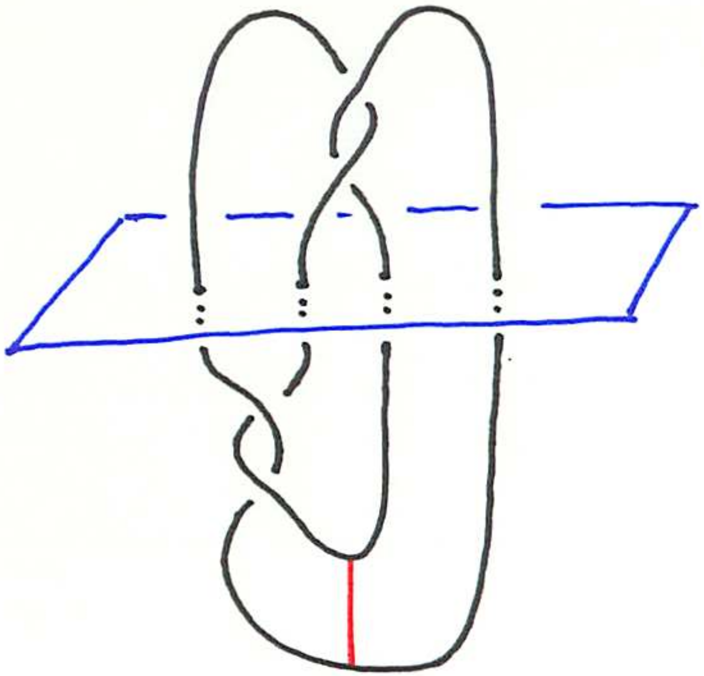
End Invariants for SL(2, ℂ)-characters of the once-punctured torus associated with Heckoid groups



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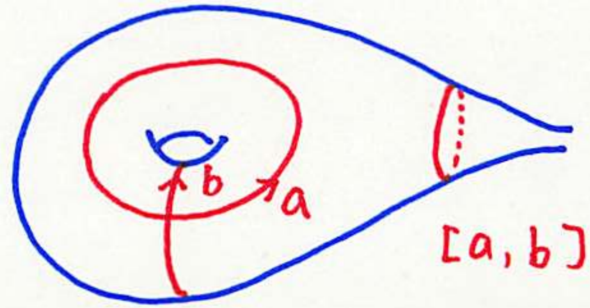
T : once-punctured torus



$\pi_1(T) = \langle a, b \mid - \rangle$

$\quad\Psi$

$[a, b]$: peripheral

Def $\rho : \pi_1(T) \to (P)SL(2, \mathbb{C})$ is type-preserving, if

(1) $\rho([a, b])$ is parabolic

(2) ρ is irreducible

$$\tilde{R} := \{ \rho : \pi_1(T) \to SL(2, \mathbb{C}) \text{ type-pres} \} / \text{conj}$$

$\downarrow \ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$$R := \{ \rho : \pi_1(T) \to PSL(2, \mathbb{C}) \text{ type-pres} \} / \text{conj}$$

Fact For ${}^\forall \rho \in \widetilde{\mathcal{R}}$, $(x, y, z) := (\mathrm{tr}\,\rho(a), \mathrm{tr}\,\rho(ab), \mathrm{tr}\,\rho(b))$

is a nontrivial Markoff triple, i.e.

$$x^2 + y^2 + z^2 = xyz$$

$$(x, y, z) \neq (0, 0, 0)$$

$$\begin{array}{ccl} \widetilde{\mathcal{R}} & \cong & \Phi := \{\text{nontrivial Markoff triples}\} \\ \downarrow & & \downarrow \\ \mathcal{R} & \cong & \Phi / \sim \end{array}$$

where $(x, y, z) \sim (x, -y, -z)$

$(-x, y, -z)$

$(-x, -y, z)$

Examples

(1) If $\rho \leftrightarrow (3,3,3)$, then ρ is Fuchsian and

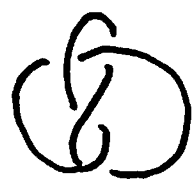
$\mathbb{H}^2 / \mathrm{Im}\,\rho$ is the hyperbolic once-punctured torus

with $\mathbb{Z}/3\mathbb{Z}$ symmetry.

(2) If $\rho \leftrightarrow \left(\frac{3+\sqrt{3}i}{3}, \frac{3-\sqrt{3}i}{3}, \frac{3+\sqrt{3}i}{3} \right)$. then

$\mathrm{Im}\,\rho \lhd \Gamma < PSL(2.\mathbb{C})$, st

$\mathbb{H}^3/\Gamma$ = punctured torus bundle over S^1 with monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$= S^3 - K$ where $K =$ 

$\Gamma / \mathrm{Im}\,\rho \cong \mathbb{Z}$

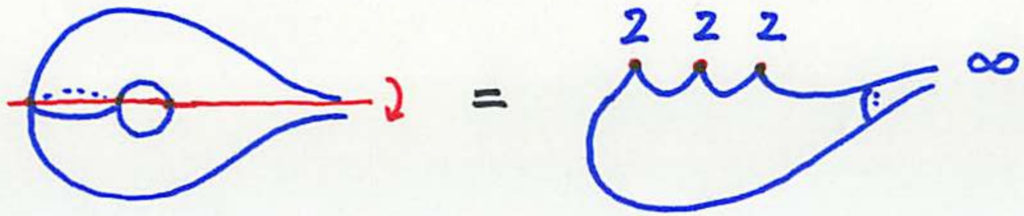
i.e $\mathbb{H}^3/\Gamma$ is the infinite cyclic cover of $S^3 - K$

(3) If $\rho \leftrightarrow (x, ix, 0)$ for some $x \in \mathbb{C}^*$, then

$\mathrm{Im}\,\rho$ is commensurable with

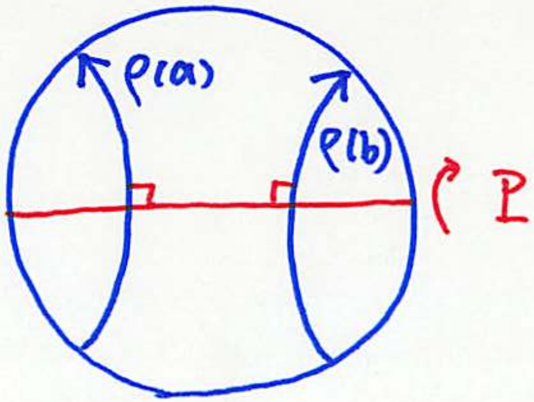
$$G_\omega := \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \right\rangle \quad \text{where } \omega = -x^2$$

(proof)

- $\mathcal{O} :=$ 

$\pi_1(T) \triangleleft \pi_1(\mathcal{O})$ index 2 subgroup

Then any $\rho \in R$ extends to a rep $\pi_1(\mathcal{O}) \to PSL(2,\mathbb{C})$



$P\,\rho(a)\,P^{-1} = \rho(a)^{-1}$

$P\,\rho(b)\,P^{-1} = \rho(b)^{-1}$

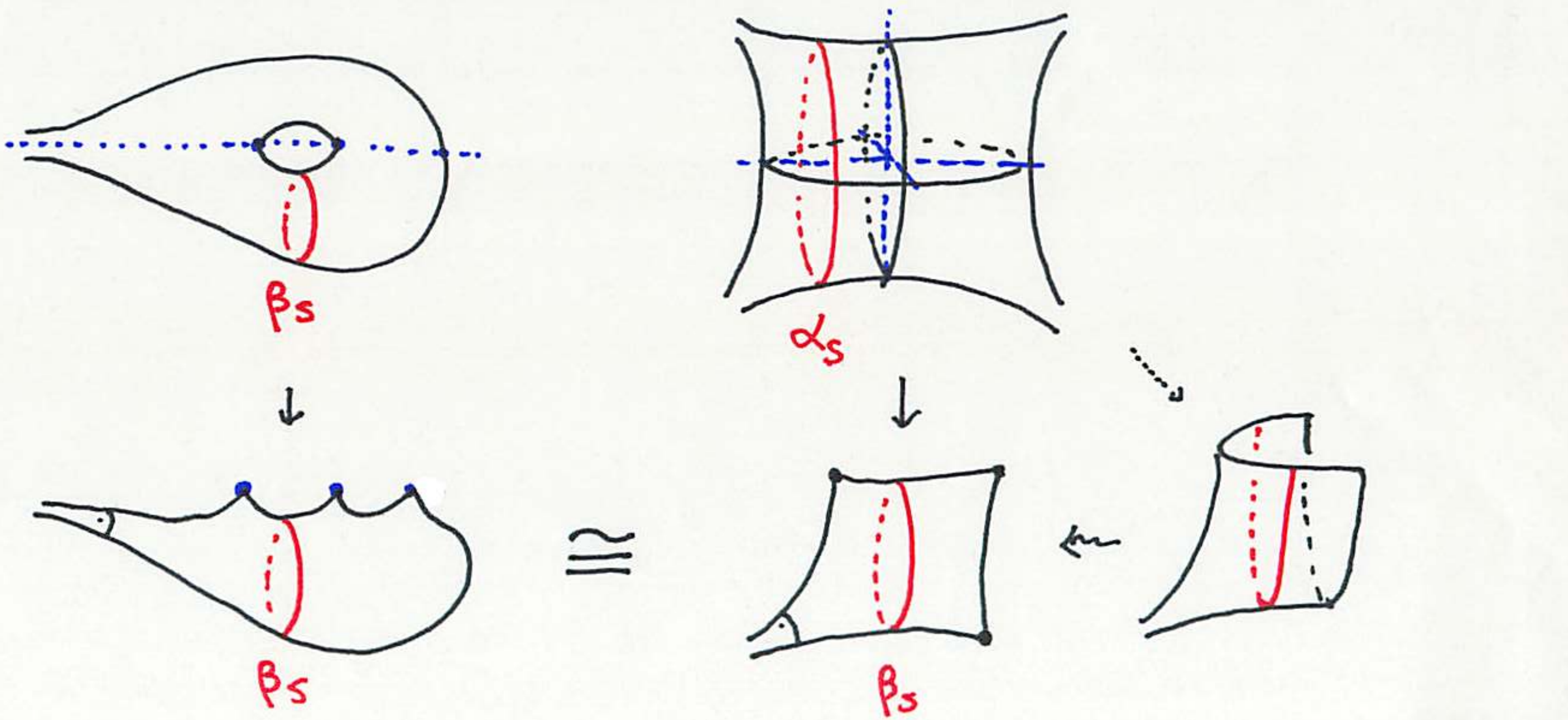
$P^2 = 1$

- S := 4-punctured sphere, then

$\pi_1(S) \lhd \pi_1(\mathcal{O})$ with $\pi_1(\mathcal{O}) / \pi_1(S) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

Moreover $\pi_1(T) \lhd \pi_1(\mathcal{O}) \rhd \pi_1(S)$

$\beta_s^2 \longleftrightarrow \beta_s^2 \longleftrightarrow \alpha_s$ for any $s \in \hat{\mathbb{Q}}$



- So any $\rho \in \mathcal{R}$ determines $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$

