$z = f(x, y)$ の極値問題.

$f(a, b)$ が極大・極小.

$\Longrightarrow f_x(a, b) = f_y(a, b) = 0$

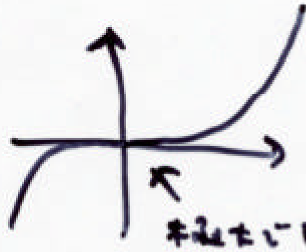
(a, b) は停留点

stationary point

逆は真ならず

1変数　$f(t) = t^3$

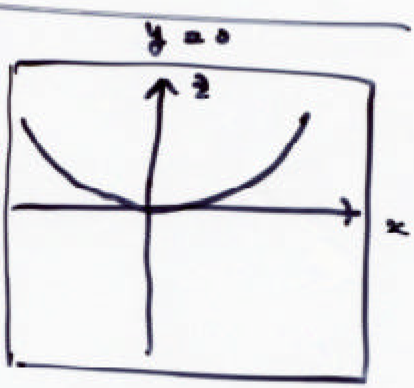
$f'(t) = 3t^2 \quad f'(0) = 0$



2変数.　$z = x^2 - y^2$

$z_x = 2x, \quad z_y = -2y$

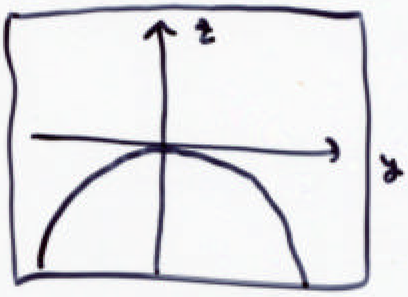
$\rightsquigarrow (0, 0)$ は停留点



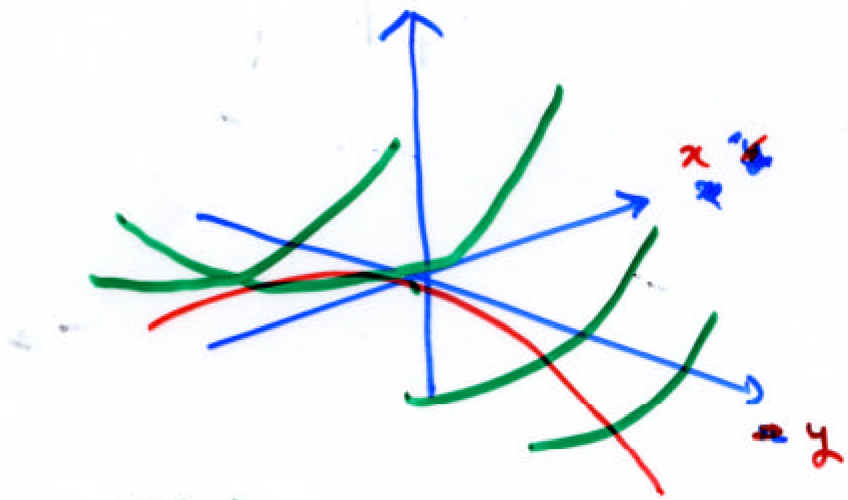
$y = 0$ の断面　$z = x^2$

cross-section

$x = 0$ の断面　$z = -y^2$



$\rightsquigarrow (0, 0)$ は極大でも極小でもない



鞍点
73.

saddle point

le col　　平蔵のコル

1変数に戻る.　94 page

$f : (a, b) \longrightarrow \mathbb{R}$　$\boxed{C^2 級}$

(i) $f'(c) = 0,\ f''(c) > 0$

$\Longrightarrow f(c)$ 極小値.

(ii) $f'(c) = 0,\ f''(c) < 0$

$\Longrightarrow f(c)$ 極大値.

各点 f', f'' が存在し連続

連続 continuous

$\left[\begin{array}{l} f'' : 連続 \\ f''(c) > 0 \end{array}\right]$

$\rightsquigarrow$ 2変数ではどうなるか?

