06月23日

(1) 関数

$$z = x^2 + xy + y^2 - 4x - 2y$$

の停留点を求めてください。

$\hookrightarrow z_x = z_y = 0$

$$\begin{aligned} z_x &= 2x + y - 4 = 0 \\ z_y &= x + 2y - 2 = 0 \end{aligned}$$

を解いて$x = 2,\ y = 0$を得ます。停留点は

$$(x, y) = (2, 0)$$

です。

(2) 関数

$$z = xy(2 - x - y)$$

を求めてください。

の停留点

$$\begin{aligned} z_x &= y\{1 \cdot (2 - x - y) + x \cdot (-1)\} \\ &= y(2 - 2x - y) = 0 \\ z_y &= x\{1 \cdot (2 - x - y) + y \cdot (-1)\} \\ &= x(2 - x - 2y) = 0 \end{aligned}$$

$$AB = 0 \iff A = 0 \text{ or } B = 0$$

が求める条件です。ここで

$$z_x = 0 \Leftrightarrow y = 0 \text{ or } 2x + y = 2$$

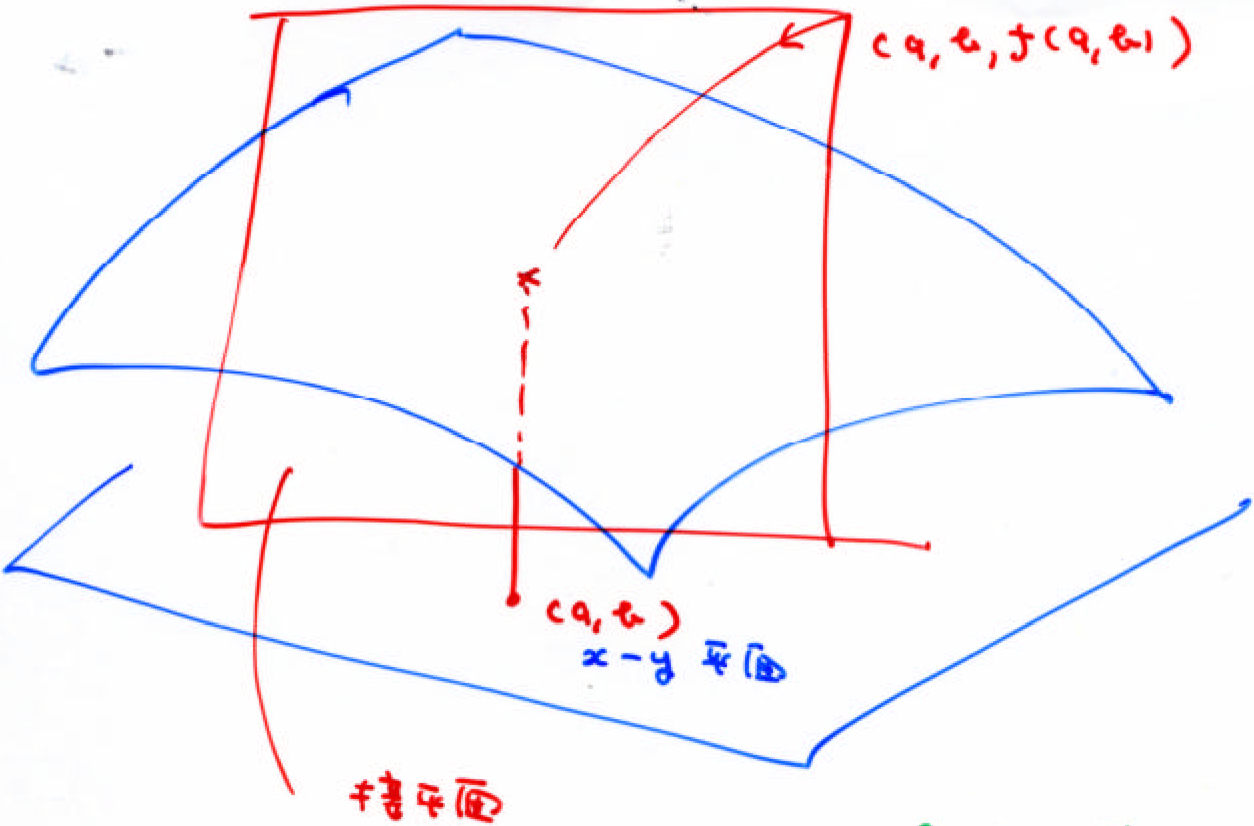
$$z_y = 0 \Leftrightarrow x = 0 \text{ or } x + 2y = 2$$

