Voluntarily Separable Repeated Prisoner's Dilemma with Reference Letters*

Takako Fujiwara-Greve† Masahiro Okuno-Fujiwara‡ Nobue Suzuki§

This version: June 21, 2009

Abstract: We consider voluntarily separable repeated Prisoner's Dilemma in which a pair of players meet randomly and repeatedly play Prisoner's Dilemma only by mutual agreement. While the literature dealt with the case of no information flow across partnerships, we consider the case that players can issue a "reference letter" to verify at least that the partnership entered the cooperation phase. We show that such reference letters can be voluntarily provided by the partners even at some cost, and that the sheer existence of a letter shortens the trust-building periods and thus improves social efficiency.

Key words: Voluntary Separation; Prisoner's Dilemma; Cooperation; Information; Random Matching.
JLE classification: C 73; D 83.

---

*We are grateful to Atsushi Kajii, Colin Rowat, Yasushi Tsuru, Nick Feltovich, Jack Stecher, and Partha Gangopadhyay for useful comments. This work was supported by JSPS Grant-in-Aid for Scientific Research (C), No.19530158. The usual disclaimer applies.

†Corresponding author: Department of Economics, Keio University. 2-15-45 Mita, Minatoku, Tokyo 108-8345 JAPAN. E-mail: takakofg@econ.keio.ac.jp. Phone & Fax: + 81 - 3 - 5427 - 1295.

‡Graduate School of Economics, University of Tokyo. 7-3-1 Hongo, Bunkyoku, Tokyo 113-0033 JAPAN. E-mail: mfujiwar@e.u-tokyo.ac.jp.

§Faculty of Economics, Komazawa University. 1-23-1 Komazawa, Setagayaku, Tokyo 154-8525 JAPAN. E-mail: nsuzuki@komazawa-u.ac.jp.

1 Introduction

In a large anonymous society, nurturing cooperation is difficult as one can always betray the partner's trust and run away to find a new partner. When the society is anonymous, such player's misconduct will cause no difference to the future play with a new partner. In more technical terminology, if one faces a random matching society to play a one-shot Prisoner's Dilemma with a randomly matched player, playing Defect is a dominant strategy unless a defector can be sanctioned in the future partnership.

The literature has proposed two ways to sanction runaway defectors. One is to inform future partner(s) of the player's record of past conducts. Another is to start new partnerships always with low payoff.

There are a few known mechanisms that transmits information to future partners. Perhaps the best known is the credit card system, where the credit card company keeps record of members' credit status and informs it to the randomly matched partner upon request. The credit card system is a centralized mechanism that tracks members' credit, like the model of Okuno-Fujiwara and Postlewaite (1995) and Matsushima (1990). Except for the necessary transaction cost, this system can achieve an efficient allocation because it can sanction cheaters by depriving them of a good credit status.

Milgrom, North and Weingast (1990) modeled a similar mechanism of the medieval law merchants that deters cheating and nurtures trust. A law merchant provides a service of answering a merchant's query of whether his current partner has "unpaid judgement". Unpaid judgement implies that this partner has cheated in the previous trading and has not paid the fine that would have nullified the bad record. Being informed that the partner has unpaid judgement, the merchant can defect against the partner as a sanction. However, these institutional systems may be quite costly to operate and also require that the actions within partnerships be verifiable to the third party.

There is also an implicit system to infer a player's past from observable information such as the reason for job search. If a worker is looking for a new job because of the plant closure or moving for family reasons, (s)he is better treated in the job market than those looking for a job because of being fired. (See Gibbons and Katz, 1991.) This is because the plant closure or family moving indicates that the worker did not defect in the previous workplace, while firing suggests a misconduct. However, not all situations have such inferable information.

At the other extreme, when there is no information flow from one partnership to another, gradual cooperation or trust building is advocated. Reduction of the payoff of newly formed partnerships serves as a punishment on defectors. (See Carmichael and MacLeod, 1997, Datta, 1996, Fujiwara-Greve and Okuno-Fujiwara, 2009, Ghosh and Ray, 1996, and Kranton, 1996.) There are many trust-building mechanisms in the society, such as gift exchange at the beginning of relationships analyzed in Carmichael and MacLeod (1997) and implicit bond-posting by low wage for young workers as shown in Lazear (1981). However, the trust-building system also reduces payoffs of players who lost partners for exogenous reasons.

In this paper, we consider a decentralized information transmission system from a previous partner who observes private actions within the reltionship to potential future partners. Specifically, we extend the voluntarily separable repeated Prisoner's Dilemma of Fujiwara-Greve and Okuno-Fujiwara (2009), henceforth Greve-Okuno, to incorporate a system of reference letters. Each player can choose to issue a reference letter for the current partner at a small cost, attesting that (s)he has cooperated within the partnership. Only the existence of a reference letter is verifiable. We show that mutual letter giving is sustained by possible future punishment of not getting a letter from the current partner, under the voluntary continuation of the partnership. Then newly matched partners can adjust whether to start cooperating right away or not depending on the letter status of each other.

We show that it becomes an evolutionary Nash equilibrium that partners pay the cost to provide a reference letter to each other and the trust-building periods are weakly reduced not only for letter-holding pairs but also those without letters. Moreover, although each player is assumed to start his (her) life with no letter, the reference letter system improves the lifetime average payoff, as compared to the no information flow case of Greve-Okuno (2009). Therefore, an additional information (even at a small cost) can be provided voluntarily and improves efficiency in the anonymous society.[1]

In sum, our contribution is two-fold. Theoretically, we extend the voluntarily separable

[1] Kandori (1992) shows that additional information improves social efficiency in repeated games with imperfect monitoring, but the voluntary provision of information is not addressed. See also the concluding remarks.

repeated game model by Greve-Okuno (2009) to include costly information transmission. In addition, we provided a rationale to the reference letter system, or more generally, a voluntary information transmission system by boundedly rational players in a random matching society.

This paper is organized as follows. In Section 2, we formulate the voluntarily separable repeated Prisoner's Dilemma with reference letters. In Section 3, we give details of the long-run and average payoffs of a player and define a Nash equilibrium. In Section 4, we introduce letter-based trust-building strategies. In Section 5, we derive an evolutionary Nash equilibrium in which newly matched players cooperate immediately if and only if both have a reference letter. By contrast to symmetric trust-building equilibria in Greve-Okuno (2009), we show that this equilibrium has weakly shorter trust-building periods even among players without reference letters. Finally the social welfare as the average payoff of players is shown to be greater than that of the no-information-flow case in Greve-Okuno (2009). Section 6 concludes the paper with some remarks.

2 Model

Consider a large society of a continuum of players of measure 1. The time is discrete. At the end of each period, each player exits from the society for an exogenous reason (which we call a "death") with probability $1-\delta$ (where $0 < \delta < 1$). If a player dies, a new player is "born" into the society, keeping the population size constant. In each period, players without a partner (including the newly born players) enter the random matching pool and form pairs to play the *Voluntarily Separable Repeated Prisoner's Dilemma with Reference Letters* as follows.

In the matching pool, each player is either with a reference letter (status Y) or without a reference letter (status N). We assume that a newly born player has no reference letter. Randomly matched players observe whether the new partner holds a reference letter or not but not the past actions of each other in past matches.

