The Core Equals Strong Equilibria in a Public Goods Economy

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We show that in a public goods economy with a linear production technology, the core is precisely the set of allocations at strong equilibria of a certain strategic game. The game is simple enough and involves virtually no specific mechanism, describing what and how much commodities players are willing to transfer each other and to contribute to public goods provision, which is just a concise description of a procedure to arrive at an allocation in such an economy. This result suggests that the two solution concepts can be in fact very close each other in such an economic situation.
JEL Classification Numbers: C7, H41.

Key Words: core, strong equilibrium, public goods economy, β-core

1. INTRODUCTION

As is well known, the core of a game is a solution concept that implies the situation at which no coalition of players can make itself better off. The strong equilibrium of a strategic game, on the other hand, also describes a similar situation *given that the strategies of the complementary coalition are held fixed* (Aumann, 1959). The two concepts are thus closely related; so that one might consider that the existence is also closely related each other. However, a general condition for the existence of a strong equilibrium is not known; whereas, that of the core is well known (Shapley,1967; Scarf,1967,1971). In fact, as Bernheim, Peleg and Whinston (1987) note, the strong equilibrium is almost always nonexistent, which was one of the motivations for developing the concept of coalition-proof Nash equilibria.

In the literature, however, there can be found several studies dealing with strong equilibria applied to specific socio-economic situations. Some examples are Kalai, Postlewaite and Roberts (1979), Peleg (1984), Greenberg and Weber (1983), or more recent Nishihara (1999). Among others, Kalai, Postlewaite and Roberts (1979) (KPR,for short), is one of the earliest applications to an economy with public goods. They showed that the core of that economy coincides with the set of allocations at strong equilibria of a game defined on the economy under an explicit mechanism of coalition formation. This of course implies that the game has a strong equilibrium because the core of the economy is nonempty.

In this paper, we shall further proceed with showing the equivalence of the core and the strong equilibria under the same economic environment; but in a more primitive level than KPR and any other similar results in the mechanism design literature. For this purpose, we define a simple strategic game with virtually no specific rules or mechanisms to obtain outcomes. In our game, rather than proposing final allocations, players choose the strategies prescribing what and how much goods to transfer each other and contribute to the public goods provision under the resource constraint. This is just a concise description of a procedure to arrive at an allocation in this economy.

In the special case where the production possibility is absent, the game reduces to a slightly modified version of the pure exchange strategic game appeared in Scarf (1971) (see also Mas-Colell,1985). In this sense, our game is a natural extension of the pure exchange game to that of a production economy with or without public goods, which would in turn suggest that the core and the strong equilibria can be in fact very close each other in these typical economic situations. These results further imply that another core concept, the β-core of the games corresponding to these economies are nonempty because the strong equilibria are contained in the β-core by definition.

2. THE MODEL

We consider a public goods economy consisting of n consumers, m private goods and l public goods. Let $N = \{1, ..., n\}$ be a finite set of agents. Each agent $i \in N$ is characterized by (X_i, u_i, ω_i), where $X_i \subseteq \mathbb{R}^m_+ \times \mathbb{R}^l_+$ is i's consumption set, $u_i : X_i \to \mathbb{R}$ is i's utility function, and $\omega_i \in \mathbb{R}^m_+$ is i's initial endowment of goods. We denote by $(x_i, y_i) \in X_i$ a commodity bundle of agent i. The first components $x_i = (x_i^1, ..., x_i^m)$ are private goods and the second components $y_i = (y_i^1, ..., y_i^l)$ are public goods. The available technology is represented by the production possibility set $Y \subseteq \mathbb{R}^{m+l}$. Positive components of elements in Y are outputs, while inputs are negative. Formally, a public goods economy is defined as $(N, X_i, u_i, \omega_i, Y)$. We make the usual assumption that

a. $(\omega_i, 0) \in X_i$,

b. if for all $y, \bar{y} \in \mathbb{R}^l_+$, $(x, y) \in X_i$ and $\bar{y} \geq y$ then $(x, \bar{y}) \in X_i$, and
c. Y is additive [i.e. $\forall z, z' \in Y \Rightarrow z + z' \in Y$] and $(0, 0) \in Y$.