ですから

$$\begin{aligned}
& z_x = z_y = 0 \\
\Leftrightarrow\ & y = x = 0 \text{ or } (y = 0 \text{ and } x + \overbrace{2y}^{=0} = 2) \\
& \quad \text{or } (x = 0 \text{ and } \underbrace{2x}_{=0} + y = 2) \\
& \qquad \text{or } (2x + y = 2 \text{ and } x + 2y = 2) \\
\Leftrightarrow\ & (x, y) = (0, 0) \text{ or } (x, y) = (2, 0) \\
& \quad \text{or } (x, y) = (0, 2) \text{ or } (x, y) = \left(\frac{2}{3}, \frac{2}{3}\right)
\end{aligned}$$

を得ます。

$z = f(x, y)$



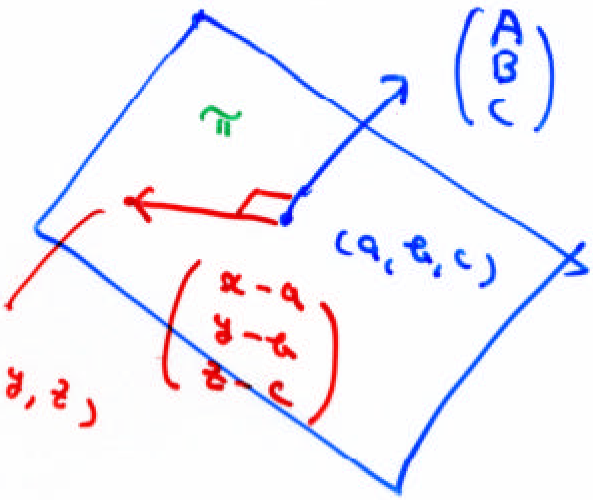
平面
plane

$\pi \leftrightarrow p$

Π
product

normal direction

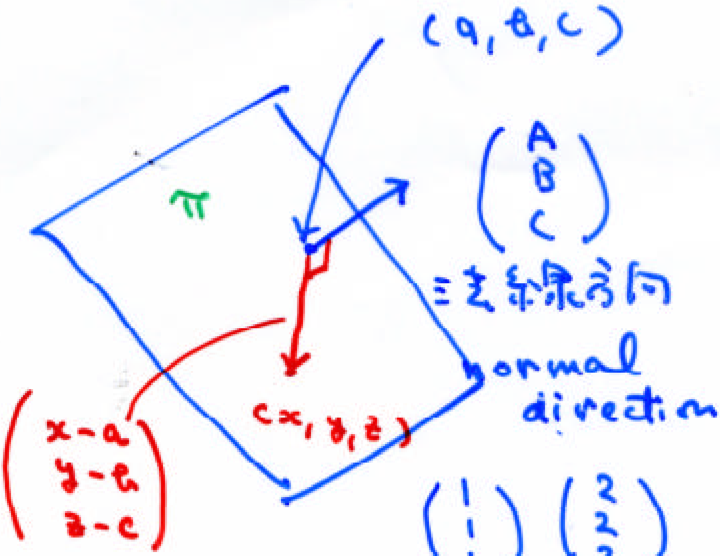
法線方向



平面の方程式

$$\begin{pmatrix}x-a\\y-b\\z-c\end{pmatrix}\cdot\begin{pmatrix}A\\B\\C\end{pmatrix}=0$$

内積
(inner product)



$\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\begin{pmatrix}2\\2\\2\end{pmatrix}$

poser une question.

$$\pi:\ A(x-a)+B(y-b)+C(z-c)=0$$

(例) $-2x+3y+4z=3$

$\begin{pmatrix}-2\\3\\4\end{pmatrix}$ が法線方向

接平面を考えるときは $C\neq0$ としてよい.