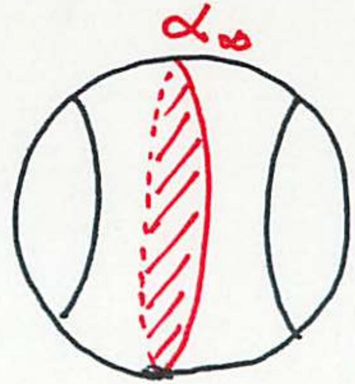
Moreover, $\operatorname{tr} \rho(\beta_{1/0}) = 0$

$\Leftrightarrow$ $\rho(\beta_{1/0})$ is elliptic of order 2

$\Leftrightarrow$ $\rho(\alpha_{1/0}) = \rho(\beta_{1/0}^2) = 1$

$\Leftrightarrow$ ρ descends to a representation

$$\begin{array}{ccc} \pi_1(S) / \langle\langle \alpha_{1/0} \rangle\rangle & \to & PSL(2.\mathbb{C}) \\ \text{\rotatebox{90}{$\cong$}} & & \vee \\ \pi_1(B^3 - t(\infty)) & \to & G_\omega = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \right\rangle \end{array}$$



- If $|\omega| > 2$, then G_ω lies in the Riley slice of Schottky space, ie

$$G_\omega \ : \ \text{discrete free}$$

$$\mathbb{H}^3 \cup \Omega(G_\omega) / G_\omega \cong B^3 - t(\infty)$$

- If $\omega = \dfrac{1+\sqrt{3}i}{2}$, then $\mathbb{H}^3 / G_\omega \cong S^3 - K$ with $K =$ 

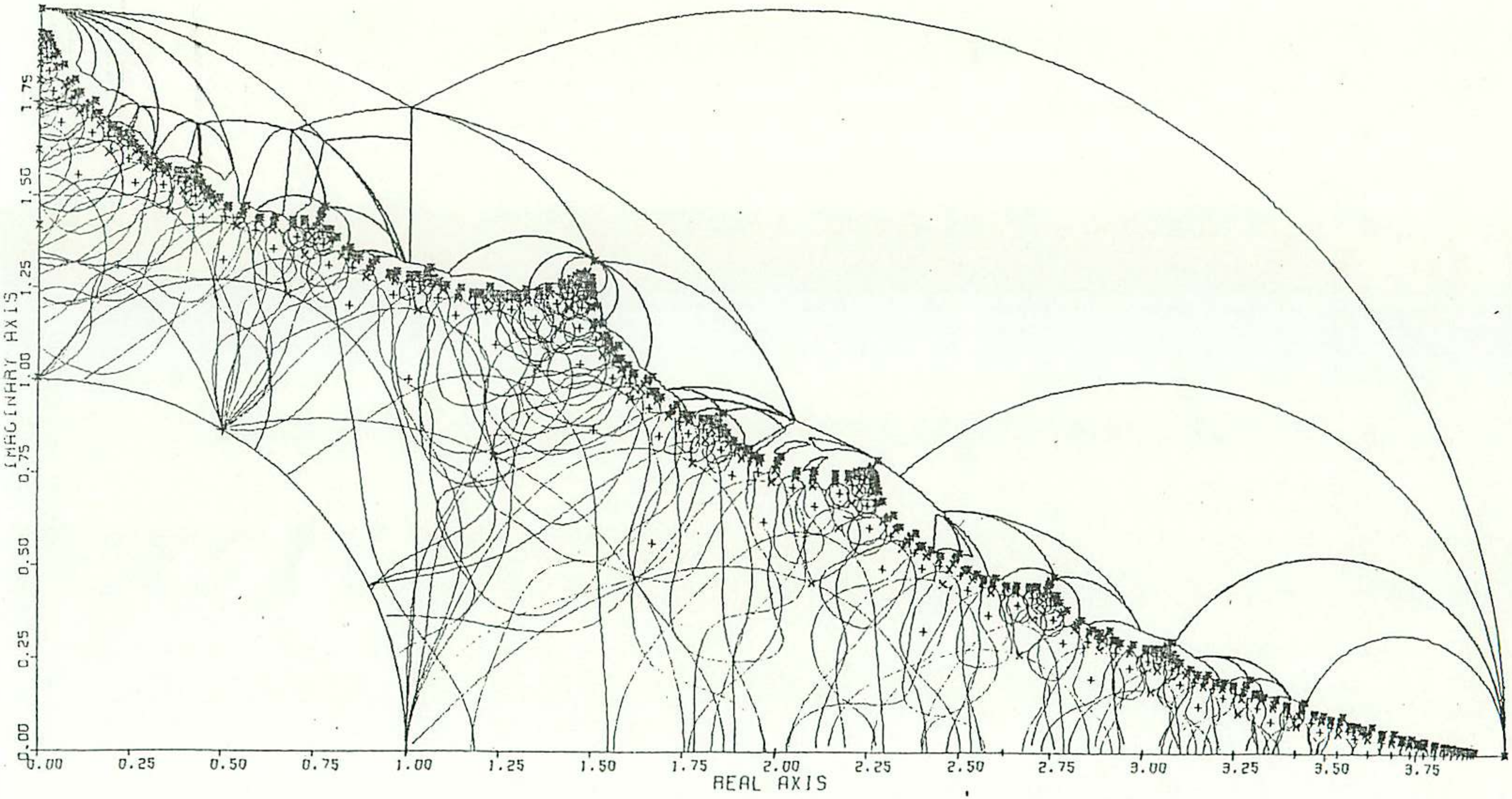
[Akiyoshi - S- Wada - Yamashita]

There is a natural path which joins a given 2-bridge knot group to the Riley slice.

Riley's pioneering exploration of two parabolic groups

GROUPS GENERATED BY TWO PARABOLICS A AND B(W) FOR W IN THE FIRST QUADRANT

W IS MARKED BY +, CROSS, OR * ACCORDING AS G(W) IS A PELL OR REAL HECKOID GROUP, A NON-REAL HECKOID GROUP, OR A CUSP GROUP.
EACH CONTOUR IS A LEVEL CURVE ABS C21(T)(W)=1 FOR SOME WORD T IN A, B, AND IS TERMINATED AT THE AXES OR UNIT CIRCLE.
INSIDE EACH CONTOUR G(W) IS INDISCRETE WHEN C21(T).NE.0 .
OUTSIDE THE *'S G(W) IS FREE, DISCRETE, NON-RIGID.
THESE GROUPS LIE IN CELLS WHERE THE GROUPS OF EACH CELL HAVE SIMILAR FORD DOMAINS.



[Agol 2002]

For $\omega \in \mathbb{C}^*$, G_ω is discrete iff

(0) G_ω : free Kleinian group

(1) $\pm\omega \in \{ 4\cos^2 \pi/p \mid p \geq 3 \}$

(2) $G_\omega \cong$ 2-bridge knot/link group

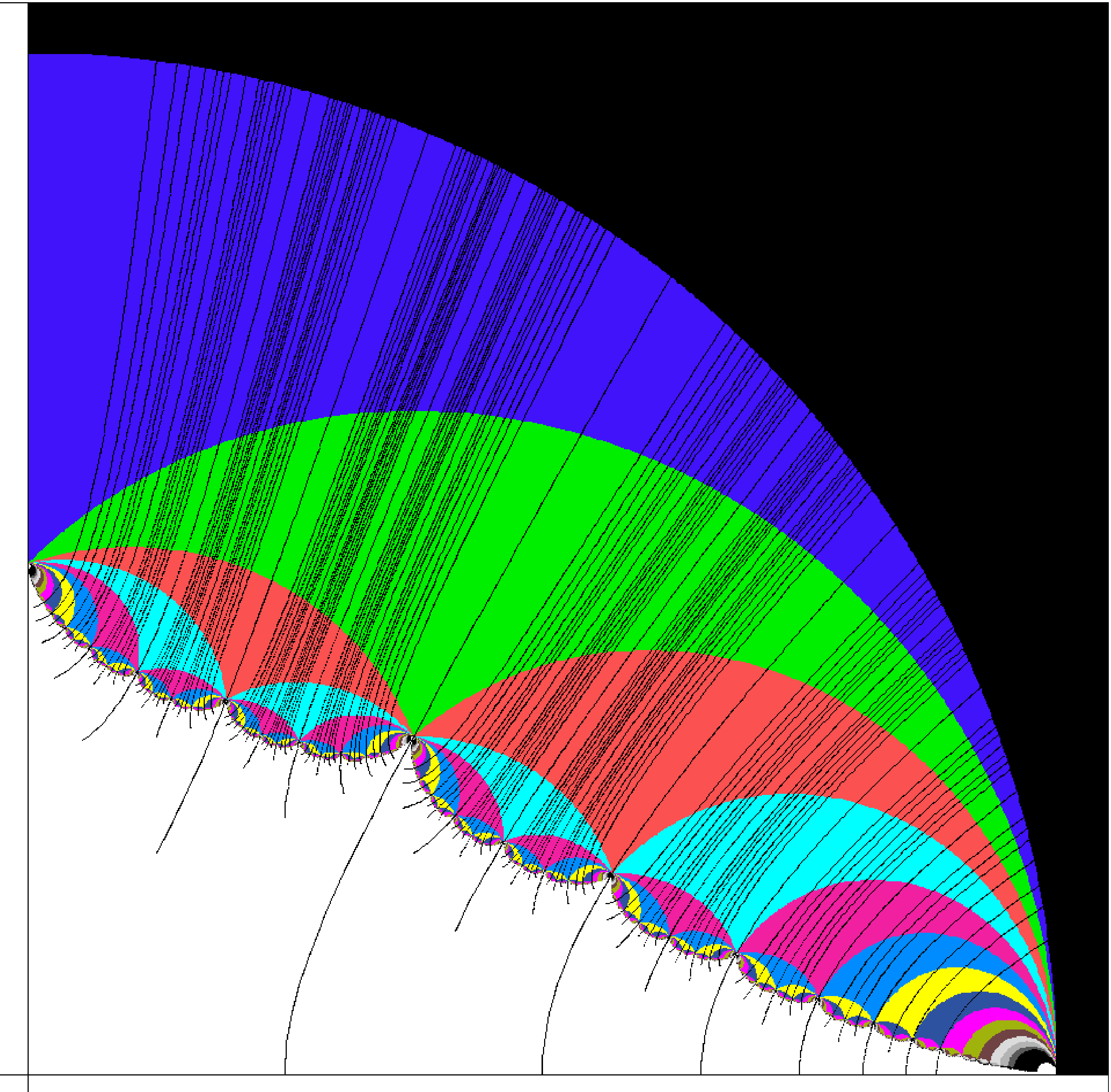
$\cong \pi_1(S^3 - K(r)) \qquad r = q/p \quad (q \not\equiv \pm 1 \bmod p)$

(3) $G_\omega \cong$ Heckoid group $H(r:d)$ for a 2-bridge link $K(r)$

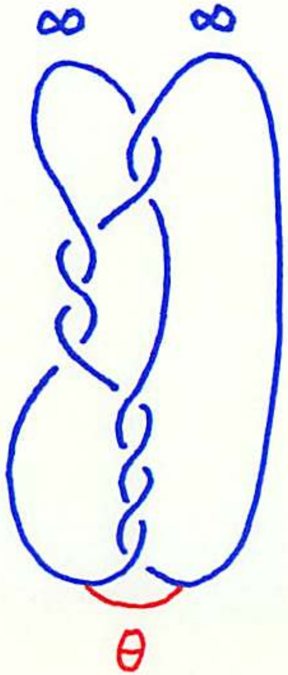
$=$ Orbifold fundamental group

of the Heckoid orbifold $\mathcal{O}(r;d)$

Remark The terminology "Heckoid group" was introduced by [Riley, 1992]



[Akiyoshi - S - Wada - Yamashita 2007] announced :



- $^{\exists}\{C(r;\theta)\}_{0 \le \theta \le 2\pi}$

  continuous family of hyperbolic cone manifolds,

  except when "$r \neq \pm 1/p$ in $\mathbb{Q}/\mathbb{Z}$ and $\theta = 2\pi$".