Based only on the letter status of each other, newly matched players play the Prisoner's Dilemma (see Table 1) by choosing action C or D simultaneously. The actions in the Prisoner's Dilemma are observable only by the partners. At the end of a period, each player has two decisions to make: continuation decision whether to keep the partnership (action k) or

P1 \ P2	C	D
C	c, c	ℓ, g
D	g, ℓ	d, d

Table 1: Prisoner's Dilemma ($g > c > d > \ell$)

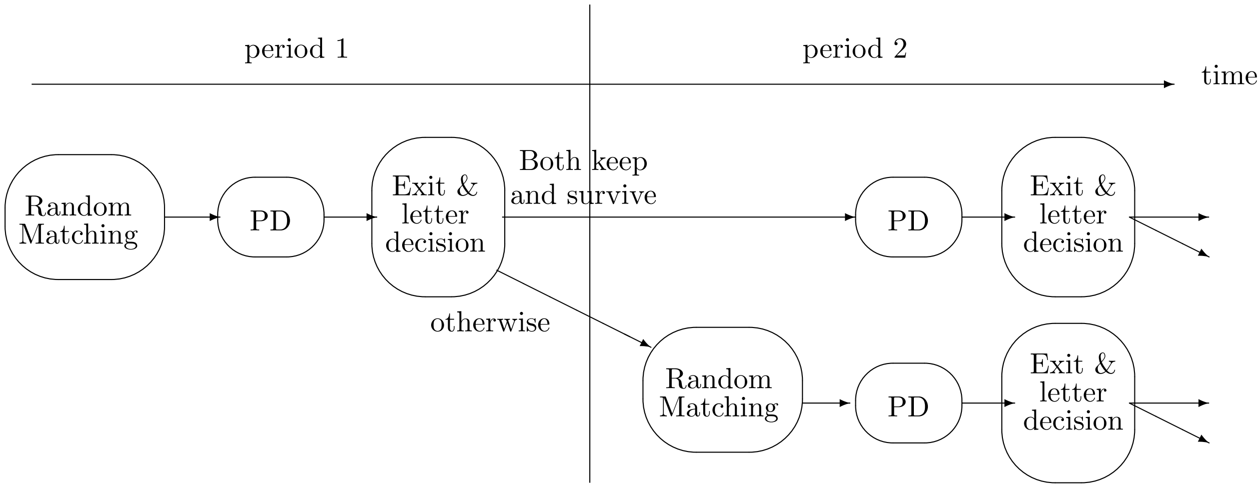

Figure 1: Outline of the Dynamic Game

end it (action e) and letter decision[2] whether to give a reference letter to the partner (action y) or not (action n). These decisions are made simultaneously by the partners.

If at least one player chooses e, the partnership ends and both players (if they survive) go to the random matching pool in the next period. If both players choose action k, unless one of them dies, they stay as partners and play the Prisoner's Dilemma in the next period, skipping the matching process. If the partner dies, the surviving player goes to the random matching pool in the next period. Whether one is able to carry a reference letter to the matching pool depends on the partner's choice. The letter, if exists, is assumed to be valid only one period so that only the most recent decision by the partner matters.[3] A sequential decision model in which players choose the letter action after the continuation decision can be analyzed similarly but complicates the exposition, hence we do not consider it in this paper. Figure 1 illustrates the outline of the dynamic game.

[2]Okuno-Fujiwara et al. (2007) consider a model in which reference letters are generated according to an exogenous rule, based on actions in the Prisoner's Dilemma.

[3]By this assumption we avoid the issue of a Y-status player skipping a match with a N-status player to wait for another Y-player in the matching pool. See concluding remarks for a discussion of this skipping possibility.

We assume that there is a small cost $\epsilon > 0$ (see Section 5.3 for the range of ϵ that supports our equilibrium) to issue a reference letter.[4] There is no cost if the player does not issue a letter. Therefore, there is a social trade-off that the valuable information to the receiver of the letter must be voluntarily paid by the issuer.

A related model is the gift-giving model by Carmichael and MacLeod (1997). However, in their model, randomly matched players have choice to give costly gifts to each other only at the time of matching, and this is essentially the same as our "trust-builiding" (to be defined in Section 4).

The one-shot payoff in the Prisoner's Dilemma is shown in the Table 1. Assume that $g > c > d > \ell$ and $2c \geq g + \ell$. The latter is for simplicity and to make the symmetric action profile (C, C) efficient. Since each player continues the dynamic game with probability δ, this δ is the effective discount factor.

To define the set of pure strategies formally, let $t = 1, 2, \ldots$ denote the periods in the current partnership. For each t, define $H_t := \{Y, N\}^2 \times [\{C, D\}^2 \times \{y, n\}^2]^{(t-1)}$ as the set of partnership histories at the beginning of the t-th period of a partnership. If it is a new match ($t = 1$), then only the status of the reference letters is the partnership history.

Definition 1. *A pure strategy s specifies the following $(x_t, y_t, z_t)_{t=1}^{\infty}$.*

For each $t = 1, 2, \ldots,$

(a) $x_t : H_t \to \{C, D\}$ specifies an action in the Prisoner's Dilemma;

(b) $y_t : H_t \times \{C, D\}^2 \to \{k, e\}$ specifies a continuation decision based on the partnership history and the action profile in the current period;

(c) $z_t : H_t \times \{C, D\}^2 \to \{y, n\}$ specifies whether to issue a reference letter to the current partner based on the partnership history and the action profile in the current period.

Two remarks are in order. First, continuation decisions are observable, but only (k, k) will lead to the continuation of the match and thus one cannot vary future actions based on

[4] An alternative model would be that the receiver pays the cost of a letter from the partner, to verify that he did not defect in the cooperation phase. It will be easier to sustain cooperation under this model since there is an incentive for the receiver to pay the cost more than the issuer does, because of the shortened trust-building phase in the future match. There is related repeated game literature on information purchase. See Ben-Porath and Kahneman (2003), Miyagawa et al. (2008) and Flesch and Perea (2009).

different combinations of the continuation decisions. Therefore without loss of generality the continuation decisions do not enter the relevant partnership history. Second, the above definition assumes that a player use the same strategy in every new match, since each partnership starts with only the letter status.[5]

Let $\boldsymbol{S}$ be the set of pure strategies and $\mathcal{P}(\boldsymbol{S})$ be the set of all probability distributions over $\boldsymbol{S}$. For simplicity, we assume that each player is endowed with a pure strategy, which is also natural in an evolutionary game.

3 Payoff Structure and Nash Equilibrium

We consider stability of stationary strategy distributions in the matching pool. Although the strategy distribution in the matching pool may be different from the distribution in the entire society, if the former is stationary, the distribution of various states of matches is also stationary, thanks to the stationary death process. (See Greve-Okuno, 2009.)

When a strategy $s \in \boldsymbol{S}$ is matched with another strategy $s' \in \boldsymbol{S}$, the *expected length* of the match is denoted as $L(s, s')$ and is computed as follows. Notice that even if s and s' intend to maintain the match, it will only continue with probability δ^2. Suppose that the partnership of s and s' would last $T(s, s')$ periods, if no death occurs. Then

$$L(s, s') := 1 + \delta^2 + \delta^4 + \cdots + \delta^{2\{T(s,s')-1\}} = \frac{1 - \delta^{2T(s,s')}}{1 - \delta^2}.$$

The *expected total discounted value of the payoff stream of s within the match with s'* is denoted as $V(s, s')$. The *average per period payoff* that s expects to receive within the match with s' is denoted as $v(s, s')$. Clearly,

$$v(s, s') := \frac{V(s, s')}{L(s, s')}, \text{ or } \; V(s, s') = L(s, s')v(s, s').$$

Next we show the structure of the lifetime and average payoff of a player endowed with strategy $s \in \boldsymbol{S}$ in the matching pool, waiting to be matched randomly with a partner. When a strategy distribution in the matching pool is $p \in \mathcal{P}(\boldsymbol{S})$ and is stationary, we write the *expected total discounted value of payoff streams s* expects to receive during his lifetime as

[5] Kandori (1992a) and Ellison (1994) consider strategies that utilize one's own past history in random matching games. It is possible to include such strategies in our analysis, but since the set of players is a continuum, there is no Nash equilibrium that can start the "contagion of defection" to sustain cooperation.

$V(s;p)$ and the average per period payoff s expects to receive during his lifetime as

$$v(s;p) := \frac{V(s;p)}{L} = (1-\delta)V(s;p),$$

where $L = 1 + \delta + \delta^2 + \cdots = \frac{1}{1-\delta}$ is the expected lifetime of s.