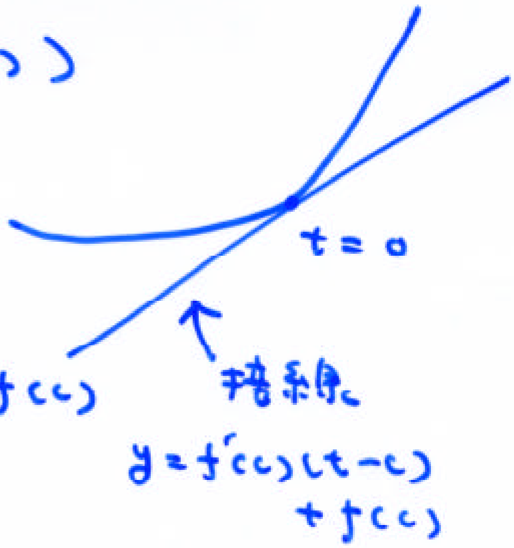
$\longrightarrow f''(t) > 0$

t は c の近く

91 page の定理 4.8

$f : (a, b) \longrightarrow \mathbb{R}$

① $f''(t) > 0 \quad (t \in (a, b))$



$\Longrightarrow$ $f(t) >$ $f'(c)(t - c) + f(c)$
$(t \neq c)$

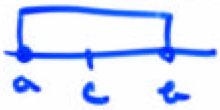
f は下に凸

② $f''(t) < 0 \quad (t \in (a, b))$

$\Longrightarrow$ $f(t) < f'(c)(t - c) + f(c)$
$(t \neq c)$

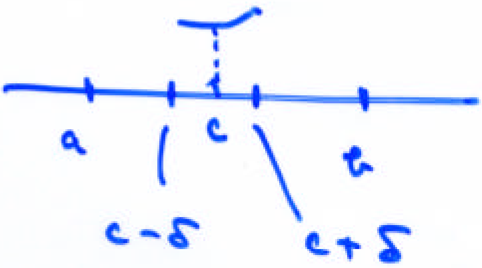
連続関数の性質

$g : (a, b) \longrightarrow \mathbb{R}$ 連続.



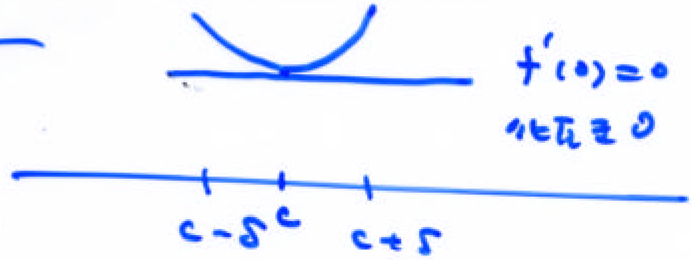
$t = c \quad g(c) > 0$

$\Longrightarrow$ $t = c$ の近くで $g(t) > 0$
$(c - \delta < t < c + \delta)$
となるような $\delta > 0$ がある.



$f''(c) > 0$, f'' : 連続

$\rightsquigarrow$ $f''(t) > 0$ $(c-\delta < t < c+\delta)$



$f(t) > f(c)$ $\begin{pmatrix} t \neq c \\ c-\delta < t < c+\delta \end{pmatrix}$

$f(c)$ は極小.

<u>前提</u>

$f_x(a,b) = f_y(a,b) = 0$

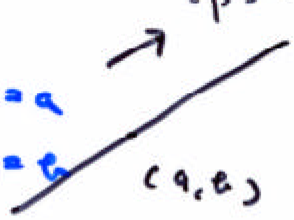
f: C^2 級.

($f, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$ 全て存在し <u>連続</u>

$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \neq \vec{0}$

$x(t) = a + \alpha t$ $\quad x(0) = a$

$y(t) = b + \beta t$ $\quad y(0) = b$



$F(t) = f(x(t), y(t))$

$$F'(t) = f_x(x(t), y(t)) \cdot \boxed{x'(t)}^{=\alpha} + f_y(x(t), y(t)) \cdot \boxed{y'(t)}^{=\beta}$$

$$F'(0) = \boxed{f_x(a,b)}^{=0}\alpha + \underbrace{\boxed{f_y(a,b)}}_{0=}\beta = 0$$