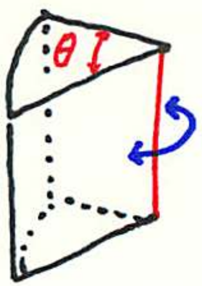
- The holonomy representation

  $\rho_\theta : \pi_1(\,|C(r:\theta)| - \text{cone axis}) \to PSL(2.\mathbb{C})$

  has a discrete image iff $\theta = 2\pi/d$ $\left(d \in \frac{1}{2}\mathbb{N}_{\geq 2}\right)$

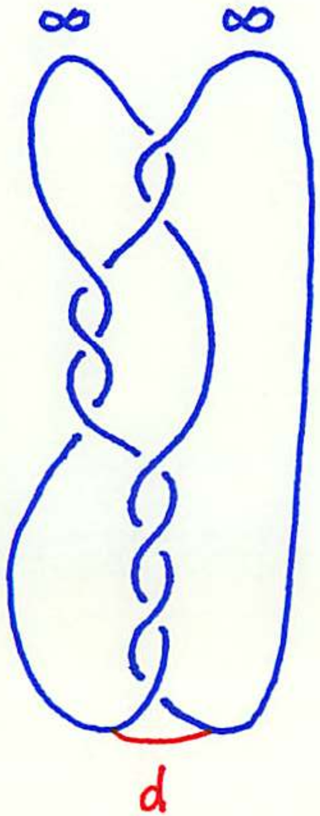
- The Heckoid group $H(r;d) = \mathrm{Im}\,\rho_\theta$

  where $\theta = 2\pi/d$



cone singularity

Heckoid "orbifold" $\mathcal{O}(r:d)$ for a 2-bridge link $K(r)$



- $r = [2,3,4] = \cfrac{1}{2+\cfrac{1}{3+\frac{1}{4}}} = \frac{13}{30} \in \mathbb{Q}-\mathbb{Z}$

  $\left(r \notin \mathbb{Z} \Leftrightarrow K(r) \neq \bigcirc \,,\ \bigcirc\bigcirc \right)$

- $d \in \frac{1}{2}\mathbb{N}_{\geqq 3} = \left\{ \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \cdots \right\}$
- $|\mathcal{O}(r;d)| = S^3 - K(r)$, $\infty \leftrightarrow$ "parabolic locus"

If $d \in \mathbb{N}_{\geqq 2}$, then $\mathcal{O}(r:d)$ is a hyperbolic orbifold

ie $\mathcal{O}(r;d) = \mathbb{H}^3 / H(r:d)$, $H(r;d) = \pi_1^{orb}(\mathcal{O}(r:d))$

If $d \notin \mathbb{N}$, then $\mathcal{O}(r;d)$ is not even an orbifold.

But $H(r:d) =$ "$\pi_1^{orb}(\mathcal{O}(r:d))$" makes sense.

D : Farey tessellation

$$D^{(0)} = \hat{\mathbb{Q}} = \{\text{essential simple loops on } T\}/\text{isotopy}$$

$$\begin{array}{ccc} \hat{\mathbb{Q}} & & \{\ldots\}/\text{isotopy} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ s & \longleftrightarrow & \beta_s \end{array}$$

Each $\rho \in \tilde{R}$ determines a **Markoff map**

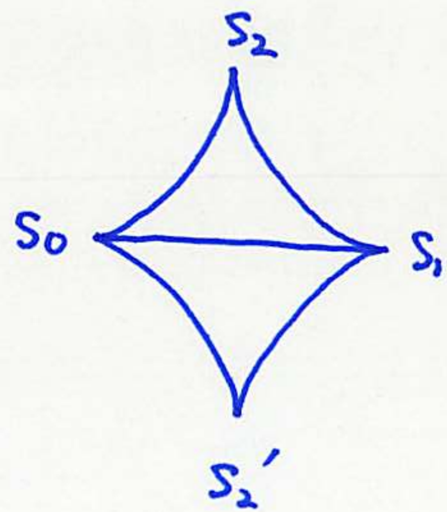
$$\phi : D^{(0)} \longrightarrow \mathbb{C}$$

$$s \longmapsto \mathrm{tr}(\rho(\beta_s))$$

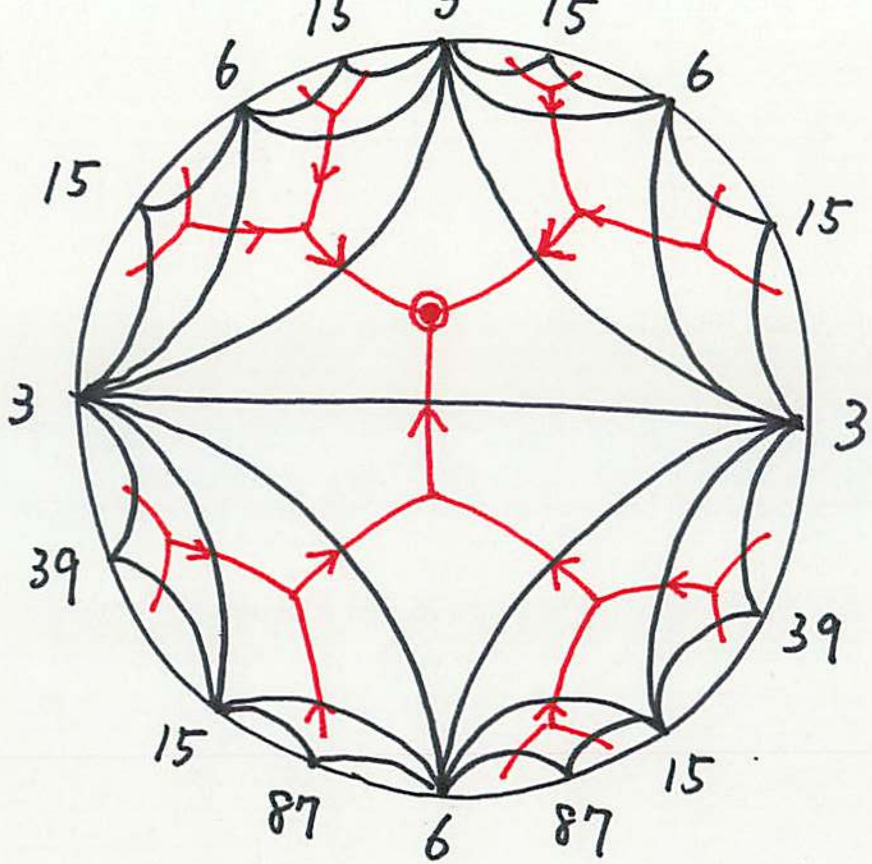
(i) $(x, y, z) := (\phi(s_0), \phi(s_1), \phi(s_2))$

is a Markoff triple

(ii) $z + w = xy$ where $w = \phi(s_2')$

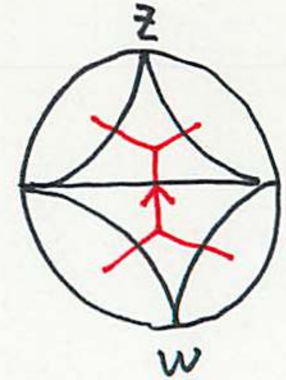


Integral Markoff map



∃ unique sink

where



$|z| < |w|$

[Bowditch] [Tan - Wong - Zhang]

- $\lambda \in \partial \mathbb{H}^2 = \hat{\mathbb{R}}$ is an **end invariant** of ρ, if $\exists K > 0$, $\exists \{r_n\}$ distinct elements of $\hat{\mathbb{Q}}$, st

  (i) $r_n \to \lambda$

  (ii) $|\phi(r_n)| < K$ for $\forall n$ (Recall $\phi(r_n) = tr(\rho(\beta_n))$)

- $\mathcal{E}(\rho) := \{$ end invariants of $\rho \} \subset \hat{\mathbb{R}}$

Examples

(1) If $\rho \leftrightarrow (3, 3, 3)$, then $\mathcal{E}(\rho) = \emptyset$

(2) If ϕ is real (ie $\phi(\hat{\mathbb{Q}}) \subset \mathbb{R}$), then ρ is Fuchsian and hence $\mathcal{E}(\rho) = \emptyset$.

(3) If ρ is quasi-fuchsian, then $\mathcal{E}(\rho) = \emptyset$

Conjecture (Bowditch)

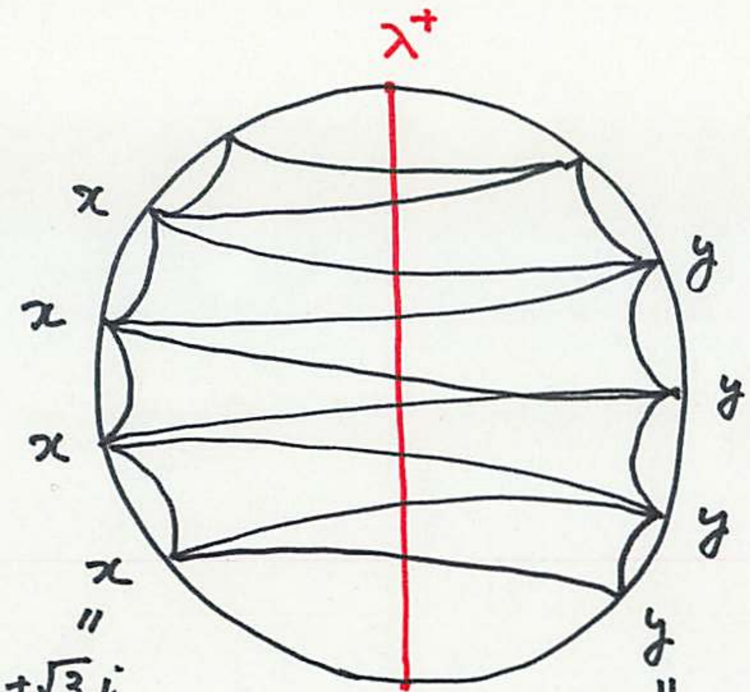
ρ is quasi-fuchsian iff $\mathcal{E}_B(\rho) = \emptyset$,

where $\mathcal{E}_B(\rho) = \mathcal{E}(\rho) \cup \{ s \in \hat{\mathbb{Q}} \mid \phi(s) = \pm 2 \}$