Given a stationary distribution p in the matching pool, we can write $V(s;p)$ as a recursive equation. If p has a finite/countable support, then we can write

$$\begin{aligned} V(s;p) &= \sum_{s' \in supp(p)} p(s') \Big[V(s,s') \\ &\quad + [\delta(1-\delta)\{1+\delta^2+\cdots+\delta^{2\{T(s,s')-2\}}\} + \delta^{2\{T(s,s')-1\}}\delta] V(s;p) \Big], \end{aligned} \tag{1}$$

where $supp(p)$ is the support of the distribution p, the sum $\delta(1-\delta)\{1+\delta^2+\cdots+\delta^{2\{T(s,s')-2\}}\}$ is the probability that s loses the partner s' before $T(s,s')$, and $\delta^{2\{T(s,s')-1\}}\delta$ is the probability that the match continued until $T(s,s')$ and s survives at the end of $T(s,s')$ to go back to the matching pool. Thanks to the stationarity of p, the continuation payoff after a match ends is always $V(s;p)$ for any reason.

Let $L(s;p) := \sum_{s' \in supp(p)} p(s')L(s,s')$. By computation,

$$\begin{aligned} V(s;p) &= \sum_{s' \in supp(p)} p(s') \Big[V(s,s') + \{1-(1-\delta)L(s,s')\}V(s;p) \Big] \\ &= \sum_{s' \in supp(p)} p(s')V(s,s') + \{1 - \frac{L(s;p)}{L}\}V(s;p). \end{aligned} \tag{2}$$

This recursive structure (2) is utilized many times in the following analysis. Note that, by solving (2) explicitly, the average payoff is shown to be a nonlinear function of p:

$$v(s;p) = \frac{V(s;p)}{L} = \frac{\sum_{s' \in supp(p)} p(s')V(s,s')}{\sum_{s' \in supp(p)} p(s')L(s,s')} = \sum_{s' \in supp(p)} p(s') \frac{L(s,s')}{L(s;p)} v(s,s'). \tag{3}$$

If p is a strategy distribution consisting of a single strategy s', then $v(s;p) = v(s,s')$.

Definition 2. *Given a stationary strategy distribution in the matching pool* $p \in \mathcal{P}(\boldsymbol{S})$, $s \in \boldsymbol{S}$ *is a* **best reply against** p *if for all* $s' \in \boldsymbol{S}$,

$$v(s;p) \geq v(s';p),$$

and is denoted as $s \in BR(p)$.

Definition 3. *A stationary strategy distribution in the matching pool $p \in \mathcal{P}(\boldsymbol{S})$ is a* **Nash equilibrium** *if, for all $s \in \text{supp}(p)$, $s \in BR(p)$.*

Similarly, we can define neutrally stable distribution as a stationary strategy distribution such that no mutant strategy can invade if they enter with a sufficiently small measure.[6] However, when a mutant strategy enters into the population with a positive measure, the distribution of letter status in the matching pool changes in a complex way over time. There will be pairs of strategies that last only certain number of periods, even if they do not die, while other pairs may last as long as they both live. Hence the distribution of letter status depends not only on the strategy distribution but also on the time periods. This makes the analysis of consistency between the strategy distributions and the flow of players into the matching pool very difficult, while symmetric Nash equilibrium allows us a clear and explicit derivation of stationary strategy distribution which is consistent with the flow of players into the matching pool. (See Section 5.1.) Therefore for simplicity we focus on symmetric Nash equilibria, but with the meaning of an evolutionary stability.

4 Trust-building Strategies

Greve-Okuno (2009) showed that if a symmetric strategy distribution is a Nash equilibrium, then it must be a strategy that plays D (but keeps the partnership if (D, D) is observed in the current period) for some initial periods of a partnership, which is called a *trust-building* strategy. (For the intuition of this result, see the beginning of Section 5.) We also focus on this class of strategies in this paper, but since we have additional letter information, we extend the definition of a trust-building strategy as follows.

Let $T : \{Y, N\}^2 \to \{0, 1, 2, \ldots\}$ be a function that specifies the number of periods that a newly formed partnership plays (D, D), depending on the letter status combination $q \in \{Y, N\}^2$.

Definition 4. *For each $T : \{Y, N\}^2 \to \{0, 1, 2, \ldots\}$, the* **letter-based T-trust-building strategy** *(written as c_T) is defined as follows: for each partnership with the initial letter*

[6] As discussed in Greve-Okuno (2009), the evolutionary stable strategy (ESS) concept is too weak in the extensive-form game model, since all strategies which differ from the incumbents off the play path can enter the distribution under the ESS concept.

status combination q,

(i) for any $t \leqq T(q)$, play D and choose (k, n) for any observation of the action profile in the Prisoner's Dilemma in the current period;

(ii) for any $t \geqq T(q)+1$, play C and choose k if and only if (C, C) is observed in the current period. As for the letter decision, choose y if and only if either (i) it is $t = T(q)+1$ and (C, C) is observed in the current period, or (ii) it is $t \geqq T(q)+2$, (y, y) was observed in $t-1$, and (C, C) was observed in the current period.

The idea of this strategy is that there is a possible "trust-building phase" for $T(q)$ periods, depending on the initial letter combination, and after that, cooperation and a letter are given if and only if the partner also did so recently. Since the letter decision and continuation decision are simultaneous, a player can "punish" the partner who did not issue a letter during the cooperation phase only in the next period of the match provided that it continues. This is a rather weak form of punishment, but, as we show below, the symmetric strategy combination still becomes an equilibrium for small enough cost of issuing a letter $\epsilon > 0$. Action D in the Prisoner's Dilemma after $T(q)$-th period will be punished by exit (action e) and no letter (action n).

5 Efficiency Improvement by Reference Letters

In the previous literature of endogenously repeated games (e.g., Ghosh and Ray, 1996; Greve-Okuno, 2009; and Kranton, 1996), it was assumed that there is no information flow across partnerships. Then in any symmetric Nash equilibrium the players cannot play C in the first period of a match. If such strategies constitute the population distribution, then a strategy that plays D in the first period of a match and ends the partnership without issuing a letter can earn g in every match, which is the highest one-shot payoff. The key was that under no information flow one cannot selectively punish a deviator.

By contrast, in our model, players can choose different initial actions depending on the letter status of the new partner. We investigate whether we can reduce the trust-building periods to zero for (Y, Y)-pair of players, while not prolonging the trust-building phase for other types of matches. If that holds in an equilibrium, the sheer existence of reference letter improves the efficiency of the society, although one's past actions are not verifiable. Thus, this model replaces the "anonymous punishment" of trust-building by the "personalized punishment" by reference letters.

Below we focus on T-functions such that $T(Y,Y) = 0$ and $T(q) = T$ for any $q \neq (Y,Y)$ for some $T \in \mathbf{N}$, where $\mathbf{N}$ is the set of natural numbers (positive integers). By a slight abuse of the notation we call the letter-based strategy with this T function as c_T-strategy with a $T \in \mathbf{N}$.

5.1 Consistent Distribution in the Matching Pool

In this subsection, we fix an arbitrary $T \in \mathbf{N}$ and derive a unique stationary distribution of Y and N status players in the matching pool, which is consistent with the stationary c_T-strategy distribution. Let the measure of players in the matching pool be 1 and the fraction of Y-status players be α. We show that this α must satisfy a certain condition under the stationary strategy distribution of c_T.

By the random matching process, the fraction of *pairs* of players such that both have reference letters, (called a YY*-pair*) is $\frac{1}{2}\alpha^2$. The fraction of pairs in which exactly one player has a reference letter (called a NY*-pair*) is $\alpha(1-\alpha)$, since there are two ways that a pair is of this form. The fraction of pairs in which none has a reference letter (called a NN*-pair*) is $\frac{1}{2}(1-\alpha)^2$. The probability that these pairs continue for t periods is the probability that both players survive (with probability δ^2) for t consecutive periods. Thus, the fraction of YY-pairs that continued for t periods is $\frac{1}{2}\alpha^2\delta^{2t}$, that of NY-pairs that continued for t periods is $\alpha(1-\alpha)\delta^{2t}$, and that of NN-pairs that continued for t periods is $\frac{1}{2}(1-\alpha)^2\delta^{2t}$.