$C=0$ $\quad \begin{pmatrix}A\\B\\0\end{pmatrix} \mathbin{/\!/}$ x-y平面

$\rightsquigarrow$ $\pi \perp$ x-y平面

$C\neq0$ だから

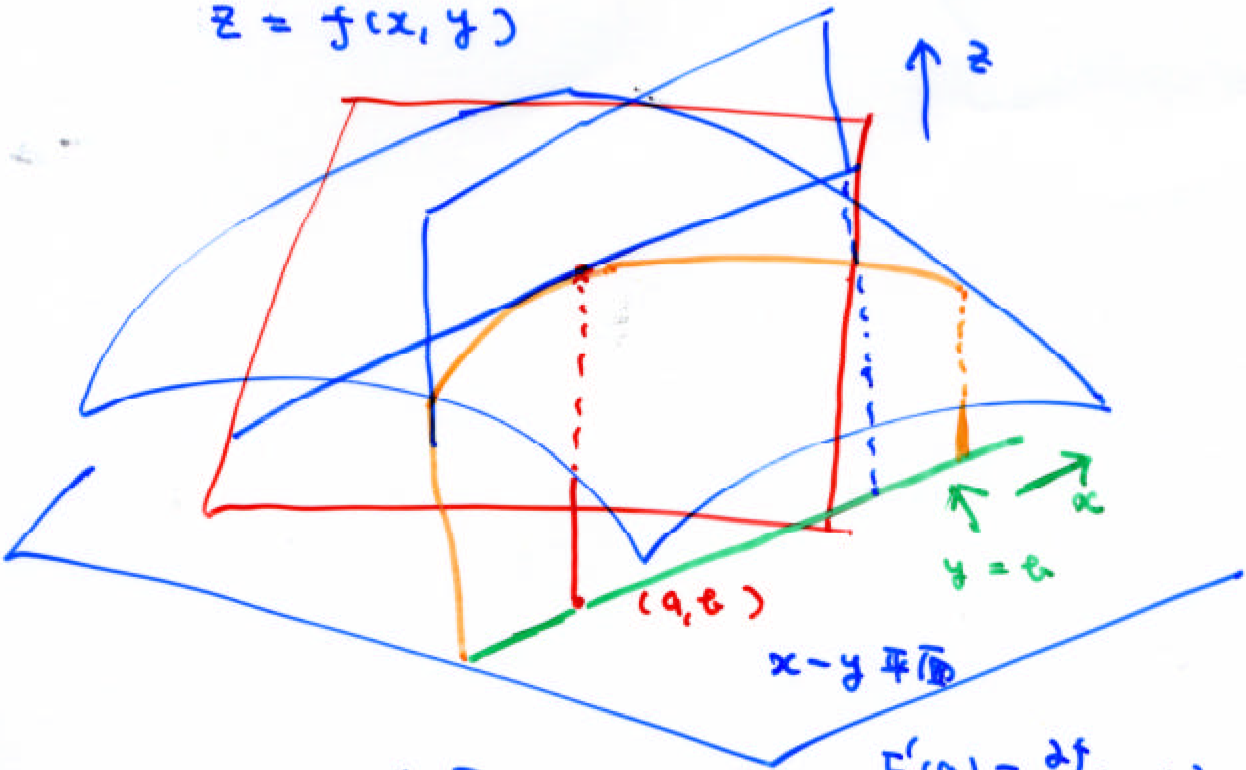
$$z=\underbrace{-\frac{A}{C}}_{=\alpha}(x-a)\underbrace{-\frac{B}{C}}_{=\beta}(y-b)+c$$

$$\boxed{z=\alpha(x-a)+\beta(y-b)+c}$$

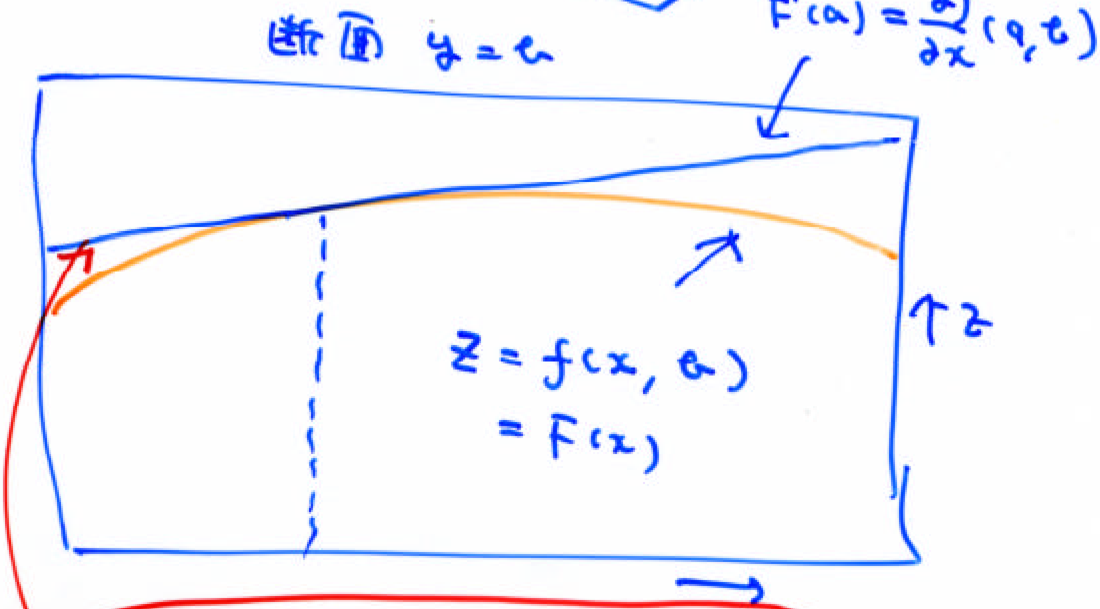
として α,β を求める.