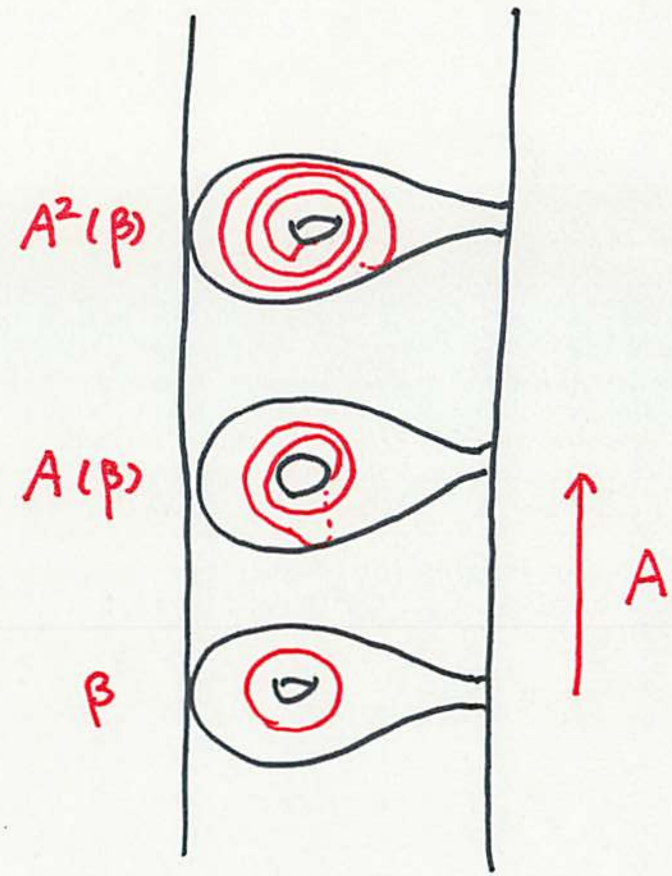
ρ is geometrically finite iff $\mathcal{E}(\rho) = \emptyset$.

(4) If $\rho \leftrightarrow \left(\frac{3+\sqrt{3}i}{2}, \frac{3-\sqrt{3}i}{2}, \frac{3+\sqrt{3}i}{2}\right)$ then

$\mathcal{E}(\rho) = \{\lambda^\pm\}$, where $\lambda^\pm$ is the eigen values of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$



$x = \frac{3+\sqrt{3}i}{2}$ $\quad$ $y = \frac{3-\sqrt{3}i}{2}$



$\{A^n(\beta)\}$ have the same length

$\rho : \pi_1(T) \hookrightarrow PSL(2, \mathbb{C})$ discrete faithful

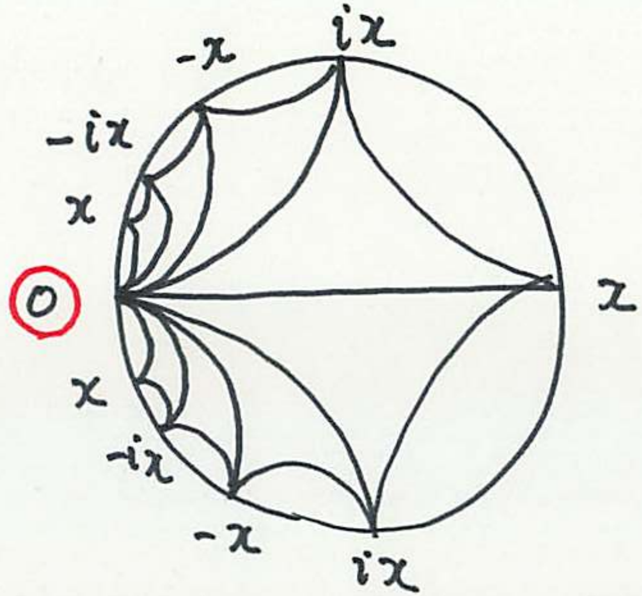
Then $\mathcal{E}(\rho) = \{$ ending lamination of a geom. infinite end $\}$

$$= \begin{cases} \{\nu^-, \nu^+\} & \text{if } \rho \text{ is doubly degenerate} \\ \{\nu\} & \text{if } \rho \text{ is singly degenerate} \\ \emptyset & \text{if } \rho \text{ is geom. finite} \end{cases}$$

Question If $\rho \in R$ has $\mathcal{E}(\rho) = \{\nu^-, \nu^+\}$, then is ρ discrete faithful representation which is doubly degenerate ?

(5) If $\rho \leftrightarrow (\phi(0/1), \phi(1/1), \phi(1/0)) = (x, ix, 0)$,

then $1/0 \in \mathcal{E}(\rho)$



If $\omega = -x^2$ lies in the Riley slice, then $\mathcal{E}(\rho) = \{1/0\}$

If $\omega = \dfrac{1+\sqrt{3}i}{2}$, then $\mathcal{E}(\rho)$ is equal to the

limit set $\Lambda(\Gamma)$ of a cetrain Fuchsian group. and so

$\mathcal{E}(\rho) \cong$ Cantor set.

[Tan - Wong - Zhang]

If ρ is discrete in the sense that

$\phi(\hat{\mathbb{Q}})$ is a discrete subset of $\mathbb{C}$,

and if $\mathcal{E}(\rho)$ has at least 3 elements,

then $\mathcal{E}(\rho)$ is a Cantor set or $\hat{\mathbb{R}}$.

Conjecture [T W Z]

Suppose $\mathcal{E}(\rho)$ has at least two accumulation point.

Then either $\mathcal{E}(\rho)$ is a Cantor set or $\hat{\mathbb{R}}$.

Conjecture [T W Z]

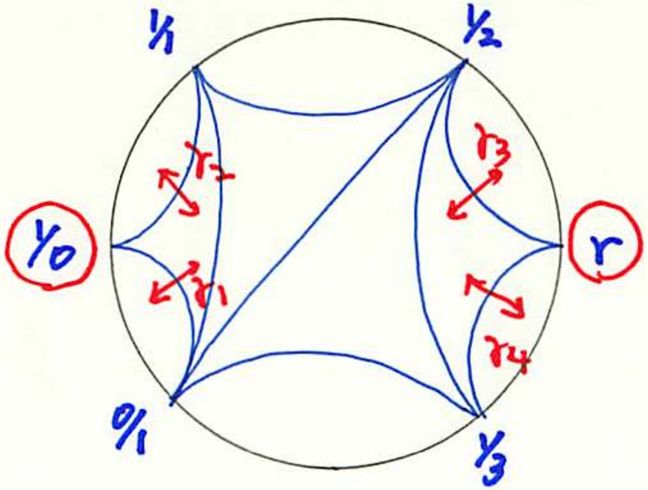
Suppose that $\mathcal{E}(\rho) = \mathcal{E}(\rho')$, $\mathcal{E}(\rho)$ has at least two elements,

and $\mathcal{E}(\rho) \neq \hat{\mathbb{R}}$. Then $\rho = \rho'$ up to sign change auto and conj.

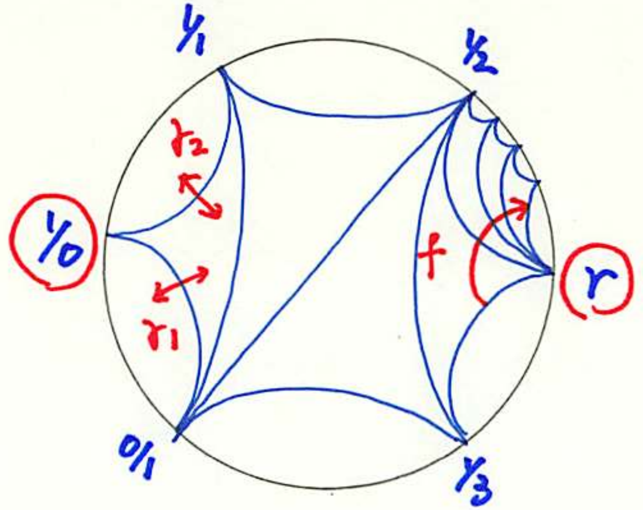
Main Theorem

(1) If ρ corresponds to the discrete faithful reprensentation of $\pi_1(S^3 - K(r))$ for a hyperbolic 2-bridge link $K(r)$, then $\mathcal{E}(\rho) = \Lambda(\hat{\Gamma}_r)$, the limit set of $\hat{\Gamma}_r < \mathrm{Aut}(D) < \mathrm{Isom}\,\mathbb{H}^2$.

(2) If ρ corresponds to the discrete faithful representation of an even Heckoid group $H(r:d)$, $r \in \mathbb{Q} - \mathbb{Z}$, $d \in \mathbb{N}_{\geqq 2}$, then $\mathcal{E}(\rho) = \Lambda(\Gamma(r:d))$



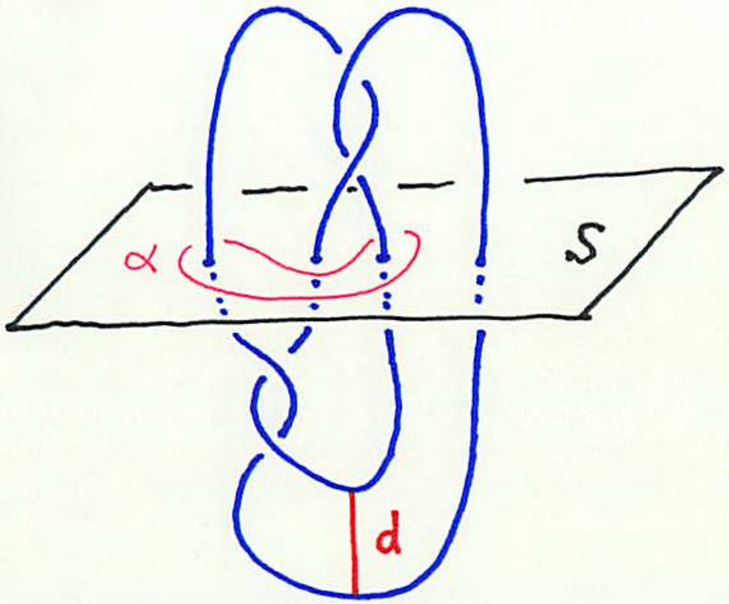
$\hat{\Gamma}_r = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle$



f : shift by 2 d blocks

(d = 2)

$\Gamma(r:n) = \langle \gamma_1, \gamma_2, f \rangle$



S : level 4-punctured sphere in $\mathcal{O}(r; d)$

$\alpha \subset S$ essential simple loop

Problem

(1) For an essential simple loop α in S, when is it null-homotopic in $\mathcal{O}(r:d)$?

" " peripheral or torsion " " ?

(2) For two essential simple loops in S when are they homotopic in $\mathcal{O}(r:d)$?

Main Theorem

For even Heckoid group $H(r;d)$ $r \in \mathbb{Q} - \mathbb{Z}$, $d \in \mathbb{N}_{\geq 2}$ the following hold.