In each period, these pairs with $t = 1, 2, 3, \ldots$ periods of duration co-exist in the society. The total fraction of YY-pairs in a period is $\frac{1}{2}\alpha^2\{1 + \delta^2 + \cdots\}$. Among these, those who go back to the matching pool are the ones that the partner died, since, for YY-pairs, c_T-strategy prescribes that matched players issue reference letters to each other in every period of the partnership. The probability of exactly one player dies in a partnership is $\delta(1-\delta)$ and there are two cases. (See Table 2.) Therefore, the fraction of Y-status players entering the matching pool from YY-pairs is

$$Y(Y,Y) := 2\delta(1-\delta)\frac{1}{2}\alpha^2\{1 + \delta^2 + \cdots\} = \frac{\delta(1-\delta)\alpha^2}{1-\delta^2}.$$

	Live (δ)	Die ($1-\delta$)
Live (δ)	keep	Y, N
Die ($1-\delta$)	N, Y	N, N

Table 2: Patterns and probabilities of players entering the matching pool from a YY-pair or a pair after trust building

Dead players in YY-pairs will generate newly born players with N status in the matching pool. The probability that exactly one player dies is $2\delta(1-\delta)$ and the probability that both die is $(1-\delta)^2$. Hence the fraction of N-status players entering the matching pool from YY-pairs is

$$N(Y,Y) := \{2\delta(1-\delta)+2(1-\delta)^2\}\frac{1}{2}\alpha^2\{1+\delta^2+\cdots\} = \frac{(1-\delta)\alpha^2}{1-\delta^2}.$$

Next, consider players going to the matching pool from NY-pairs or NN-pairs. The total fraction of these types of pairs in a period is $\{\alpha(1-\alpha)+\frac{1}{2}(1-\alpha)^2\}\{1+\delta^2+\cdots\}$. From these types of pairs, during the trust building phase ($t \le T$), two N-status players are generated in the matching pool regardless of the reason of dissolution, and no Y-status player is generated. The probability that a partnership dissolves is $1-\delta^2$. Therefore, the fraction of N-status players entering the matching pool from $q \neq (Y,Y)$-pairs during their trust-building phase is

$$\begin{aligned} N(q;TB) &:= 2(1-\delta^2)\{\alpha(1-\alpha)+\frac{1}{2}(1-\alpha)^2\}\{1+\delta^2+\cdots+\delta^{2(T-1)}\} \\ &= \frac{(1-\delta^2)(1-\alpha^2)(1-\delta^{2T})}{1-\delta^2}. \end{aligned}$$

After the trust-building phase, the fraction of Y-status players and N-status players entering the matching pool are the same as those from YY-pairs. Let $Y(q;CP)$ be the fraction of Y-players entering the matching pool from the cooperation phase for $q \neq (Y,Y)$. It is expressed as

$$\begin{aligned} Y(q;CP) &:= 2\delta(1-\delta)\{\alpha(1-\alpha)+\frac{1}{2}(1-\alpha)^2\}\{\delta^{2T}+\delta^{2(T+1)}+\cdots\} \\ &= \frac{\delta(1-\delta)(1-\alpha^2)\delta^{2T}}{1-\delta^2}. \end{aligned}$$

The fraction of newly born players (with N-status) from $q \neq (Y,Y)$ pairs during the cooperation phase is

$$\begin{aligned} N(q;CP) &:= \{2\delta(1-\delta)+2(1-\delta)^2\}\{\alpha(1-\alpha)+\frac{1}{2}(1-\alpha)^2\}\{\delta^{2T}+\delta^{2(T+1)}+\cdots\} \\ &= \frac{(1-\delta)(1-\alpha^2)\delta^{2T}}{1-\delta^2}. \end{aligned}$$

In total, the overall fraction of Y-status players entering the matching pool is

$$Y(Y,Y)+Y(q;CP)=\frac{\delta\{\delta^{2T}+\alpha^2(1-\delta^{2T})\}}{1+\delta}=:f(\alpha).$$

If there exists a fixed point $\alpha^* \in (0,1)$ such that $f(\alpha^*)=\alpha^*$, i.e.,

$$\delta^{2T+1}+\alpha^{*2}\delta(1-\delta^{2T})=\alpha^*(1+\delta), \tag{4}$$

this is the stationary fraction of Y-status players consistent with the stationary c_T-distribution.

Lemma 1. *For any $\delta \in (0,1)$ and any $T \in \mathbf{N}$, there exists a unique $\alpha^* \in (0,1)$ such that $f(\alpha^*)=\alpha^*$, that is, the stationary symmetric strategy distribution of the c_T-strategy is consistent with the flow of players over time.*

Proof of Lemma 1. Since $f(\alpha)-\alpha$ is a continuous function of α and $f(0)-0=\frac{\delta^{2T+1}}{1+\delta}>0$ and $f(1)-1=\frac{\delta}{1+\delta}-1<0$, by the Intermediate Value Theorem, there exists $\alpha^* \in (0,1)$ such that $f(\alpha^*)-\alpha^*=0$. Moreover, such α^* is unique because $f(\alpha)-\alpha$ is convex in α.[7] □

In the following we assume that when the society consists of the symmetric c_T-strategy, this α^* is the fraction of Y-status players in the matching pool.

Remark 1. *For any $\delta \in (0,1)$, α^* is a strictly decreasing function of T.*

Proof of Remark 1. $f(\alpha)$ can be written as

$$f(\alpha)=\frac{\delta}{1+\delta}[(1-\alpha^2)\delta^{2T}+\alpha^2].$$

Hence $\delta \in (0,1)$ implies that $f(\alpha)$ is strictly decreasing in T, which also implies that $f(\alpha)-\alpha$ is. Thus as T increases, the intersection between $f(\alpha)-\alpha$ and the horizontal axis decreases.

□

[7] The function $f(\alpha)-\alpha$ may be increasing above some $\alpha \in (0,1)$ but since it is convex and $f(1)-1<0$, it cannot exceed 0 within $(0,1)$ once it became negative. It is also easy to check that α^* satisfies $N(Y,Y)+N(q;TB)+N(q;CP)=1-\alpha^*$ as well.

5.2 Expected Payoff

Let $V(Y,Y)$ (resp. $V(q)$) denote the total expected payoff of a player starting a new partnership with the letter status combination of (Y,Y) (resp. $q \neq (Y,Y)$). The (on-path) values $V(Y,Y)$ and $V(q)$ are formulated as follows. First, if a player starts in a YY-pair, he cooperates, writes a letter, and gets a reference letter in any period. After a dissolution, which must be due to the death of the partner, the continuation value is $(1-\alpha^*)V(q)+\alpha^* V(Y,Y)$. Hence $V(Y,Y)$ satisfies the following recursive equation. (Recall the general computation method in (2).)

$$\begin{aligned} V(Y,Y) &= (c-\epsilon)(1+\delta^2+\cdots)+\delta(1-\delta)(1+\delta^2+\cdots)[(1-\alpha^*)V(q)+\alpha^* V(Y,Y)] \\ &= \frac{c-\epsilon}{1-\delta^2}+\frac{\delta(1-\delta)}{1-\delta^2}[(1-\alpha^*)V(q)+\alpha^* V(Y,Y)]. \end{aligned} \tag{5}$$