In this paper, we denote by $((x_1, y_1), ..., (x_n, y_n))$ the generic element of $\prod_{i \in N} X_i$. Nonempty subsets of N are called coalitions. For each coalition S, $(x_i, y_i)_{i \in S}$ and $(x_i, y_i)_{i \notin S}$ will be denoted by (x_S, y_S) and (x_{-S}, y_{-S}), respectively. Further, $(x_S(i), y_S(i))$ or $(x_S, y_S)(i)$ will be the projection of (x_S, y_S) onto X_i. For each coalition S, we define a subset of $\prod_{i \in N} X_i$ as

$$\widehat{X}_S = \left\{ (x_S, y_S) \in \prod_{i \in S} X_i \;\middle|\; \begin{array}{l} 1.\ \forall i \in S,\ y_i = y, \\ 2.\ \forall i \in S,\ (x_i, y_i) \in X_i, \\ 3.\ (\sum_{i \in S} x_i, y) \in Y + \{\sum_{i \in S} \omega_i\}. \end{array} \right\},$$

which will be called the set of S-feasible allocations. Let $\mathcal{T} \subset 2^N \setminus \{\emptyset\}$ be the set of all permissible coalitions.

Definition 2.1. The core of the public goods economy is defined as

$$C(\mathcal{T}) = \left\{ (x^*, y^*) \in \prod_{i \in N} X_i \;\middle|\; \begin{array}{l} 1.\ \neg\Big(\exists S \in \mathcal{T}, \exists (x_S, y_S) \in \widehat{X}_S : \\ \quad \forall i \in S,\ u_i(x_S(i), y_S(i)) > u_i(x_i^*, y_i^*)\Big) \\ 2.\ \forall S \in \mathcal{T}, (x_S^*, y_S^*) \in \widehat{X}_S \end{array} \right\}.$$

The set $\mathcal{T}$ captures the stability requirements. When $\mathcal{T} = \{\{1\}, ..., \{n\}\}$, the set of individually rational allocations equals $C(\mathcal{T})$. When $\mathcal{T}$ consists of all coalitions, $C(\mathcal{T})$ equals the core usually defined.

We now define the core and the strong equilibria in the public goods economy. To do so, we construct a strategic game corresponding to this economy. Let S_i be the strategy set of agent i given by

$$S_i = \left\{ (t_i, r_i) \in \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times l}_+ \;\middle|\; \begin{array}{l} 1.\ \forall h = 1, ..., m, \sum_{j \in N} t_{ij}^h = x_i^h \\ 2.\ (x_i, r_i) \in Y + \{\omega_i\} \end{array} \right\}.$$

We interpret that t_{ij}^h is i's transfer of the private good h to agent or player j, and r_i describes how much public goods player i is willing to contribute to their provision. Note that t_{ij}^h is not assumed nonnegative: when $t_{ij}^h < 0$, this means that player i is requesting player j of good h.

Any strategy combination $s \in \prod_{i \in N} S_i$ generates an allocation $a(s)$ given by

$$a(s) = \begin{cases} \Big((\sum_{j \in N} t_{j1}, \sum_{j \in N} r_j), ..., (\sum_{j \in N} t_{jn}, \sum_{j \in N} r_j)\Big) & \\ \qquad\qquad\qquad\qquad \text{if this belongs to } \prod_{i \in N} X_i, \\ (\omega_1, 0, ..., \omega_n, 0) \qquad\qquad\qquad \text{otherwise,} \end{cases}$$

which is our allocation rule.

