$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$ ①

固有値 $\lambda = 1, 6$

$\underline{\lambda = 1}$ $t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $(t \neq 0)$ 固有ベクトル

$\underline{\lambda = 6}$ $t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $(t \neq 0)$ 固有ベクトル

$\vec{v_1} = \begin{pmatrix} +1 \\ -2 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $P = (\vec{v_1}\ \vec{v_2})$

$AP = P\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$, P : 正則（逆行列あり）

$\boxed{A = P\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} P^{-1}}$ 対角化.

A : 対称 $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ ${}^tA = A$

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\begin{pmatrix} a & c \\ c & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

2次形式

$$= \left(\begin{pmatrix} ax+cy \\ cx+by \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$= (ax+cy)x + (cx+by)y$$

$$= ax^2 + 2cxy + by^2$$

$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \rightsquigarrow 5x^2 + 4xy + 2y^2$

$\qquad\qquad\qquad\qquad \shortparallel$

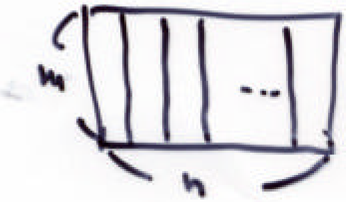
$\qquad\qquad\qquad \xi^2 + 6\eta^2$

変数変換

Keyword 2次形式, 直交行列 ....

一般的な話.

$A : m \times n \quad A = (\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n)$



$\vec{a}_j \in \mathbb{R}^m$

$$A\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n \in \mathbb{R}^m$$

$$\mathbb{R}^n \ni \vec{c} \longmapsto A\vec{c} \in \mathbb{R}^m$$

転置

$${}^t\begin{pmatrix} a & \alpha \\ b & \beta \\ c & \gamma \end{pmatrix} = \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \qquad {}^t\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \begin{pmatrix} a & \alpha \\ b & \beta \\ c & \gamma \end{pmatrix}$$

transposition $3\times 2 \xrightarrow{t} 2 \times 3$

$A : m \times n \longrightarrow {}^tA : n \times m$

$$\llcorner = \begin{pmatrix} {}^t\vec{a}_1 \\ {}^t\vec{a}_2 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix}$$

$$\vec{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{mn} \end{pmatrix} \to {}^t\vec{a}_1 = (a_{11}\ a_{21} \cdots a_{mn})$$

序奏

助走

基本的性質. (i) ${}^t({}^tA) = A$

${}^t(AB)$
$l \times m$

(ii) $A : m \times n,\ B : n \times l$

$\rightsquigarrow AB : m \times l$

${}^tB : l \times n,\ {}^tA : n \times m$

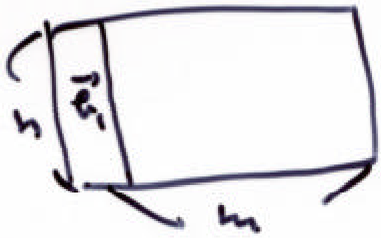
$\boxed{{}^t(AB) = {}^tB\,{}^tA}$

$\rightsquigarrow {}^tB\,{}^tA : l \times m$

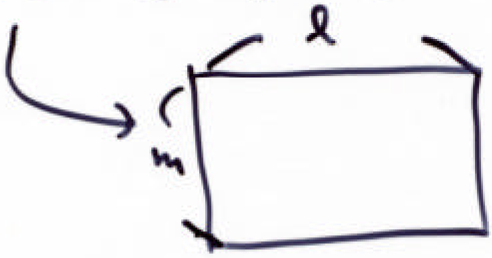
$A : m \times n \quad B : n \times l$

$B = (\vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_l)$

$A\vec{b}_1 \in \mathbb{R}^m$



$AB = (A\vec{b}_1\ A\vec{b}_2\ \cdots\ A\vec{b}_l)$



何故転置を使うか?

$A : m \times n \quad \vec{v} \in \mathbb{R}^n \quad \rightarrow \quad {}^tA : n \times m$

$\rightsquigarrow A\vec{v} \in \mathbb{R}^m, \quad \vec{w} \in \mathbb{R}^m \quad \rightarrow \quad {}^tA\vec{w} \in \mathbb{R}^n$

$$\boxed{(A\vec{v}, \vec{w}) = (\vec{v}, {}^tA\vec{w})}$$

$\mathbb{R}^m$ の内積

(証明) 1° $(\vec{a}, \vec{b}) = {}^t\vec{a}\,\vec{b}$

$$(右) = (a_1\ a_2\ \cdots\ a_k)\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix} = \sum_{k=1}^{n} a_k b_k = (\vec{a}, \vec{b})$$