(1) $\alpha_s \sim 1$ iff $s \in \Gamma(r;d)\infty$

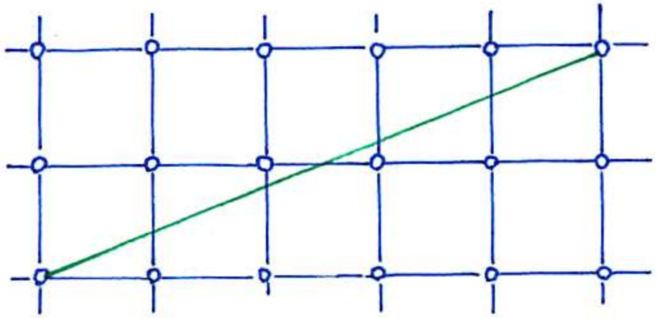
(2) α_s can never be peripheral

(3) α_s is torsion iff $s \in \Gamma(r;d)r$

(4) $\alpha_s \sim \alpha_{s'}$ iff $s' \in \Gamma(r:d) \cdot s$

$S := \mathbb{R}^2 - \mathbb{Z}^2 / \langle \pi\text{-rotations around punctures} \rangle$: 4-punctured sphere

(Conway sphere)



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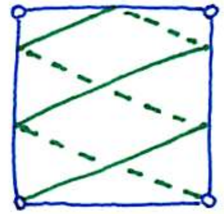
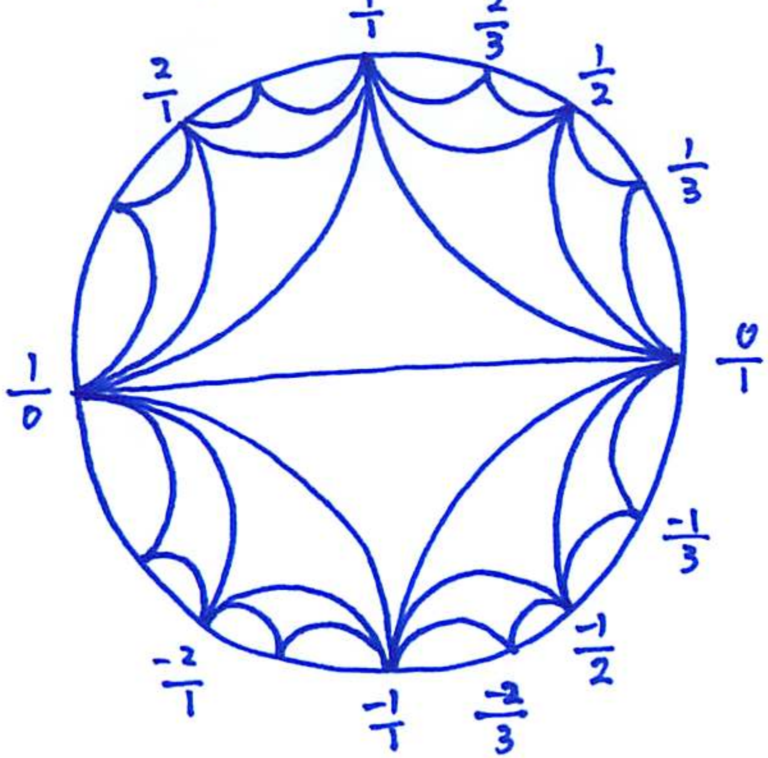


$\delta_{2/5}$

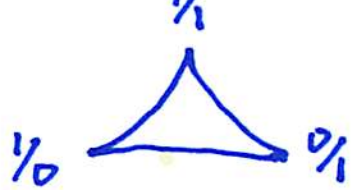
D : Farey tessellation



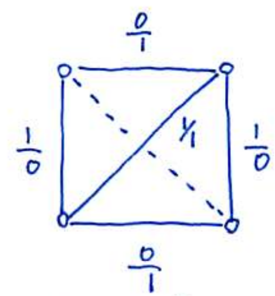
Vertex set of $D = \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\frac{1}{0}\} \ni r$

$\underset{1-1}{\leftrightarrow}$ {essential simple loops on S} $\ni \alpha_r$

$\underset{1-2}{\leftrightarrow}$ {essential simple arcs on S} $\ni \delta_r$



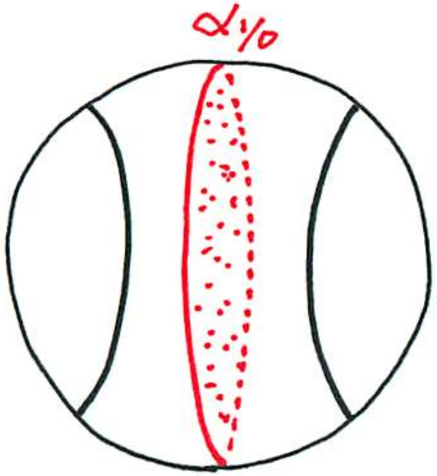
$\leftrightarrow$



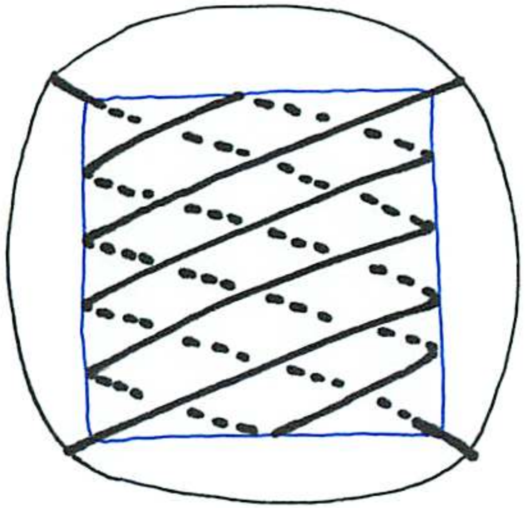
Farey triangle

ideal triangulation of S

Rational tangle $(B^3, t(r))$ of slope r :



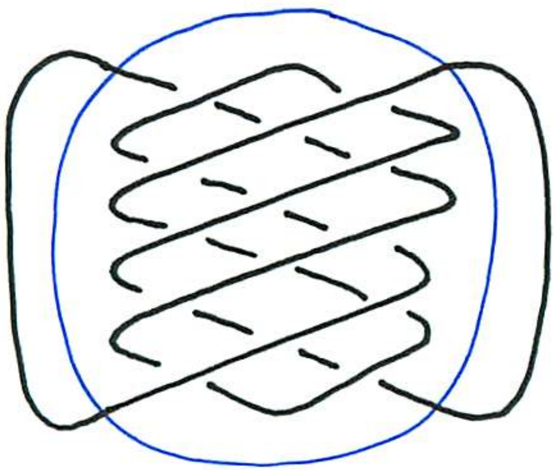
$(B^3, t(1/0))$



$(B^3, t(2/5))$

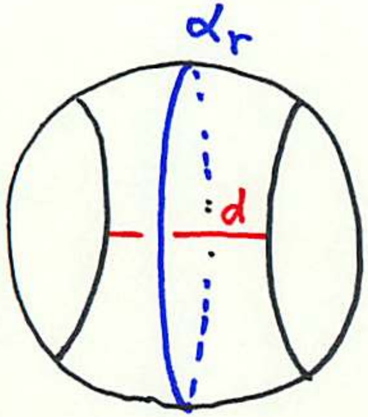
$$\pi_1(B^3 - t(r)) \cong \pi_1(S) / \langle\langle \alpha_r \rangle\rangle$$

$(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$: 2-bridge link of slope r



$$G(K(r)) := \pi_1(S^3 - K(r)) \cong \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle$$

Orbifold fundamental group $(d \in \mathbb{N}_{\geqq 2})$



$$\pi_1^{orb}((B^3, t(r):d)) \cong \pi_1(S) / \langle\!\langle \alpha_r^d \rangle\!\rangle$$

$$\mathcal{O}(r:d) = (B^3, t(\infty)) \cup (B^3, t(r);d)$$

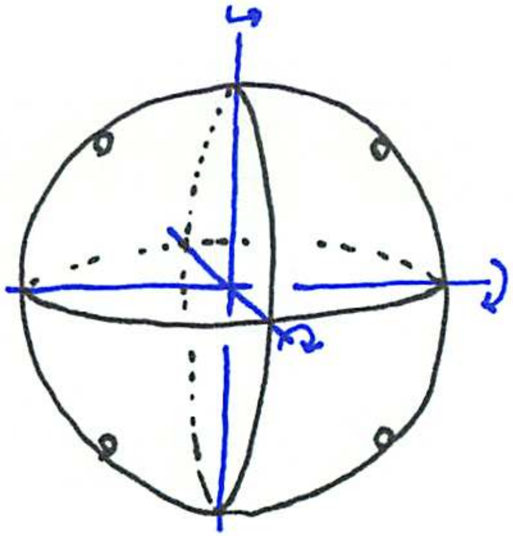
$$\begin{aligned} H(r:d) &= \pi_1^{orb}(\mathcal{O}(r,d)) \\ &\cong \pi_1(S) / \langle\!\langle \alpha_\infty, \alpha_r^d \rangle\!\rangle \\ &\cong \pi_1(B^3 - t(\infty)) / \langle\!\langle \alpha_r^d \rangle\!\rangle \\ &\cong \langle x, y \mid u_r^d = 1 \rangle \qquad (\alpha_r \leftrightarrow u_r \in \langle x, y \mid - \rangle) \end{aligned}$$

Example $H(\frac{2}{5}:d) = \langle x, y \mid (x y x y^{-1} x^{-1} y x y x^{-1} y^{-1})^d \rangle$

Mapping class group $\mathcal{M}(S) := \pi_0 \operatorname{Diff}(S)$

$$1 \to (\mathbb{Z}/2\mathbb{Z})^2 \to \mathcal{M}(S) \xrightarrow{\Phi} \underset{\substack{\cong \\ PGL(2,\mathbb{Z})}}{\operatorname{Aut}(D)} \to 1$$

$(\mathbb{Z}/2\mathbb{Z})^2$-action on S acts trivially on D.



$$\mathcal{M}(S) \overset{\Phi}{\twoheadrightarrow} \mathrm{Aut}(D)$$

$$\cup$$

$$\mathcal{M}(B^3, t(\infty)) := \pi_0 \mathrm{Diff}(B^3, t(\infty))$$

$$\cup$$

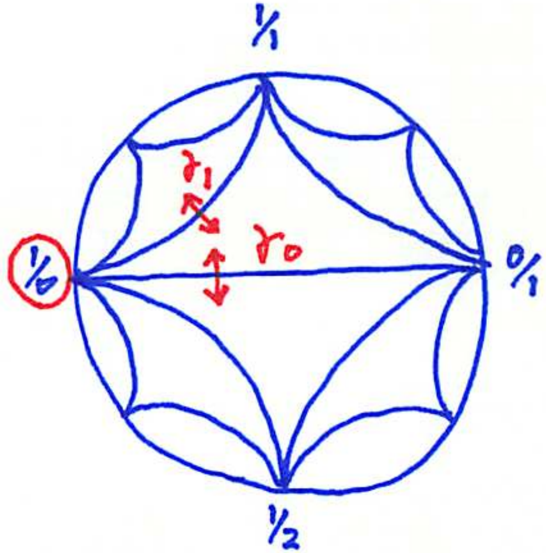
$$\mathcal{M}_0(B^3, t(\infty)) := \{ f \in \mathcal{M}(B^3, t(\infty)) \mid f_* = \mathrm{id} \in \mathrm{Out}(\pi_1(B^3 - t(\infty))) \}$$