For each strategy s, player or agent i obtains utility $v_i(s_1, ..., s_n) = u_i(a(s)(i))$ through the allocation $a(s)$. Furthermore, let $s_S = (s_i \in S_i | i \in S)$ be the strategies of a coalition S and $s_{-S} = (s_i \in S_i | i \notin S)$ be the strategies of the complementary coalition $N \setminus S$. Thus, the strategic game for the public goods economy is characterized by $G = (N, S_i, v_i)$.

Definition 2.2. The strong equilibrium of G is defined as

$$SE(\mathcal{T}) = \left\{ s^* \in \prod_{i \in N} S_i \;\middle|\; \begin{array}{r} \neg\Big(\exists S \in \mathcal{T}, \exists s_S \in \prod_{i \in S} S_i : \\ \forall i \in S, v_i(s_S, s^*_{-S}) > v_i(s^*)\Big) \end{array} \right\}.$$

The set $\mathcal{T}$ captures the strength of the strong equilibrium. When $\mathcal{T} = \{\{1\}, ..., \{n\}\}$, $SE(\mathcal{T})$ is the set of Nash equilibria. When $\mathcal{T}$ consists of all coalitions, $SE(\mathcal{T})$ equals the strong equilibria usually defined.

3. RESULTS

Before stating our main results, we just note that any strategy combination of the game generates an N-feasible allocation in the economy.

Remark 3. 1. $\forall s \in S,\ a(s) \in \widehat{X}_N$

Proof. By the definition of the allocation rule and the assumption (c), we can obtain $(\sum_{i \in N} \omega_i, 0) \in Y + \{\sum_{i \in N} \omega_i\}$. Moreover, noting Y is additive, if for all $i \in N$, $(x_i, r_i) \in Y + \{\omega_i\}$ then $(\sum_{i \in N} x_i, \sum_{i \in N} r_i) \in Y + \{\sum_{i \in N} \omega_i\}$. By the definition of S_i, we can conclude that

$$(\sum_{i \in N} \sum_{j \in N} t_{ij}, \sum_{i \in N} r_i) \in Y + \{\sum_{i \in N} \omega_i\}.$$

Hence, $a(s) \in \widehat{X}_N$. ∎

Proposition 3.1. *If $N \in \mathcal{T}$ and $(x^*, y^*) \in C(\mathcal{T})$, then the strategy combination $s(x^*, y^*)$, which is generated by (x^*, y^*), belongs to $SE(\mathcal{T})$.*

Proof. It follows from $(x^*, y^*) \in C(\mathcal{T})$ and $N \in \mathcal{T}$ that $y_i^* = y^*$ for all $i \in N$ and $(\sum_{i \in N} x_i^*, y^*) \in Y + \{\sum_{i \in N} \omega_i\}$. Since Y is additive, for each $i \in N$ we can choose a pair (x_i, y_i) satisfying $(x_i, y_i) \in Y + \{\omega_i\}$ and $(\sum_{i \in N} x_i, \sum_{i \in N} y_i) = (\sum_{i \in N} x_i^*, y^*)$.

We define a strategy $s(x^*, y^*)$ in the following way. For all $i, j \in N$, $h = 1, ..., m$ and $k = 1, ..., l$,

$$t_{ij}^h := \frac{x_i^h}{\sum_{i \in N} x_i^{*h}} x_j^{*h} \text{ and } r_i^k := y_i^k. \tag{1}$$

Then, for all $i \in N$ and $h = 1, ..., m$

$$\sum_{j \in N} t_{ij}^h = x_i^h.$$

Hence, $(t_i, r_i) \in S_i$ for all $i \in N$. Moreover, for all $j \in N$, $h = 1, ..., m$ and $k = 1, ..., l$

$$\sum_{i \in N} t_{ij}^h = x_j^{*h} \text{ and } \sum_{i \in N} r_i^k = y^{*k}.$$

We can conclude $a(s(x^*, y^*))(i) = (\sum_{j \in N} t_{ji}, \sum_{j \in N} r_j) \in X_i$.