接平面 tangential plane.

$z = f(x, y)$



断面 $y = b$



接平面と$\{y = b\}$の交わり.

接平面

$$z = \alpha(x-a) + \beta(\underbrace{y-b}_{y=b}) + \underbrace{\boxed{f(a,b)}}_{=\,c}$$

$(a, b, f(a,b))$ を通る.

$y = b$ の断面

$$z = \alpha(x-a) + f(a,b)$$

傾き

$$\boxed{\alpha = \frac{\partial f}{\partial x}(a,b)}$$

同様に $\{x = a\}$ の断面を考えると

$$\boxed{\beta = \frac{\partial f}{\partial y}(a,b)}$$

公式

$$\boxed{z = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + f(a,b)}$$

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例1 $z = xy$ at $(1, 1, 1)$

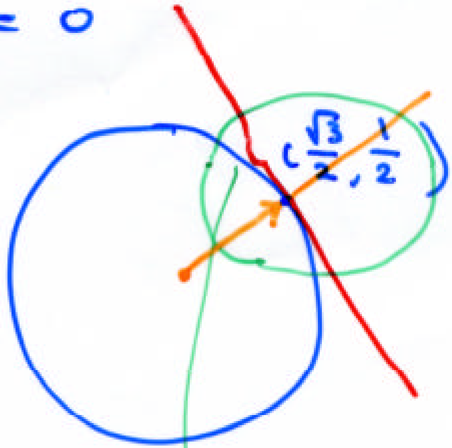
$z_x = y, \ z_y = x$

$(1,1)$

$\rightsquigarrow z_x = 1, \ z_y = 1$

$z = 1(x-1) + 1(y-1) + 1$

134 $g(x,y) = x^2 + y^2 - 1 = 0$



接線の方程式

$$x^2 + \varphi(x)^2 - 1 = 0$$

恒等的に成立

$\Phi \quad \varphi$

$$2x + 2\varphi(x)\varphi'(x) = 0$$

$x = \frac{\sqrt{3}}{2}$ を代入

$$2 \times \frac{\sqrt{3}}{2} + 2\underbrace{\varphi\left(\frac{\sqrt{3}}{2}\right)}_{=\frac{1}{2}} \cdot \varphi'\left(\frac{\sqrt{3}}{2}\right) = 0$$

$$\sqrt{3} + \varphi'\left(\frac{\sqrt{3}}{2}\right) = 0 \rightsquigarrow \boxed{\varphi'\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}}$$