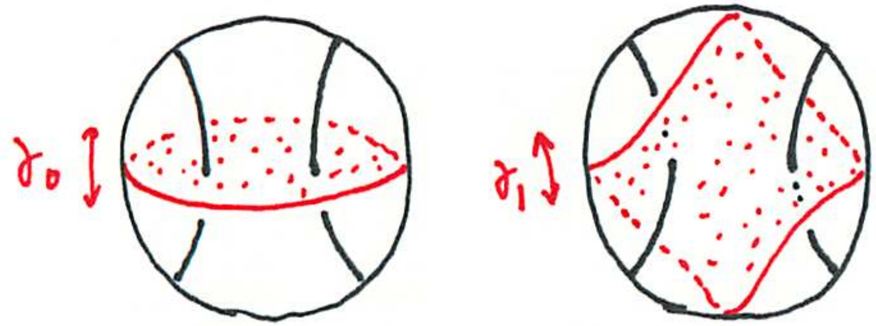
Observation

$$\mathrm{Aut}(D)$$

$$\cup$$

$$\Gamma_\infty := \Phi(\mathcal{M}_0(B^3, t(\infty))) = \left\langle \begin{array}{l} \text{reflections in the edges of } D \\ \text{with endpoint } \infty \end{array} \right\rangle$$

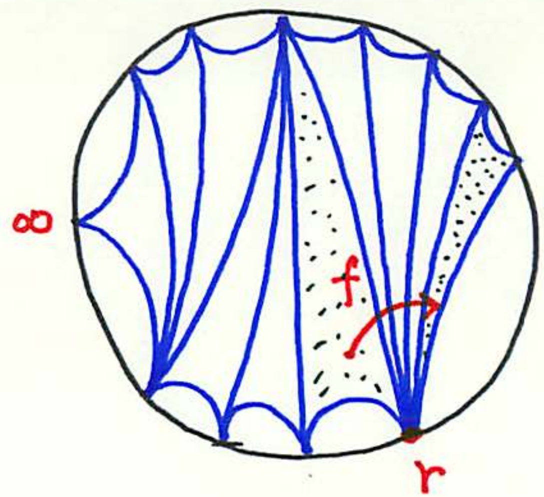




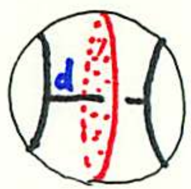
$$\mathcal{M}(S) \xrightarrow{\ \bar{\Phi}\ } \mathrm{Aut}(\mathcal{D})$$
$$\cup$$
$$\mathcal{M}(B^3, t(r); d) = \pi_0 \, \mathrm{Diff}(B^3, t(r) : d)$$
$$\cup$$
$$\mathcal{M}_0(B^3, t(r); d) = \{ f \in \mathcal{M}(B^3, t(r); d) \mid f_* = id \in \mathrm{Out}(\pi_1^{orb}(B^3, t(\infty) : d)) \}$$

Observation

$$\mathrm{Aut}(\mathcal{D})$$
$$\cup$$
$$\Gamma_r(d) := \bar{\Phi}(\mathcal{M}_0(B^3, t(r); d))$$
$$= \left\langle \underbrace{\text{parabolic transformation}}_{f} \text{ of } \mathcal{D}, \atop \text{centered at } r, \text{ by } 2d \text{ blocks} \right\rangle$$
$$\cong \mathbb{Z}$$

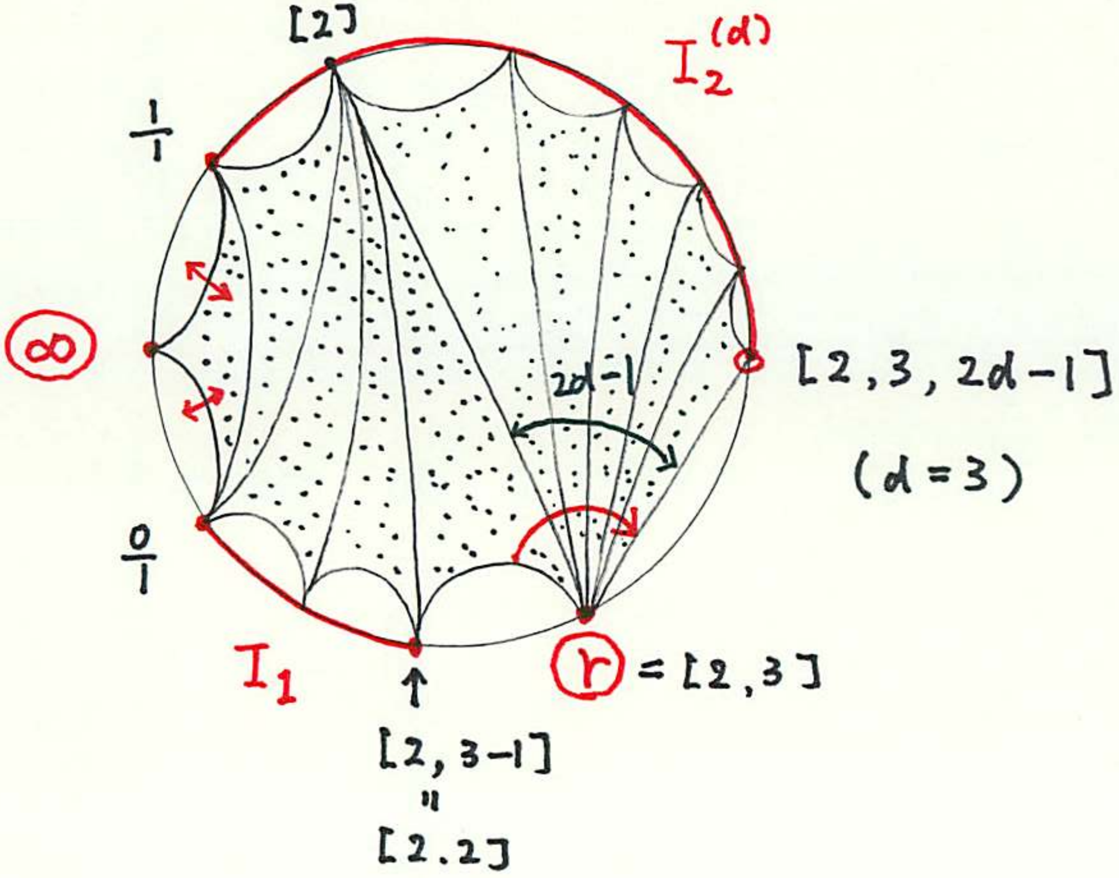


- f is induced by the d-th power of the Dehn twist along the meridian disk



$(d = 2)$

$$\Gamma(r;d) := \langle \Gamma_\infty, \Gamma_r(d) \rangle \subset \mathrm{Aut}(D)$$
$$\cong \Gamma_\infty * \Gamma_r(d) \cong D_\infty * \mathbb{Z}$$



- The limit set $\Lambda(\Gamma(r;d))$
  $=$ the closure of $\Gamma(r;d)\{\infty, r\}$
- $I_1 \cup I_2^{(d)}$ is a fundamental domain of the action of $\Gamma(r;d)$ on $\Omega(\Gamma(r,d)) = \partial \mathbb{H}^2 - \Lambda(\Gamma(r;d))$

Lemma

(1) For any $s \in \hat{\mathbb{Q}}$, there is a unique $s_0 \in I_1 \cup I_2^{(d)} \cup \{\infty, r\}$, st $s \in \Gamma(r;d)\{s_0\}$.

(2) $\alpha_s \sim \alpha_{s_0}$ in $\mathcal{O}(r;d)$.

(proof)

(1) follows from the fact that $I_1 \cup I_2^{(d)} \cup \{\infty, r\} = \bar{D} \cap \partial \mathbb{H}^2$ for some fundamental domain D for $\Gamma(r:d)$.

(2) follows from the following facts:

(i) If $s = \gamma(s_0)$ for some $\gamma \in \Gamma_\infty$, then $\alpha_s \sim \alpha_{s_0}$ in $B^3 - t(\infty)$ and so in $\mathcal{O}(r;d)$.

(ii) If $s = \gamma(s_0)$ for some $\gamma \in \Gamma_r(d)$, then $\alpha_s \sim \alpha_{s_0}$ in $(B^3, t(r); d)$ and so in $\mathcal{O}(r:d)$

$\square$

Proof of if part of Main Theorem

Suppose $S \in \Gamma(r;d)\{\infty\}$.

Then, by Lemma, $\alpha_S \sim \alpha_\infty$ in $\mathcal{O}(r;d)$.

On the other hand, $\alpha_\infty \sim 1$ in $B^3 - t(\infty)$ and so in $\mathcal{O}(r:d)$.

Hence $\alpha_S \sim 1$ in $\mathcal{O}(r:d)$.

$\square$

The only if part of Main Theorem is equivalent to:

Theorem

If $s \in I_1 \cup I_2^{(d)}$, then $\alpha_s \neq 1$ in $O(r;d)$.

(Idea of the proof)

Apply the small cancellation theory to the one-relator presentation

$$H(r:d) = \langle x, y \mid (u_r)^d \rangle$$

where u_r is a cyclically reduced word in $\{x, y\}$ representing the simple loop α_r.

Small Cancellation Theory

[Dehn] Consider :

length $4g$

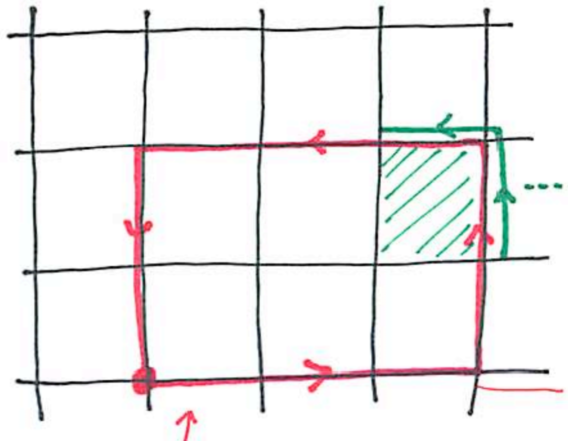
$$\pi_1(F_g) = \langle x_1, y_1, \cdots, x_g, y_g \mid \prod_{i=1}^{g} [x_i, y_i] \rangle \qquad (g \geqq 2)$$

If a cyclically reduced word $w \in \langle x_1, y_1, \cdots, x_g, y_g \rangle$ represents a trivial element in $\pi_1(F_g)$, then the cyclic word w contains a subword of the cyclic word represented by the relator $\prod_{i=1}^{g} [x_i, y_i]$ (or its inverse) of length $\geqq 2g+1 = \frac{1}{2}(\text{length of } \prod_{i=1}^{g} [x_i, y_i]) + 1$.

