$$A = P\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}{}^tP$$

${}^tP$も直交.

$${}^tP \cdot {}^t({}^tP) = {}^tP\,P = I_2$$

$${}^t({}^tP)\,{}^tP = P\,{}^tP = I.$$

$${}^t({}^tP)\,{}^tP = {}^tP\,{}^t({}^tP) = I_2$$

$$(A\vec{v}, \vec{v}) = \cancel{\left({}^tPAP\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\vec{v},\right.}$$

$$\left(P\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}{}^tP\vec{v}, \vec{v}\right)$$

$$= \left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}{}^tP\vec{v}, {}^tP\vec{v}\right)$$

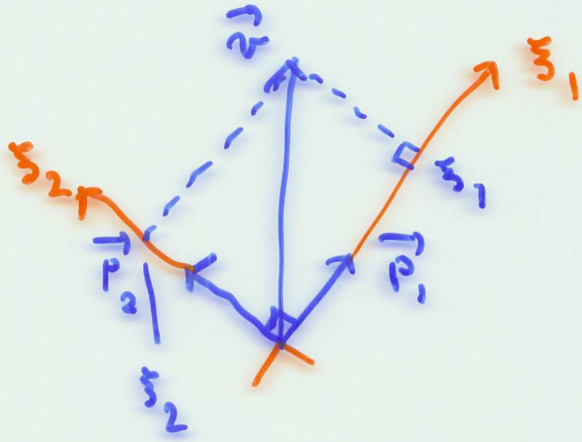
$${}^tP\vec{v} = \vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \qquad \vec{v} = P\vec{\xi}$$

$$= \left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} \alpha\xi_1 \\ \beta\xi_2 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}\right)$$

$$= \alpha\xi_1^2 + \beta\xi_2^2$$

$$\vec{v} = P\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = (\vec{p}_1\ \vec{p}_2)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$
$$= \xi_1\vec{p}_1 + \xi_2\vec{p}_2$$



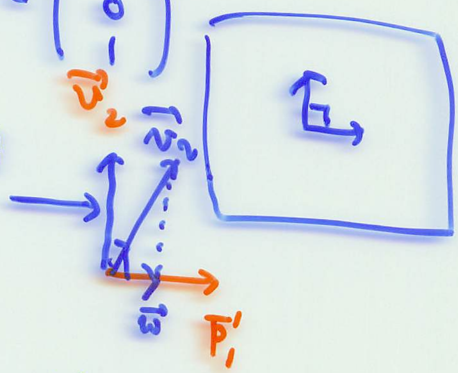
$$A=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix} \qquad \Phi_A(\lambda) = (\lambda-1)^2(\lambda-4)$$

$\underline{V(1)}$ $\quad I_3 - A = \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \to \begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$

$x+y+z=0$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-y-z\\y\\z\end{pmatrix} = y\underbrace{\begin{pmatrix}-1\\1\\0\end{pmatrix}}_{\vec{v}_1} + z\underbrace{\begin{pmatrix}-1\\0\\1\end{pmatrix}}_{\vec{v}_2}$$

$$\vec{p}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$$



$\vec{w}$: $\vec{v}_2$ の $\vec{p}_1$ 方向への正射影.

$$= \left(\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$\vec{p}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\-1\\2\end{pmatrix}$$

$$\vec{v}_2 - \vec{w} = \begin{pmatrix}-1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1\\-1\\2\end{pmatrix}$$

$\underline{V(4)}$ $\quad 4I_3 - A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \xrightarrow[1r \leftrightarrow 3r]{} \begin{pmatrix} -1 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & -1 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow \cdots \leftarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix} \longleftarrow \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & -1 \\ 2 & -1 & -1 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x &= z \\ y &= z \end{aligned} \qquad \begin{array}{r} 2 \ -1 \ -1 \\ -\ )\ 2 \ \ 2 \ -4 \\ \hline \end{array}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{p}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\|\vec{p}_1\| = \|\vec{p}_2\| = \|\vec{p}_3\| = 1$$

$$(\vec{p}_1, \vec{p}_2) = 0 \qquad (\vec{p}_1, \vec{p}_3) = (\vec{p}_2, \vec{p}_3) = 0$$

$\underline{\text{定理}}$ A：対称. α, β 固有値

$\alpha \neq \beta$.

$$\left.\begin{aligned} A\vec{v}_1 &= \alpha \vec{v}_1, \\ A\vec{v}_2 &= \beta \vec{v}_2 \end{aligned}\right. \longrightarrow (\vec{v}_1, \vec{v}_2) = 0$$

$$P = (\vec{p}_1\, \vec{p}_2\, \vec{p}_3)$$

$${}^tPP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leadsto {}^tP = P^{-1}$$

$$P\,{}^tP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AP = P\begin{pmatrix} 1 & & \\ & 1 & \\ & & 4 \end{pmatrix}$$

$${}^tPAP = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 4 \end{pmatrix} \qquad P : 直交$$

$$\Longleftrightarrow \ {}^tPP = P\,{}^tP = I_3$$

定ぎ.

$A = \begin{pmatrix} 3 & -1 & -2 \\ -1 & 3 & 2 \\ -2 & 2 & 6 \end{pmatrix}$ を直交行列で対角化.

$$\Phi_A(\lambda) = (\lambda - 2)^2(\lambda - 8)$$