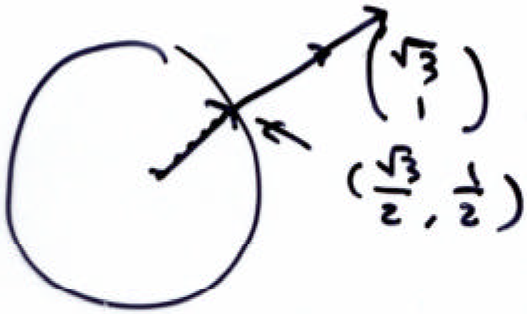
$$y = \varphi(x) = \sqrt{1-x^2}$$

$$y' = \frac{1}{2}\frac{1}{\sqrt{1-x^2}}(-2x) = \frac{-x}{\sqrt{1-x^2}}$$

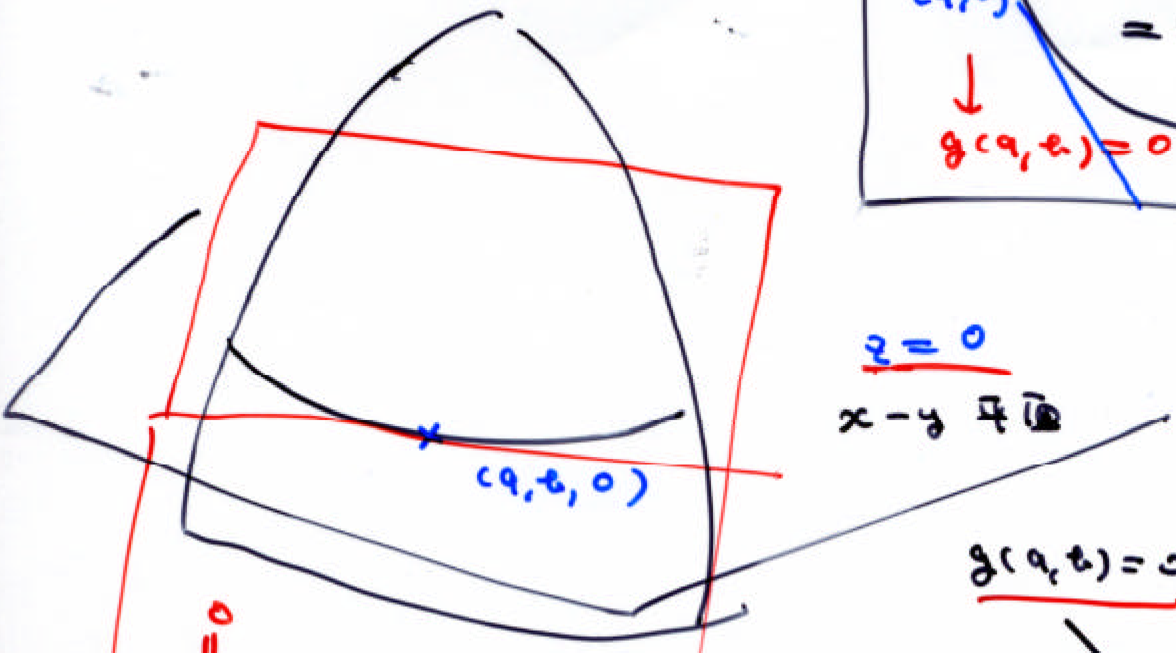
$g = x^2 + y^2 - 1 = 0$

$g_x = 2x, \quad g_y = 2y.$ $\quad \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$$2 \times \frac{\sqrt{3}}{2}\left(x - \frac{\sqrt{3}}{2}\right) + 2 \times \frac{1}{2}\left(y - \frac{1}{2}\right) = 0$$



$z = g(x, y)$



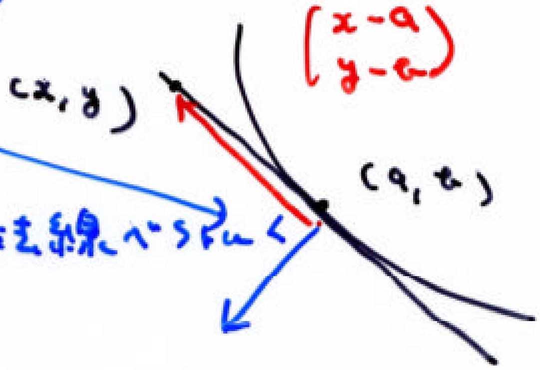
$$\underset{\substack{0\\ \parallel}}{z} = g_x(a, b)(x - a) + g_y(a, b)(y - b) + 0$$

接線 = 接平面と x − y 平面の交わり.

$$\boxed{g_x(a, b)(x - a) + g_y(a, b)(y - b) = 0}$$

$$\begin{pmatrix} g_x(a, b) \\ g_y(a, b) \end{pmatrix} \cdot \begin{pmatrix} x - a \\ y - b \end{pmatrix} = 0$$

nabla ‖
$\nabla(g)(a, b)$
勾配ベクトル　法線ベクトル



$$g = x^2 + y^2 - 1 = 0 \qquad \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$g_x = 2x, \quad g_y = 2y$$

$$g_x = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}, \quad g_y = 2 \times \frac{1}{2} = 1$$

$$\boxed{\sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(y - \frac{1}{2}\right) = 0}$$

$$g_x(a,b)(x-a) + \underset{\neq 0 \text{ とする.}}{g_y}(a,b)(y-b) = 0$$

$$y = -\frac{g_x(a,b)}{g_y(a,b)}(x-a) + b.$$

傾きは $\quad -\dfrac{g_x(a,b)}{g_y(a,b)}$

上の例 $\quad = -\dfrac{\sqrt{3}}{1} = -\sqrt{3}.$

I $z = x^4 + y^4 - 4(x-y)^2$ の
停留点を求める.

II $z = x e^{x+y}$ の $(0,0)$ における
接平面.

$e^x \cdot e^y$

III $x^2 - xy + y^2 - 3 = 0$ の
$(1,2)$ における接線

I $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$x^2 - xy + y^2 = (x - \frac{1}{2}y)^2 + \frac{3}{4}y^2$

$x^2 - xy + y^2 = 0 \iff x = \frac{1}{2}y,\ y = 0$

$\iff x = y = 0$