定理 f : C^2級 のとき

$$(f_x)_y = (f_y)_x$$

証明はしない。

$F''(0)$ の符号をみてみる　　$f_{xy} = (f_x)_y$

$$F''(t) = \alpha(f_{xx}\cdot\alpha + f_{xy}\cdot\beta) + \beta(f_{yx}\cdot\alpha + f_{yy}\cdot\beta)$$
$$= \alpha^2 f_{xx} + 2\alpha\beta f_{xy} + \beta^2 f_{yy}.$$

$$\boxed{F''(0) = \alpha^2 f_{xx}(a,b) + 2\alpha\beta f_{xy}(a,b) + \beta^2 f_{yy}(a,b)}$$

定理 定符

$f(x,y) = Ax^2 + 2Cxy + By^2$　$\begin{pmatrix} A & C \\ C & B \end{pmatrix}$ が定める2次形式

$A > 0$ / $A < 0$, $\boxed{AB - C^2} > 0$　($= \det\begin{pmatrix} A & C \\ C & B \end{pmatrix}$)

$\Longrightarrow$ $f(x,y) > 0$ / < 0　$(x,y) \neq (0,0)$

$A = f_{xx}(a,b)$　　$x \longleftrightarrow \alpha$

$B = f_{yy}(a,b)$　　$y \longleftrightarrow \beta$

$C = f_{xy}(a,b)$

定理の証明

$A > 0$

$$* = A\left(x^2 + \frac{2C}{A}xy + \frac{B}{A}y^2\right)$$

$$= A\left(x + \frac{C}{A}y\right)^2 + By^2 - \frac{C^2}{A}y^2$$

$$= \underset{\underset{0}{\vee}}{\boxed{A}}\ \underset{\underset{0}{\vee\!\shortparallel}}{\left(x + \frac{C}{A}y\right)^2} + \underset{\underset{0}{\vee}}{\boxed{\frac{AB - C^2}{A}}}\ \underset{\underset{0}{\vee\!\shortparallel}}{y^2} \geqq 0$$

$$* = 0 \iff x + \frac{C}{A}y = y = 0$$

$$\iff x = y = 0$$

(注)

$$(x, y) \neq (0, 0) \Longrightarrow * > 0$$

2

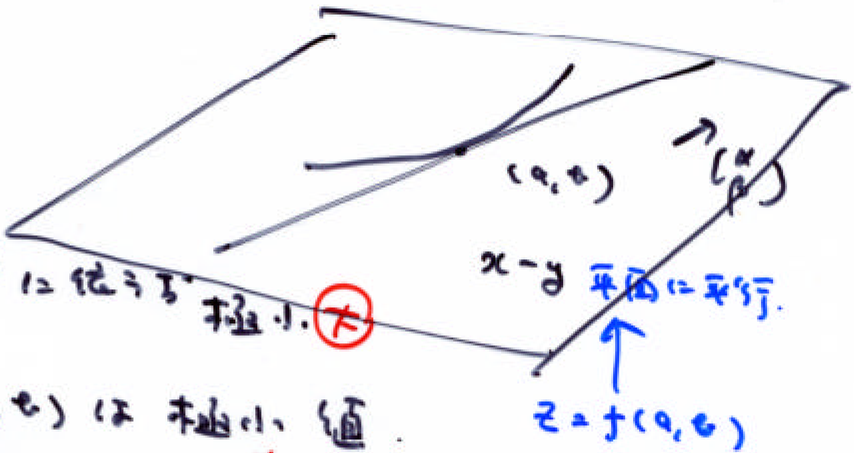
・ $\left\{ F_{xx}(a,b) \underset{<}{>} 0,\quad F_{xx} \cdot F_{yy} - F_{xy}^2 > 0 \right.$

$\Rightarrow \quad F''(0) \underset{<}{>} 0 \quad \left(\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \neq \vec{0} \text{ に依らず}\right)$

・ $F'(0) = 0$

$\boxed{\text{平均値の定理} \\ \text{連続性}}$

$\Rightarrow \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \neq \vec{0}$ に依らず極小 (大)



$f(a, b)$ は極小値
大.

f_{xx} etc. の連続性.

定理

$$f_x(a,b) = f_y(a,b) = 0$$

$$f_{xx}(a,b) > 0$$
$$< 0$$

$$\begin{vmatrix} f_{xx}(a,b) & f_{yx}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix} > 0$$

$$\Rightarrow f(a,b) \text{ 極小, 大}$$

小テスト

・ $z = x^3 + y^3 + 6xy$ 　(0,0)を除く.
　の極大・極小を調べる