(Thus we can find a shorter cyclic word w' st $[w] = [w']$ in $\pi_1(F_g)$.)

(Idea of Dehn's theorem)

If $g=1$, $\pi_1(F_1) = \langle x, y \mid x y \bar{x} \bar{y} \rangle$



subword of length 2 of the relator

$x\, x\, x\, y\, \boxed{y\, \bar{x}}\, \bar{x}\, \bar{x}\, \bar{y}\, \bar{y}$

lift of a cyclic word representing the trivial element

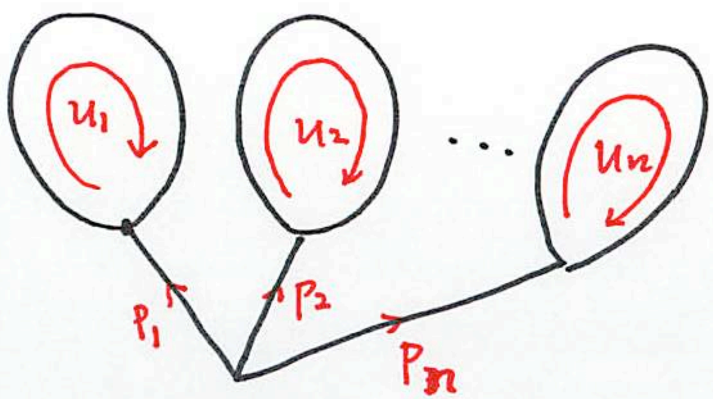
- In small cancellation theory "van-Kampen diagram" plays the role of the "region" in the universal cover bounded by the lift of a word representing trivial element.

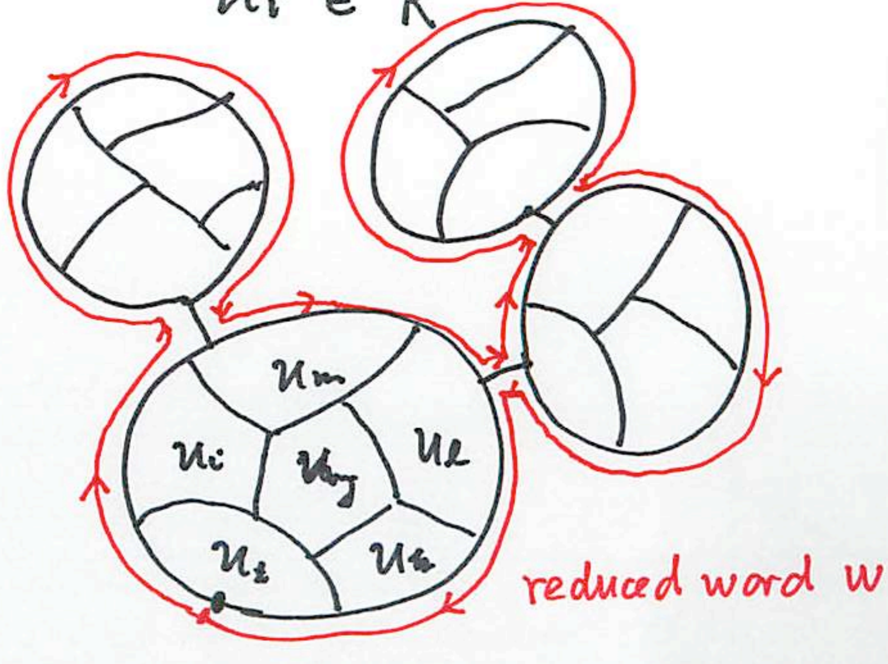
Van-Kampen Diagram

- $G = \langle x, y \mid u \rangle$ $\quad$ u : cyclically reduced
- $R := \{ \text{cyclic permutations of } u, u^{-1} \} \subset \langle x, y \rangle$
- A cyclically reduced word $w \in \langle x, y \rangle$ represents $1 \in G = \langle x, y \mid u \rangle$

$$\Leftrightarrow \quad \underset{\substack{\uparrow \\ \text{reduced}}}{w} = \underset{\substack{\uparrow \\ \text{non-reduced}}}{\prod_{i=1}^{n} p_i u_i p_i^{-1}} \in \langle x, y \rangle \qquad p_i \in \langle x, y \rangle, \quad u_i \in R$$

$\Leftrightarrow$





Def (M, ϕ) is a van-Kampen diagram over $\langle x, y \mid u \rangle$, if

M: a map, ie 2-dim cell complex embedded in $\mathbb{R}^2$,
which is simply connected

ϕ: $\{\text{oriented edges of } M\} \to \langle x, y \rangle$

(i) $\phi(e^{-1}) = \phi(e)^{-1}$

(ii) For each 2-cell D of M

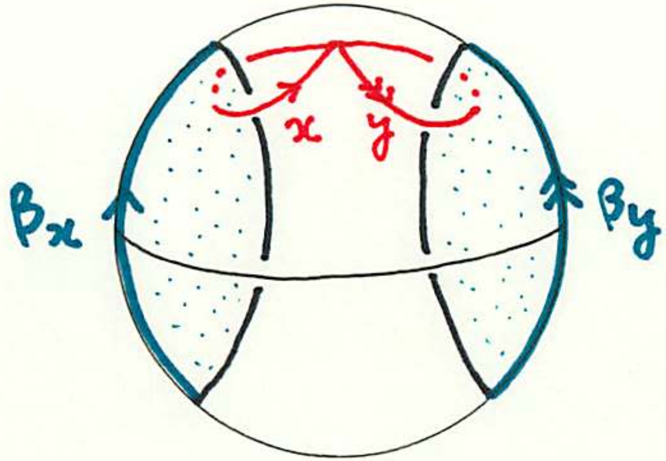
$$\underset{\displaystyle \phi(e_1)\cdot\phi(e_2)\cdot\cdots\cdot\phi(e_n)}{\phi(\partial D) \overset{\shortparallel}{}} \in R = \{\text{cyclic permutations of } u^{\pm 1}\}$$

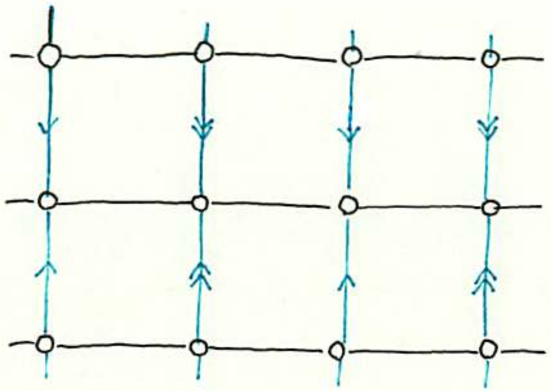
$\phi(e_1)\cdot\phi(e_2)\cdot\cdots\cdot\phi(e_n)$ is cyclically reduced

(iii) For the "boundary cycle" $\partial M = e_1 \cdots e_m$

$\phi(\partial M) := \phi(e_1) \cdots \phi(e_m)$ is cyclically reduced.

Prop A cyclically reduced word $w \in \langle x, y \rangle$ represents $1 \in \langle x, y \mid u \rangle$ iff there is a van-Kampen diagram M st $w \equiv \phi(\partial M)$.

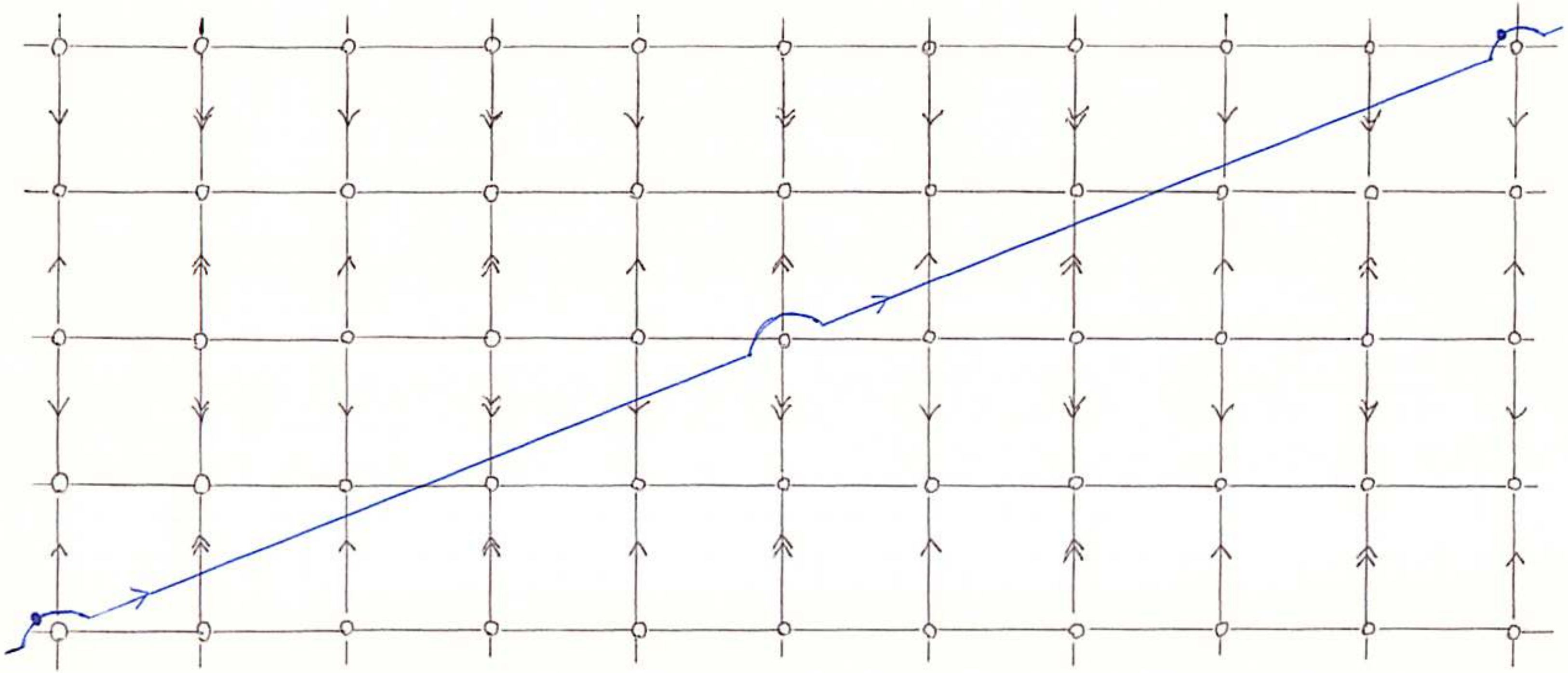




For a loop $\alpha \subset S = \partial(B^3 - t(\infty))$,

$[\alpha] \in \pi_1(B^3 - t(\infty)) = \langle x, y \mid - \rangle$

is obtained by "reading" the intersection of α with the arcs β_x and β_y.



$$[\alpha_{2/5}] := u_{2/5} = x \cdot y x \bar{y} \bar{x} \cdot y \cdot x y \bar{x} \bar{y}$$

$$= x y x \cdot \bar{y} \bar{x} \cdot y x y \cdot \bar{x} \bar{y}$$

$S(2/5) := S(u_{2/5}) := (3, 2, 3, 2)$ S-sequence

$CS(2/5) := ((3, 2, 3, 2))$ Cyclic S-sequence

Observation

- $u_{2/5} = x\, y\, x\, \bar{y}\, \bar{x}\, y\, x\, y\, \bar{x}\, \bar{y}$ is alternating, i.e. x and y apper alternatively.
- $u_{2/5}$ is determined by its S-sequence $(3, 2, 3, 2)$ and the initial letter x.
- Any alternating word w with $S(w) = S(2/5)$ is conjugate to $u_{2/5}$ or $\bar{u}_{2/5}$.

$$x\, y\, x\, \bar{y}\, \bar{x}\, y\, x\, y\, \bar{x}\, \bar{y} = u_{2/5}$$
$$y\, x\, y\, \bar{x}\, \bar{y}\, x\, y\, x\, \bar{y}\, \bar{x} \sim u_{2/5}$$
$$\bar{x}\, \bar{y}\, \bar{x}\, y\, x\, \bar{y}\, \bar{x}\, \bar{y}\, y\, x \sim \bar{u}_{2/5}$$
$$\bar{y}\, \bar{x}\, \bar{y}\, x\, y\, \bar{x}\, \bar{y}\, \bar{x}\, y\, x \sim \bar{u}_{2/5}$$

- Conjugacy class of $\{u_{2/5}, \bar{u}_{2/5}\}$ is determined by $CS(2/5)$.