Second, consider a player starting in a pair with the letter combination $q \neq (Y,Y)$. During the trust-building phase, if the partner dies, the player goes back to the matching pool without a reference letter and thus $V(q)$ is the continuation payoff. After the trust-building phase, the continuation payoff becomes $(1-\alpha^*)V(q)+\alpha^* V(Y,Y)$. Hence,

$$\begin{aligned} V(q) &= d\{1+\delta^2+\cdots+\delta^{2(T-1)}\}+\delta(1-\delta)\{1+\delta^2+\cdots+\delta^{2(T-1)}\}V(q) \\ &\quad +(c-\epsilon)\{\delta^{2T}+\delta^{2(T+1)}+\cdots\}+\delta(1-\delta)\{\delta^{2T}+\delta^{2(T+1)}+\cdots\}[(1-\alpha^*)V(q)+\alpha^* V(Y,Y)] \\ &= \frac{1-\delta^{2T}}{1-\delta^2}d+\frac{\delta(1-\delta)(1-\delta^{2T})}{1-\delta^2}V(q) \\ &\quad +\frac{\delta^{2T}}{1-\delta^2}(c-\epsilon)+\frac{\delta(1-\delta)\delta^{2T}}{1-\delta^2}[(1-\alpha^*)V(q)+\alpha^* V(Y,Y)]. \end{aligned} \tag{6}$$

Solving (5) and (6) simultaneously, we explicitly obtain

$$V(Y,Y) = \frac{(1+\delta^{2T+1})(c-\epsilon)+(1-\alpha^*)\delta(1-\delta^{2T})d}{(1-\delta)\{1+\delta-\alpha^*\delta(1-\delta^{2T})\}} \tag{7}$$

$$V(q) = \frac{\delta^{2T}(1+\delta)(c-\epsilon)+\{1+(1-\alpha^*)\delta\}(1-\delta^{2T})d}{(1-\delta)\{1+\delta-\alpha^*\delta(1-\delta^{2T})\}} \tag{8}$$

Hence

$$V(Y,Y)-V(q) = \frac{(1-\delta^{2T})(c-\epsilon-d)}{(1-\delta)\{1+\delta-\alpha^*\delta(1-\delta^{2T})\}} > 0, \tag{9}$$

as long as $\epsilon < c-d$. Also, notice that a newly born player has N status, and thus each player's lifetime expected payoff is $V(q)$. We summarize the above result as follows.

Lemma 2. *For any $\delta \in (0,1)$ and any $T \in \mathbf{N}$, under the symmetric strategy distribution of c_T-strategy, $V(Y,Y) > V(q)$ if and only if $\epsilon < c-d$.*

Note also that the average payoff functions are expressed as follows.

$$v(Y,Y) = \frac{(1+\delta^{2T+1})(c-\epsilon)+(1-\alpha^*)\delta(1-\delta^{2T})d}{1+\delta-\alpha^*\delta(1-\delta^{2T})}; \tag{10}$$

$$v(q) = \frac{\delta^{2T}(1+\delta)(c-\epsilon)+\{1+(1-\alpha^*)\delta\}(1-\delta^{2T})d}{1+\delta-\alpha^*\delta(1-\delta^{2T})}. \tag{11}$$

Lemma 3. *For any $\delta \in (0,1)$, any $T \in \mathbf{N}$ and any $\epsilon < c-d$, $v(q)$ is strictly decreasing in T.*

Proof of Lemma 3. From (11) and (4), $v(q)$ can be rewritten as

$$v(q) = \frac{\alpha^*}{\delta^{2T+1}}\left[\delta^{2T}(1+\delta)(c-\epsilon)+\{1+\delta-\alpha^*\delta\}(1-\delta^{2T})d\right].$$

By rearranging and using (4), we have

$$\begin{aligned}
v(q) &= \frac{\alpha^*}{\delta^{2T+1}}\left[\delta^{2T}(1+\delta)(c-\epsilon)+(1+\delta)(1-\delta^{2T})d-\alpha^*\delta(1-\delta^{2T})d\right] \\
&= \frac{\alpha^*}{\delta^{2T+1}}\left[(1+\delta)\delta^{2T}(c-\epsilon-d)+(1+\delta)d-\alpha^*\delta(1-\delta^{2T})d\right] \\
&= \frac{1}{\delta^{2T+1}}\left[\alpha^*(1+\delta)\delta^{2T}(c-\epsilon-d)+\alpha^*(1+\delta)d-\alpha^{*2}\delta(1-\delta^{2T})d\right] \\
&= \frac{1}{\delta^{2T+1}}\left[\alpha^*(1+\delta)\delta^{2T}(c-\epsilon-d)+\alpha^*(1+\delta)d-\{\alpha^*(1+\delta)-\delta^{2T+1}\}d\right] \\
&= \frac{1}{\delta^{2T+1}}\left[\alpha^*(1+\delta)\delta^{2T}(c-\epsilon-d)+\delta^{2T+1}d\right] \\
&= \frac{1}{\delta}\left[\alpha^*(1+\delta)(c-\epsilon-d)+\delta d\right].
\end{aligned} \tag{12}$$

Since α^* is decreasing in T, so is $v(q)$, when $\epsilon < c-d$. □

5.3 Nash Equilibrium

We investigate conditions to make c_T-strategy constitute a symmetric Nash equilibrium and compare the associated equilibrium payoff with theose when there is no reference letter (Greve-Okuno, 2009). By the dynamic programming, it is sufficient to determine conditions that no strategy that differ from c_T-strategy in one step obtains a higher payoff than c_T's.

Although there are many decision nodes to consider, we show that, for sufficiently small ϵ, the necessary and sufficient condition is that strategies that choose D in the Prisoner's

Dilemma during the cooperation phase (against the partner who would play C) obtains no higher payoff. The intuition is as follows.

First, during the trust-building phase (relevant only for non-YY-pairs), choosing an action different from D would not improve the payoff.

Second, when the c_T-strategy is supposed to keep the partnership, strategies that end the partnership will not yields a higher payoff than c_T does. This is straightforward by considering the continuation payoff from the next period on: If the players are in the trust-building phase, ending the partnership gives the continuation payoff of $V(q)$, while following c_T-strategy that keeps the partnership gives more than $V(q)$ since the players have less periods to build trust. During the cooperation phase, the continuation payoff of the c_T-strategy is $V(Y,Y)$, while ending the partnership gives $(1-\alpha^*)V(q) + \alpha^* V(Y,Y) < V(Y,Y)$, if $\epsilon < c - d$ by Lemma 2.

Third, consider not issuing a letter after seeing (C,C) (and (y,y) when relevant) in the cooperation phase.[8] This saves the cost ϵ in one period but the player would not receive a letter from the partner in the next period if the partnership continues. When ϵ is sufficiently small, the delayed punishment is enough to make the deviant strategy worse off than the c_T-strategy, as shown below.

Lemma 4. *For any $\delta \in (0,1)$ and any $T \in \mathbf{N}$, strategies that do not issue a letter on the play path during the cooperation phase obtain less long-run payoff than that of c_T-strategy if and only if*

$$0 < \epsilon \leqq \frac{\delta^3 \alpha^* (1-\delta^{2T})(c-d)}{1+\delta-\alpha^*\delta(1-\delta^{2T})(1+\delta^2)} = \bar{\epsilon}(T). \tag{13}$$

Proof of Lemma 4. See Appendix.

Note that the upper bound $\bar{\epsilon}(T)$ is not necessarily a monotone function of T, since the stationary fraction α^* of YY-pairs depends on T. However, it turns out that this upper bound is always smaller than $c-d$.

Lemma 5. *For any $\delta \in (0,1)$ and any $T \in \mathbf{N}$, $\bar{\epsilon}(T) < c - d$.*

[8] Issuing a letter during the trust-building phase is obviously costly, and thus a player will not deviate in this way.