Suppose $s(x^*, y^*) \notin SE(\mathfrak{T})$. That is,

$$\exists S \in \mathfrak{T}, \exists s_S \in \prod_{i \in S} S_i : \forall i \in S, v_i(s_S, s(x^*, y^*)_{-S}) > v_i(s(x^*, y^*)).$$

By the definition of v_i,

$$\forall i \in S, \; u_i(a(s_S, s(x^*, y^*)_{-S})(i)) > u_i(a(s(x^*, y^*))(i)) = u_i(x_i^*, y^*).$$

This contradicts the assumption that $(x^*, y^*) \in C(\mathfrak{T})$. ▮

Proposition 3.1 shows that the strategy combination generated by the core allocation is in the strong equilibria.

Proposition 3.2. *Suppose u_i is monotone increasing in $y \in \mathbb{R}_+^l$ for all $i \in N$. If $s^* \in SE(\mathfrak{T})$, then $a(s^*) \in C(\mathfrak{T})$.*

Proof.
Suppose $a(s^*) = ((\sum_{j \in N} t_{j1}^*, \sum_{j \in N} r_j^*), ..., (\sum_{j \in N} t_{jn}^*, \sum_{j \in N} r_j^*)) \notin C(\mathfrak{T})$. That is,

$$\exists S \in \mathfrak{T}, \exists (x_S, y_S) \in \widehat{X}_S : \forall i \in S, u_i(x_S(i), y_S(i)) > u_i(\sum_{j \in N} t_{ji}^*, \sum_{j \in N} r_j^*),$$

where $y_S(i) = y_S$ for all $i \in S$. Noting that $(x_S, y_S) \in \widehat{X}_S$ and Y is additive, we can choose a pair $(\tilde{x}_i, \tilde{y}_i) \in Y + \{\omega_i\}$ for all $i \in S$ satisfying

$\sum_{i\in S}\tilde{x}_i = \sum_{i\in S} x_S(i)$ and $\sum_{i\in S}\tilde{y}_i = y_S$. We define a strategy $(t,r)\in\prod_{i\in S} S_i$ using the allocation $a(s^*)$ and (x_S, y_S). For all $i\in S$, $j\in N$, $h=1,...,m$ and $k=1,...,l$, define:

$$t_{ij}^h := \frac{\tilde{x}_i^h}{\sum_{j\in S} x_S^h(j)}(x_S^h(j) - \sum_{k\in N\setminus S} t_{kj}^{*h}),\ \text{and}\ r_i^k := \tilde{y}_i^k, \tag{2}$$

where $x_S^h(j) = \sum_{i\in N} t_{ji}^{*h}$ for all $j\in N\setminus S$. For all $i\in N\setminus S$, $j\in N$, $h=1,...,m$ and $k=1,...,l$, let $r_i^k := r_i^{*k}$ and $t_{ij}^h := t_{ij}^{*h}$.

Then, for all $i\in S$ and for all $h=1,...,m$,

$$\begin{aligned}
\sum_{j\in N} t_{ij}^h &= \frac{\tilde{x}_i^h}{\sum_{j\in S} x_S(j)}\left(\sum_{j\in S} x_S^h(j) + \sum_{j\in N\setminus S} x_S^h(j) - \sum_{i\in N\setminus S}\sum_{j\in N} t_{ij}^{*h}\right)\\
&= \frac{\tilde{x}_i^h}{\sum_{j\in S} x_S(j)}\left(\sum_{j\in S} x_S^h(j) + \sum_{j\in N\setminus S}\sum_{i\in N} t_{ji}^{*h} - \sum_{i\in N\setminus S}\sum_{j\in N} t_{ij}^{*h}\right)\\
&= \tilde{x}_i^h.
\end{aligned}$$