右 $= (\vec{v}, {}^tA\vec{w})$ $=$

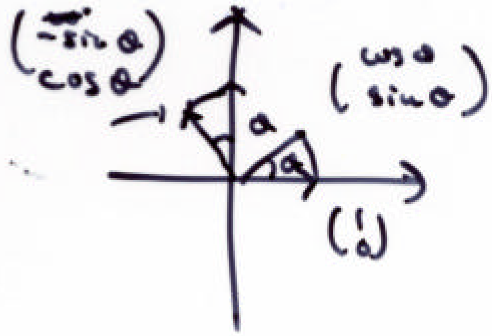
2° $(左) = {}^t(A\vec{v})\vec{w} = ({}^t\vec{v}\ {}^tA)\vec{w} \underset{\text{結合則}}{=} {}^t\vec{v}({}^tA\vec{w})$

$$P = \left(\boxed{\begin{matrix}\cos\theta \\ \sin\theta\end{matrix}} \; \boxed{\begin{matrix}-\sin\theta \\ \cos\theta\end{matrix}} \right)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto P\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$



$$\left(P\begin{pmatrix} x \\ y \end{pmatrix}, P\begin{pmatrix} x' \\ y' \end{pmatrix} \right) = \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right)$$

$${}^tPP = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{= P}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$P\,{}^tP = I_2$ ← 各自計算する.

$$(P\vec{v}, P\vec{w}) = (\vec{v}, \underbrace{{}^tPP}_{= I_2}\vec{w}) = (\vec{v}, \vec{w})$$

$${}^t({}^tP) = P$$

$$({}^tP\vec{v}, {}^tP\vec{w}) = (\vec{v}, \underbrace{P\,{}^tP}_{= I_2}\vec{w}) = (\vec{v}, \vec{w})$$

P, ${}^tP$ は 内積を保つ

$P\,{}^tP = I_2$, ${}^tPP = I_2$ が成立したら OK.

$$A=\begin{pmatrix}5&2\\2&2\end{pmatrix}$$

$$\vec{v}_1=\begin{pmatrix}1\\-2\end{pmatrix},\quad \vec{v}_2=\begin{pmatrix}2\\1\end{pmatrix}\qquad \boxed{(\vec{v}_1,\vec{v}_2)=0}$$

↑ 一般的に成立

$$\vec{p}_1=\frac{1}{\sqrt5}\begin{pmatrix}1\\-2\end{pmatrix},\quad \vec{p}_2=\frac{1}{\sqrt5}\begin{pmatrix}2\\1\end{pmatrix}$$ 固有ベクトル

$$P=(\vec{p}_1\ \vec{p}_2)=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$$

$$\cos\theta=\frac{1}{\sqrt5},\quad \sin\theta=-\frac{2}{\sqrt5}$$

$$AP=P\begin{pmatrix}1&0\\0&6\end{pmatrix}\ \longrightarrow\ \underline{{}^tPAP=\begin{pmatrix}1&0\\0&6\end{pmatrix}}$$

$$\left(A\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right)=\left({}^tPA\begin{pmatrix}x\\y\end{pmatrix},\ {}^tP\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$=\left({}^tPA\underbrace{P\,{}^tP}_{=I_2}\begin{pmatrix}x\\y\end{pmatrix},\ {}^tP\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$\left.\begin{array}{l}{}^tPP=P\,{}^tP=I_2.\\ {}^tP=P^{-1}\end{array}\right\}$$ に注意

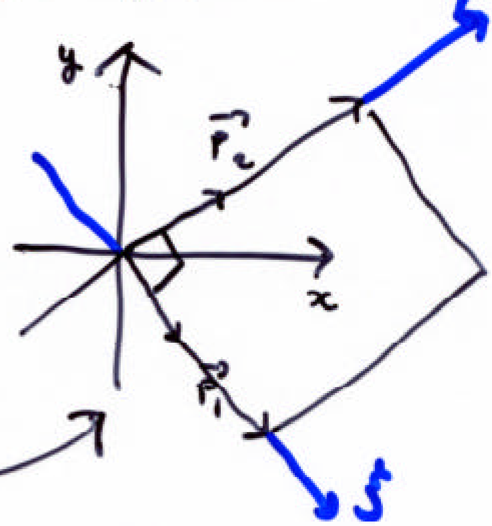
$$\begin{pmatrix}\xi\\\eta\end{pmatrix}={}^tP\begin{pmatrix}x\\y\end{pmatrix}\qquad =\left(\begin{pmatrix}1&0\\0&6\end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix},\begin{pmatrix}\xi\\\eta\end{pmatrix}\right)$$

$$\begin{pmatrix}x\\y\end{pmatrix}=P\begin{pmatrix}\xi\\\eta\end{pmatrix}\qquad =\left(\begin{pmatrix}\xi\\6\eta\end{pmatrix},\begin{pmatrix}\xi\\\eta\end{pmatrix}\right)=\xi^2+6\eta^2$$

$$=(\vec{p}_1\ \vec{p}_2)\begin{pmatrix}\xi\\\eta\end{pmatrix}$$

$$=\xi\vec{p}_1+\eta\vec{p}_2$$

$$\boxed{\begin{pmatrix}x\\y\end{pmatrix}=\xi\vec{p}_1+\eta\vec{p}_2}$$



$\vec{p}_1$ 方向が ξ 軸

$\vec{p}_2$ 方向が η 軸

テスト

対称行列 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ が定める

2次形式

$$\left(A \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

を簡略化する.

(A を回転の行列 P で対角化する)