Proof of Lemma 5. By computation,

$$\begin{aligned} & \bar{\epsilon}(T) < c - d \\ \iff \quad & \delta^3 \alpha^* (1 - \delta^{2T}) < 1 + \delta - \alpha^* \delta (1 - \delta^{2T})(1 - \delta^2) \\ \iff \quad & 0 < 1 + \delta - \alpha^* \delta (1 - \delta^{2T}). \end{aligned} \tag{14}$$

From (4), we can rewrite

$$1 + \delta - \alpha^* \delta (1 - \delta^{2T}) = \frac{1}{\alpha^*} \delta^{2T+1} > 0.$$

Therefore the inequality (14) holds. □

In summary, when $\epsilon \leqq \bar{\epsilon}(T)$, strategies do not earn higher payoff than the c_T-strategy does, if they differ from the c_T-strategy during the trust-building phase, continuation decision phase, and letter decision phase. The remaining deviation point is the Prisoner's Dilemma action during the cooperation phase. That is, for sufficiently small ϵ, the symmetric distribution of the c_T-strategy is a Nash equilibrium if and only if a strategy which plays D during the cooperation phase does not earn higher payoff than c_T does. This is the characterization of a Nash equilibrium.

Proposition 1. *For any $\delta \in (0, 1)$, any $T \in \mathbf{N}$ and any $\epsilon \in (0, \bar{\epsilon}(T)]$, the symmetric strategy distribution of the c_T-strategy is a Nash equilibrium if and only if*

$$g + \delta V(q) \leq \frac{c - \epsilon}{1 - \delta^2} + \frac{\delta(1 - \delta)}{1 - \delta^2} [(1 - \alpha^*) V(q) + \alpha^* V(Y, Y)]. \tag{15}$$

Proof of Proposition 1. If a player chooses D during the cooperation phase, she receives g in this period but must go to the matching pool without a reference letter. Therefore the continuation payoff is $g + \delta V(q)$. The continuation payoff of the c_T-strategy is

$$V(Y, Y) = \frac{c - \epsilon}{1 - \delta^2} + \frac{\delta(1 - \delta)}{1 - \delta^2} [(1 - \alpha^*) V(q) + \alpha^* V(Y, Y)].$$

Therefore, no strategy that differ in one step from the c_T-strategy is better off if and only if (15) holds. □

Let us compare the condition (15) with the condition for a Nash equilibrium under no information flow (Greve-Okuno, 2009). With no information flow, the continuation payoff right after a partnership dissolution is the same regardless of the cause of the dissolution, which is also the total expected payoff at the time of entering the matching pool, i.e., one's lifetime payoff. Let this be V^{NL}.[9] If there are τ periods of trust-building, then

$$\begin{aligned} V^{NL}(c_\tau) &= (1+\delta^2+\cdots+\delta^{2(\tau-1)})d + (\delta^{2\tau}+\delta^{2(\tau+1)}+\cdots)c \\ &\quad + \frac{\delta(1-\delta)}{1-\delta^2}V^{NL}(c_\tau) \\ \iff \quad V^{NL}(c_\tau) &= \frac{1-\delta^2}{1-\delta}\Big\{\frac{(1-\delta^{2\tau})}{1-\delta^2}d + \frac{\delta^{2\tau}}{1-\delta^2}c\Big\}. \end{aligned}$$

Or on average,

$$v^{NL}(c_\tau) = (1-\delta^{2\tau})d + \delta^{2\tau}c. \tag{16}$$

Note that both the long-run payoff $V^{NL}(c_\tau)$ and the average payoff $v^{NL}(c_\tau)$ are decreasing functions of the length τ of the trust-building phase.

The necessary and sufficient condition for a Nash equilibrium under no information flow is also to prevent deviation to play D during the cooperation phase;

$$\begin{aligned} & g + \delta V^{NL}(c_\tau) \leqq \frac{c}{1-\delta^2} + \frac{\delta(1-\delta)}{1-\delta^2}V^{NL}(c_\tau) \\ \iff \quad & v^{NL}(c_\tau) \leqq \frac{1}{\delta^2}\{c-(1-\delta^2)g\} =: v^{BR}. \end{aligned} \tag{17}$$

Since v^{BR} is independent of the length of the trust-building phase and $v^{NL}(c_\tau)$ is decreasing in τ, Greve-Okuno (2009) shows that there exists $\underline{\delta} > 0$ such that for any $\delta \in (\underline{\delta}, 1)$, there exists the minimum trust-building phase $\underline{\tau}(\delta) \in \Re$ such that (17) holds for any $T \in \mathbf{N}$ such that $T \geqq \underline{\tau}(\delta)$. Thus, the most efficient symmetric equilibrium of the model without reference letters consists of the trust-building strategy with T periods of trust-building phase where T is the minimum integer not less than $\underline{\tau}(\delta)$.

We show that with reference letters, the necessary trust-building phase can be (weakly) shortened even for non-(Y, Y) pairs. For every $\delta \in (\underline{\delta}, 1)$, let $T^{NL}(\delta)$ be the smallest integer not less than $\underline{\tau}(\delta)$. Since no trust-building would not be a Nash equilibrium under no information flow, $T^{NL}(\delta) \in \mathbf{N}$.

Proposition 2. *For any $\delta \in (\underline{\delta}, 1)$ and any $\epsilon \in (0, \bar{\epsilon}(\underline{\tau}(\delta))]$, there exists $T^* \in \mathbf{N}$ such that $T^* \leqq T^{NL}(\delta)$ and (15) holds, i.e., the c_{T^*}-strategy constitutes a Nash equilibrium.*

[9] NL stands for No Letter.

Proof of Proposition 2. See Appendix.

The intuition is as follows. By rearranging (15), we obtain

$$
\begin{aligned}
& g+\delta V(q) \leqq \frac{c}{1-\delta^2}+\frac{\delta(1-\delta)}{1-\delta^2}V(q)-\frac{\epsilon}{1-\delta^2}+\frac{\alpha^*\delta(1-\delta)}{1-\delta^2}[V(Y,Y)-V(q)]; \\
\Longleftrightarrow\quad & v(q)-\frac{1}{\delta^2}[\alpha^*\delta\{v(Y,Y)-v(q)\}-\epsilon] \leqq v^{BR}. \qquad (18)
\end{aligned}
$$

Note that from (20) in the proof of Lemma 4, $\epsilon \leqq \overline{\epsilon}(T)$ implies that the term $[\alpha^*\delta\{v(Y,Y)-v(q)\}-\epsilon]$ is positive.

It holds that (i) at $T=\underline{\tau}(\delta)\in\Re$, the LHS of (18) is strictly less than v^{BR} for sufficiently small ϵ, and (ii) when $T\to 0$, the LHS exceeds v^{BR}. Then by the Intermediate Value Theorem there exists $\tau^*\in\Re$ such that $\tau^*<\underline{\tau}(\delta)$ and the equilibrium condition (15) is satisfied with equality. Since $\underline{\tau}(\delta)\leqq T^{NL}(\delta)$, the claim follows.

Finally we prove that the social welfare is improved by the existence of reference letters, in the sense that the average payoff of a player (starting with no letter) is greater than that of the no-letter model.

Proposition 3. *For any $\delta\in(\underline{\delta},1)$, there exists $0<\epsilon^*\leqq\overline{\epsilon}(\underline{\tau}(\delta))$ such that for any $\epsilon\in(0,\epsilon^*)$, the average payoff $v(q)$ of c_{T^*}-strategy is greater than $v^{NL}(c_{T^{NL}})$.*

Proof of Proposition 3. See Appendix.

Note that there are two factors that give this result. First, when newly matched players both have reference letters, they can skip the trust-building phase and cooperate immediately. Second, even when some of the newly matched players does not have a reference letter, the trust-building phase is not longer than the one for no-letter model. The combination of these two factors improves the social efficiency even at a (small) cost of information transmission.

6 Concluding Remarks

We analyzed the use of voluntary information transmission in the form of reference letters in voluntarily separable repeated Prisoner's Dilemma. During the cooperation phase, if both partners cooperated, they issue reference letters to each other, and, in the matching pool, players with reference letters can start cooperation right away. We derived a necessary and sufficient condition for such strategy combination to be an evolutaionry Nash equilibrium and showed that this equilibrium has average payoff greater than any symmetric equilibrium in the no information flow model. Therefore the sheer existence of reference letters, which only signals the cause of partnership dissolution, improves the efficiency.