Hence, we can obtain $s_i = (t_i, r_i)\in S_i$ for all $i\in S$. Furthermore, for all $j\in N$,

$$\sum_{i\in S} t_{ij}^h + \sum_{i\in N\setminus S} t_{ij}^{*h} = x_j^h.$$

By the definition of v_i, for each $i\in S$

$$\begin{aligned}
v_i(s_S, s_{-S}^*) &= u_i(a(s_S, s_{-S}^*)(i))\\
&= u_i\Big(\sum_{j\in S} t_{ji} + \sum_{j\in N\setminus S} t_{ji}^*, \sum_{j\in N} r_j\Big)\\
&= u_i\Big(\sum_{j\in S} t_{ji} + \sum_{j\in N\setminus S} t_{ji}^*, \sum_{j\in S}\tilde{y}_j + \sum_{j\in N\setminus S} r_j^*\Big)\\
&\qquad\qquad (\text{definition of } r_j)\\
&= u_i\Big(x_S(i), y_S + \sum_{j\in N\setminus S} r_j^*\Big) \qquad (\text{definition of } \tilde{y}_i)\\
&\geq u_i(x_S(i), y_S) \qquad (y_S = y_S(i) \text{ and monotonicity})\\
&> u_i\Big(\sum_{j\in N} t_{ji}^*, \sum_{j\in N} r_j^*\Big) \qquad (a(s^*)\notin C(\mathfrak{T}))\\
&= u_i(a(s^*)(i))\\
&= v_i(s^*).
\end{aligned}$$

This is a contradiction to the assumption that $s^* \in SE(\mathfrak{T})$. ∎

Proposition 3.2 shows that the allocation generated by the strong equilibrium strategy is in the core. When $\mathfrak{T} = \{\{1\}, ..., \{n\}\}$, just as in KPR this assertion implies that a Nash equilibrium generates an individually rational allocation.

We summarize the results obtained so far in the following corollary.

Corollary 3.1. *Suppose u_i is monotone increasing in $y \in \mathbb{R}^l_+$ for all i. If $s^* \in SE(2^N)$ then $a(s^*) \in C(2^N)$. Conversely, if $(x^*, y^*) \in C(2^N)$ then the strategy generated by (x^*, y^*) is a strong equilibrium.*

A production economy (without public goods) and a pure exchange economy are described as special cases of the public goods economy. A production economy is characterized by $(N, X_i, u_i, \omega_i, Y)$, where $X_i \subseteq \mathbb{R}^m_+$ is i's consumption set of private goods, $u_i : X_i \to \mathbb{R}$ is i's utility function, $\omega_i \in X_i$ is i's initial endowment of goods, and $Y \subseteq \mathbb{R}^m$ is the production possibility set of private goods. The strategic game corresponding to the production economy is given by (N, S_i', v_i), where

$$S_i' = \left\{ t_i \in \mathbb{R}^{n\times m} \;\middle|\; \begin{array}{l} 1.\ \forall h = 1, ..., m, \sum_{j\in N} t_{ij}^h = x_i^h \\ 2.\ x_i \in Y + \{\omega_i\} \end{array} \right\}.$$

Note that the variable r_i is eliminated, since public goods are not produced in this economy.

A pure exchange economy is obtained as the special case of the production economy where $Y = \emptyset$. It is represented by (N, X_i, u_i, ω_i). The strategic game corresponding to this pure exchange economy is given by (N, S_i'', v_i), where

$$S_i'' = \left\{ t_i \in \mathbb{R}^{n\times m} \;\middle|\; \forall h = 1, ..., m, \sum_{j\in N} t_{ij}^h = \omega_i^h \right\}.$$

This is obtained from S_i' by setting $Y = \emptyset$.

In these cases, we can show the similar results without assuming the monotonicity of u_i as follows.

Corollary 3.2. *In the production economy $(N, X_i, u_i, \omega_i, Y)$ without public goods, we have that*

(1) If $N \in \mathfrak{T}$ and $x^* \in C(\mathfrak{T})$, then the strategy combination $s(x^*)$, which is generated by x^*, belongs to $SE(\mathfrak{T})$.

(2) If $s^* \in SE(\mathfrak{T})$, then $a(s) \in C(\mathfrak{T})$.

Proof. The proof is similar to proposition 3.1 and 3.2. Note that we do not need to define r_i. ∎

Corollary 3.3. *In the pure exchange economy (N, X_i, u_i, ω_i), we have that*

(1) If $N \in \mathfrak{T}$ and $x^* \in C(\mathfrak{T})$, then the strategy combination $s(x^*)$, which is generated by x^*, belongs to $SE(\mathfrak{T})$.
(2) If $s^* \in SE(\mathfrak{T})$, then $a(s) \in C(\mathfrak{T})$.

Proof. Essentially the same as the proof of proposition 3.1 and 3.2. The difference is that equation (1) becomes $t_{ij}^h := \dfrac{\omega_i^h}{\sum_{j\in N}\omega_j^h} x_j^{*h}$ and equation (2) becomes $t_{ij}^h := \dfrac{\omega_i^h}{\sum_{j\in S}\omega_i^h}(x_j^h - \sum_{k\in N\setminus S} t_{kj}^{*h})$. ∎

The existence of a strong equilibrium implies the existence of a β-core, a related core concept of a strategic game with coalitions. A payoff vector is said to be in the β-core if there does not exist a coalition such that for any strategies of the complementary coalition there are strategies of the coalition that makes each member better off. Thus, any payoff vector at a strong equilibrium is in the β-core, because no coalition can make its members better off given the equilibrium strategies of the complementary coalition. This fact further implies that the α-core is also nonempty, since the β-core is a subset of the α-core, which is defined to be the set of payoff vectors such that no coalition can make each of its members better off independently of the strategies of the complementary coalition. This observation is summarized as follows.

Corollary 3.4. *Denote simply by core the set of payoff vectors corresponding to $C(2^N)$.*

(1) In the public goods economy, assume that u_i is monotone increasing in $y \in \mathbb{R}_+^l$ for all $i \in N$. Then, core $\subseteq$ β-core $\subseteq$ α-core.
(2) In the production economy without public goods or the pure exchange economy, it follows that core $\subseteq$ β-core $\subseteq$ α-core.

4. CONCLUDING REMARKS

We have shown that under usual economic assumptions, the core and the strong equilibria are in fact equivalent each other in typical economic situations such as pure exchange, production and public goods economies.

The key to this result is the strategy allowing negative transfers of goods. If transfers are restricted to be nonnegative, as in the pure exchange game mentioned in Scarf (1971), some allocations will not be attained by the choice of strategies in a coalition given the strategies of the complementary coalition, which may impede the equivalence between the core and the strong equilibria.

In such a game with nonnegative transfers, however, the α (or β)$-$core might be a more appropriate solution concept, since for each coalition S there is a unique worst state that the complementary coalition contributes nothing to the consumption in S. Then, if transfers are restricted to be nonnegative, the α-core coincides with the β-core in each of the economies considered in this paper (see also Mas-Colell (1987) or Nakayama (1998)).

Another point to remark is the binding agreement that may be needed for a coalition to make a deviation from a non-equilibrium state in the game. However, this is not so restrictive as it may seem. We may argue, in the spirit of coalition-proof Nash equilibrium, that only 'credible' deviations are important. Let us say, by recursion, that a coalition C has a *credible* deviation at strategy combination s if and only if C has a deviation at s such that after the deviation no proper subcoalition of C has a *credible* deviation. The coalition-proof Nash equilibrium is one at which no coalition has a credible deviation. Thus, non-credible deviations are ones admitting further subcoalitional deviations that are credible. In this sense, a coalition may refrain from carrying out a non-credible deviations in the absence of binding agreements.

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