Most of the previous literature of the voluntarily repeated Prisoner's Dilemma assumed that there was no information flow across partnerships, while some (e.g., Okuno-Fujiwara and Postelwaite, 1995, Okuno-Fujiwara et al., 2007) studied a model in which an action-dependent "status" is attached to a player and the status is observable to anyone. This can be interpreted as a centrally managed system like the credit card bureau. Our model stands in between these models and displays a voluntary decentralized management of information.

There is related literature of repeated games with information purchase by Ben-Porath and Kahneman (2003), Miyagawa et al. (2008) and Flesch and Perea (2009). These research deals with the trade-off between the cost of additional information and the improvement of imperfect monitoring, but do not consider voluntary provision of information.

Although the signal structure in our model is quite simple that a letter exists or not, it is sufficient to adjust the length of trust-building periods as 0 or some T. It can be also interpreted as the information in the letter is of a positive tone or not. In reality, it is not easy to fully verify whether the content of information is accurate, but the tone of the message can be understood clearly. A possible extension is to include some "degree of informativeness" in the model to see the effect on the equilibrium payoffs.

There are some other interesting possible extensions. First, we note that if newly born players automatically have a reference letter, then on the play path of the c_T-strategy there is no one without a letter in the matching pool. This cannot be an equilibrium since some trust-building is needed in any equilibrium. However, if there is matching friction, i.e., a possibility of not forming a pair in the matching pool, then the "unemployment" probability serves as a punishment and thus it can become an equilibrium, by a similar logic as in Shapiro and Stiglitz (1984).

Second, recall that we have assumed that the letter is valid only for one period. In reality a letter can be valid until a new partnership is formed, and then there can be an incentive for a Y-player to "skip" a match with an N-player in the matching pool to wait for another

Y-player in the future. We have done a preliminary analysis of this "skip" equilibrium, but there are numerous inequality conditions to satisfy and it is difficult to obtain a general sufficient condition on the parameters.

Third, although we could not introduce mutations, it is possible to introduce some other perturbation. For example, we can consider a case that the reference letter gets lost with a small probability before one enters the matching pool. It is of interest to investigate the robustness of our equilibrium under this perturbation.

Appendix

Proof of Lemma 4. If a player does not issue a letter during the cooperation phase, the continuation payoff becomes

$$\begin{aligned} D &= c + \frac{\delta^2}{1-\delta^2}(c-\epsilon) + \delta(1-\delta)[(1-\alpha^*)V(q) + \alpha^* V(Y,Y)] \\ &\quad + \delta^3(1-\delta)V(q) + \delta^4 \frac{\delta(1-\delta)}{1-\delta^2}[(1-\alpha^*)V(q) + \alpha^* V(Y,Y)]. \end{aligned} \tag{19}$$

To explain, the first term c shows that the deviant strategy saved the cost ϵ in this period. The second term is the payoff within the partnership assuming that the play conforms to the c_T-strategy from the next period on. The third term is the continuation payoff in this period if the partner dies so that this player goes to the matching pool with a letter (since the current partner follows the c_T-strategy). The fourth term is the continuation payoff in the next period when the current partner punishes and dies so that this player goes back to the matching pool without a letter. The last term is the continuation payoff in later periods when the partner dies.

By contrast, the c_T-strategy's continuation payoff is

$$V(Y,Y) = \frac{c-\epsilon}{1-\delta^2} + \frac{\delta(1-\delta)}{1-\delta^2}[(1-\alpha^*)V(q) + \alpha^* V(Y,Y)].$$

By computation, $D \leqq V(Y,Y)$ is equivalent to

$$\begin{aligned} D &= c + \frac{\delta^2}{1-\delta^2}(c-\epsilon) + \frac{\delta(1-\delta)}{1-\delta^2}[(1-\alpha^*)V(q) + \alpha^* V(Y,Y)] \\ &\quad -\delta^3(1-\delta)\big\{[(1-\alpha^*)V(q) + \alpha^* V(Y,Y)] - V(q)\big\} \\ &\leqq (c-\epsilon) + \frac{\delta^2}{1-\delta^2}(c-\epsilon) + \frac{\delta(1-\delta)}{1-\delta^2}[(1-\alpha^*)V(q) + \alpha^* V(Y,Y)] \\ \iff \quad & \epsilon \leqq \delta^3(1-\delta)\alpha^*[V(Y,Y) - V(q)] \qquad (20) \\ \iff \quad & \{1+\delta-\alpha^*\delta(1-\delta^{2T})(1-\delta^2)\}\epsilon \leqq \delta^3\alpha^*(1-\delta^{2T})(c-d). \end{aligned}$$

where the last equivalence comes from (9). □

Proof of Proposition 2.

(i) At $T = \underline{\tau}(\delta) \in \Re_{++}$, for any $\epsilon \leqq \bar{\epsilon}(T)$,

$$v(q) - \frac{1}{\delta^2}[\alpha^*\delta\{v(Y,Y) - v(q)\} - \epsilon] < v^{BR}.$$

Proof of (i): From (9) and (11),

$$\begin{aligned} & v(q) - \frac{1}{\delta^2}[\alpha^*\delta\{v(Y,Y) - v(q)\} - \epsilon] - v^{BR} \\ = \; & \frac{\delta^{2T}(1+\delta)(c-\epsilon) + \{1+(1-\alpha^*)\delta\}(1-\delta^{2T})d}{1+\delta-\alpha^*\delta(1-\delta^{2T})} \\ & -\frac{1}{\delta^2}\Big[\frac{\alpha^*\delta(1-\delta^{2T})(c-\epsilon-d)}{1+\delta-\alpha^*\delta(1-\delta^{2T})}\Big] + \frac{\epsilon}{\delta^2} - v^{BR} \\ = \; & \frac{1}{\delta^2\{1+\delta-\alpha^*\delta(1-\delta^{2T})\}} \times \\ & \Big[\delta^2(1+\delta)\{\delta^{2T}c + (1-\delta^{2T})d\} - \delta^{2T+2}(1+\delta)\epsilon - \alpha^*\delta^3(1-\delta^{2T})d \\ & -\alpha^*\delta(1-\delta^{2T})(c-\epsilon-d) + \{1+\delta-\alpha^*\delta(1-\delta^{2T})\}\epsilon - v^{BR}\delta^2\{1+\delta-\alpha^*\delta(1-\delta^{2T})\}\Big] \end{aligned}$$

Notice that Greve-Okuno (2009) defines $\underline{\tau}(\delta)$ as the solution to $\delta^{2T}c + (1-\delta^{2T})d = v^{BR}$. Hence the above expression becomes

$$\begin{aligned} & \delta^2\{1+\delta-\alpha^*\delta(1-\delta^{2T})\}\Big[v(q) - \frac{1}{\delta^2}[\alpha^*\delta\{v(Y,Y)-v(q)\} - \epsilon] - v^{BR}\Big] \\ = \; & -\delta^{2T+2}(1+\delta)\epsilon - \alpha^*\delta^3(1-\delta^{2T})d \\ & -\alpha^*\delta(1-\delta^{2T})(c-\epsilon-d) + \{1+\delta-\alpha^*\delta(1-\delta^{2T})\}\epsilon + \alpha^*\delta^3(1-\delta^{2T})v^{BR} \\ = \; & \alpha^*\delta(1-\delta^{2T})\{\delta^2(v^{BR}-d) - (c-d)\} + \epsilon(1+\delta)(1-\delta^{2T+2}) \\ = \; & (1-\delta^{2T+2})\{\epsilon(1+\delta) - \alpha^*\delta(1-\delta^{2T})(c-d)\}. \end{aligned}$$

Therefore, if $\epsilon < \frac{\alpha^*\delta(1-\delta^{2T})(c-d)}{1+\delta}$, then the above equation is negative.

Finally, we show that

$$\bar{\epsilon}(T) < \frac{\alpha^*\delta(1-\delta^{2T^*})(c-d)}{1+\delta}.$$

From the definition of $\bar{\epsilon}(T)$,

$$\begin{aligned}
& \bar{\epsilon}(T) < \frac{\alpha^*\delta(1-\delta^{2T^*})(c-d)}{1+\delta} \\
\iff & \frac{\delta^3\alpha^*(1-\delta^{2T})(c-d)}{1+\delta-\alpha^*\delta(1-\delta^{2T})(1-\delta^2)} < \frac{\alpha^*\delta(1-\delta^{2T^*})(c-d)}{1+\delta} \\
\iff & \delta^2(1+\delta) < 1+\delta-\alpha^*\delta(1-\delta^{2T})(1-\delta^2) \\
\iff & \alpha^*\delta(1-\delta^{2T})(1-\delta^2) < (1+\delta)(1-\delta^2),
\end{aligned}$$

which holds from (4) (that is, $1+\delta-\alpha^*\delta(1-\delta^{2T}) = \frac{\delta^{2T+1}}{\alpha^*} > 0$). Thus, at $T = \underline{\tau}(\delta)$, for any $\epsilon \in (0, \bar{\epsilon}(T)]$,

$$v(q) - \frac{1}{\delta^2}[\alpha^*\delta\{v(Y,Y) - v(q)\} - \epsilon] < v^{BR}.$$

Note that Greve-Okuno (2009) shows that $\underline{\tau}(\delta) > 0$, which is intuitively obvious as well.

(ii) For any $\epsilon > 0$, $\lim_{T\to 0}\big[v(q) - \frac{1}{\delta^2}[\alpha^*\delta\{v(Y,Y) - v(q)\} - \epsilon]\big] > v^{BR}$.

Proof of (ii): From (12) and (10),

$$\begin{aligned}
& v(q) - \frac{1}{\delta^2}[\alpha^*\delta\{v(Y,Y) - v(q)\} - \epsilon] \\
= & \frac{1}{\delta}\big[\alpha^*(1+\delta)(c-\epsilon-d) + \delta d\big] - \frac{1}{\delta^2}\big[\alpha^*\delta\frac{(1-\delta^{2T})(c-\epsilon-d)}{1+\delta-\alpha^*\delta(1-\delta^{2T})} - \epsilon\big] \\
= & \frac{1}{\delta}\big[\alpha^*(1+\delta)(c-\epsilon-d) + \delta d\big] - \frac{1}{\delta^2}\cdot\frac{\alpha^{*2}\delta(1-\delta^{2T})(c-\epsilon-d)}{\delta^{2T+1}} + \frac{\epsilon}{\delta^2} \quad \text{(from (4))} \\
= & (c-\epsilon-d)\big\{\frac{\alpha^*(1+\delta)}{\delta} - \frac{\alpha^{*2}\delta(1-\delta^{2T})}{\delta^{2T+3}}\big\} + d + \frac{\epsilon}{\delta^2}
\end{aligned}$$

From (4), $\lim_{T\to 0}\alpha^* = \delta/(1+\delta)$, and $\lim_{T\to 0}(1-\delta^{2T}) = 0$. Therefore

$$\begin{aligned}
& \lim_{T\to 0}\big[v(q) - \frac{1}{\delta^2}[\alpha^*\delta\{v(Y,Y) - v(q)\} - \epsilon]\big] \\
= & (c-\epsilon-d) + d + \frac{\epsilon}{\delta^2} = c + \frac{1-\delta^2}{\delta^2}\epsilon > c - \frac{1-\delta^2}{\delta^2}(g-c) = v^{BR}.
\end{aligned}$$

From (i) and (ii), the Intermediate Value Theorem implies that for any $\epsilon \in (0, \bar{\epsilon}(\underline{\tau}(\delta))]$, there exists $\tau^* \in \Re$ such that $0 < \tau^* < \underline{\tau}(\delta)$ and the condition (15) is satisfied with equality. Let T^* be the smallest integer which is not less than τ^*. Then $T^* \leqq T^{NL}(\delta)$ and $T^* > 0$. □

Proof of Proposition 3. We show that if $\epsilon = 0$ and $T^* = T^{NL}$, then $v(q) > v^{NL}(c_{T^{NL}})$. From (11) and (16),

$$\begin{aligned}
v(q) - v^{NL}(c_{T^{NL}}) &= \frac{\delta^{2T^*}(1+\delta)c + \{1+(1-\alpha^*)\delta\}(1-\delta^{2T^*})d}{1+\delta-\alpha^*\delta(1-\delta^{2T^*})} - \{(1-\delta^{2T^*})d + \delta^{2T^*}c\} \\
&= \frac{1}{1+\delta-\alpha^*\delta(1-\delta^{2T^*})}\Big[\delta^{2T^*}c[(1+\delta) - \{1+\delta-\alpha^*\delta(1-\delta^{2T^*})\}] \\
&\quad +(1-\delta^{2T^*})d[\{1+(1-\alpha^*)\delta\} - \{1+\delta-\alpha^*\delta(1-\delta^{2T^*})\}]\Big] \\
&= \frac{1}{1+\delta-\alpha^*\delta(1-\delta^{2T^*})}\Big[\alpha^*\delta^{2T^*+1}(1-\delta^{2T^*})c - \alpha^*\delta^{2T^*+1}(1-\delta^{2T^*})d\Big] \\
&= \frac{\alpha^*\delta^{2T^*+1}(1-\delta^{2T^*})(c-d)}{1+\delta-\alpha^*\delta(1-\delta^{2T^*})} > 0.
\end{aligned}$$

Since $v(q)$ is decreasing in T, for any $T \leqq T^{NL}$, the above inequality holds. For sufficiently small $\epsilon > 0$, the inequality is valid. □

References

Ben-Porath E., Kahneman M., 2003. Communication in repeated games with costly monitoring. Games Econ. Behav. 44, 227-250.

Carmichael L., MacLeod B., 1997. Gift giving and the evolution of cooperation. Int. Econ. Rev. 38, 485-509.

Datta, S., 1996. Building trust. Working Paper, London School of Economics.

Ellison, G., 1994. Cooperation in the prisoner's dilemma with anonymous random matching. Rev. Econ. Stud. 61, 567-588.

Flesch, J., Perea A. 2009. Repeated games with voluntary information purchase. Games Econ. Behav. 66, 124-145.

Fujiwara-Greve, T., Okuno-Fujiwara M., 2009. Voluntarily separable repeated prisoner's dilemma. Rev. Econ. Stud. 76, 993-1021.

Ghosh, P., Ray D., 1996. Cooperation in community interaction without information flows. Rev. Econ. Stud. 63, 491-519.

Gibbons, R., Katz L., 1991. Layoffs and lemons. J. Labour Econ. 9, 351-380.

Kandori, M., 1992. The use of information in repeated games with imperfect monitoring. Rev. Econ. Stud. 59, 581-593.

Kranton, R., 1996. The formation of cooperative relationships. J. Law, Econ., Org. 12, 213-233.

Lazear, E., 1981. Agency, earnings profiles, productivity, and hours restrictions. Amer. Econ. Rev. 71, 606-620.

Matsushima, H., 1990. Long-term partnership in a repeated prisoner's dilemma with random matching. Econ. Let. 34, 245-248.

Milgrom, P., D. North, and B. Weingast. 1990. The role of institutions in the revival of trade: the law merchant, private judges and the Champagne fairs. Econ. Politics 2, 1-23.

Miyagawa, E., Miyahara, Y., Sekiguchi T. 2008. The folk theorem for repeated games with observation costs. J. Econ. Theory 139, 192-221.

Okuno-Fujiwara, M., T. Fujiwara-Greve, and Suzuki N., 2007. Social norms and voluntary cooperation. Econ. Rev. (Keizai Kenkyu) 58, 110-121. (In Japanese.)

Okuno-Fujiwara, M., Postlewaite A., 1995. Social norms and random matching games. Games Econ. Behav. 9, 79-109.

Shapiro, C., Stiglitz J., 1984. Equilibrium unemployment as a worker discipline device. Amer. Econ. Rev. 74, 433-444.