Fix $0 < r = \frac{q}{p} < 1$ and set $u := \alpha_{q/p} \in \langle x, y \rangle$

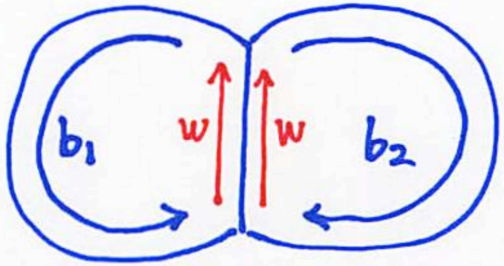
Then $G(K(r)) = \langle x, y \mid u \rangle$

Set $R := \{$ cyclic permutations of $u, u^{-1} \}$

<u>Def</u> A reduced word $w \in \langle x, y \rangle$ is a piece of R

$\Leftrightarrow$ $\exists$ distint elements $w_1, w_2 \in R$

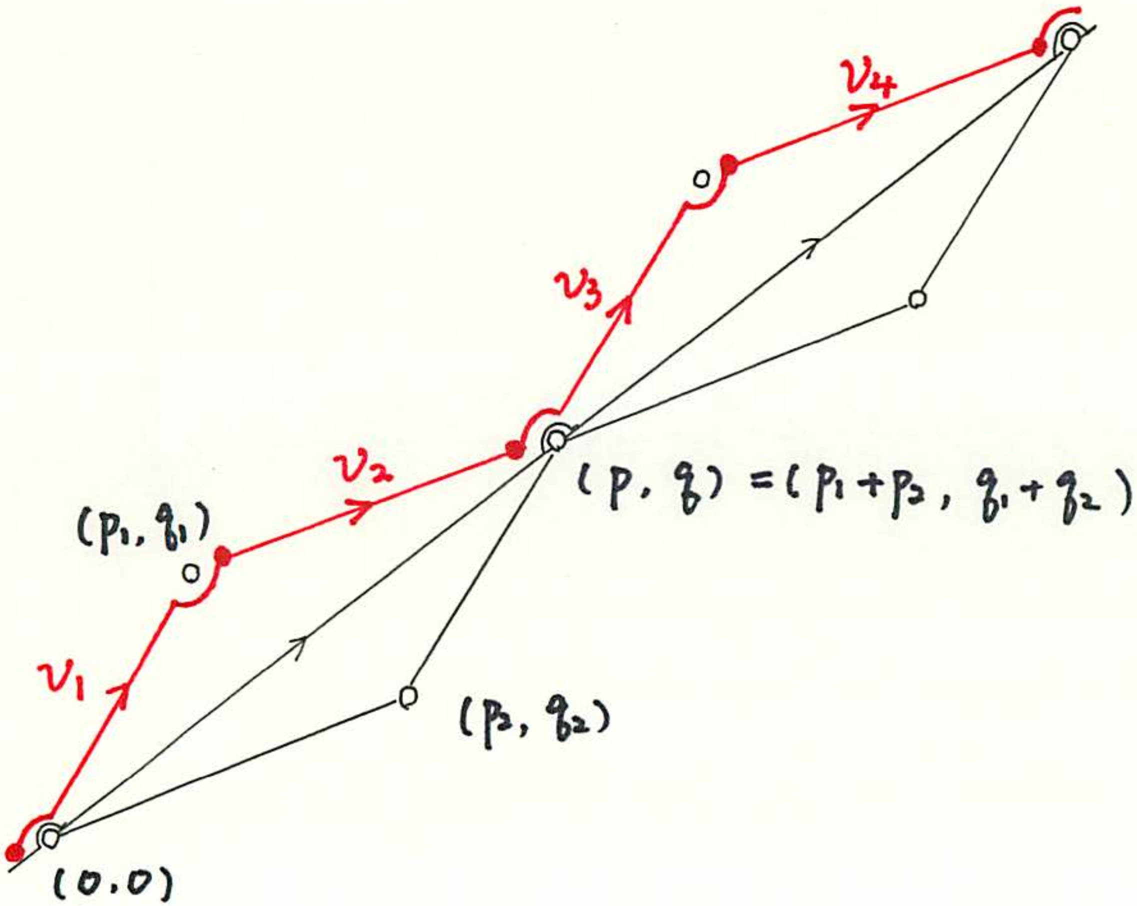
st $w_1 \equiv w b_1$, $w_2 \equiv w b_2$ as reduced words



in van-Kampen diagram

Natural decomposition of the word $u_{q/p} = [\alpha_{q/p}]$

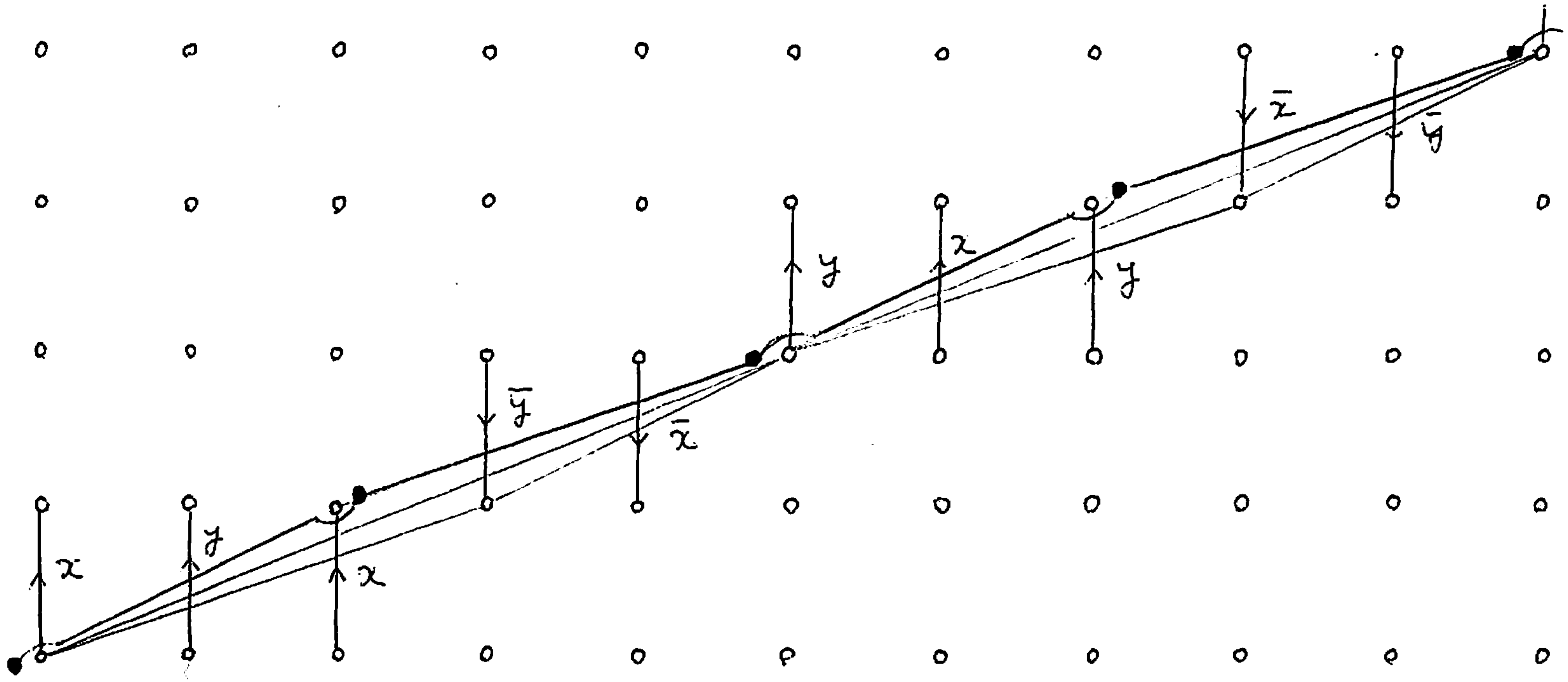
q_i/p_i $(i=1,2)$: Farey neighbors of q/p st $\dfrac{q_2}{p_2} < \dfrac{q}{p} < \dfrac{q_1}{p_1}$



The decomposition $u_{q/p} = v_1 v_2 v_3 v_4$ $\left(|v_1| = |v_3| = p_1+1,\ |v_2| = |v_4| = p_2-1\right)$

plays a crutial role.

Example $\frac{2}{5} = \frac{1}{2} \oplus \frac{1}{3}$



$$u_{2/5} = \alpha_{2/5} = \underbrace{x\,y\,x}_{v_1}\ \underbrace{\bar{y}\ \bar{x}}_{v_2}\ \underbrace{y\,x\,y}_{v_3}\ \underbrace{\bar{x}\ \bar{y}}_{v_4}$$

$$v_1,\ v_3 \leftrightarrow 1/2 \quad , |v_1| = |v_3| = 2+1 = 3$$

$$v_2,\ v_4 \leftrightarrow 1/3 \quad , |v_2| = |v_3| = 3-1 = 2$$

In general $S(u_p) = (S(v_1), S(v_2), S(v_3), S(v_4)) = (S_1, S_2, S_1, S_2)$

Key Lemma (Complete characterization of the pieces)

Suppose $d(\infty, r) \geqq 3$, and recall the decomposition

$u = v_1\, v_2\, v_3\, v_4$ arising from $\frac{q}{p} = \frac{q_1}{p_1} \oplus \frac{q_2}{p_2}$. Then:

(a) No piece can contain v_1 or v_3

(b) No piece can contain v_2 or v_4 in its "interior",

i.e. $v_{1e}\, v_2\, v_{3b}$ is not a piece,

where v_{1e} is a non-empty terminal subword of v_1

v_{3b} : : initial : : v_3.

(c) v_2 and v_4 are pieces.

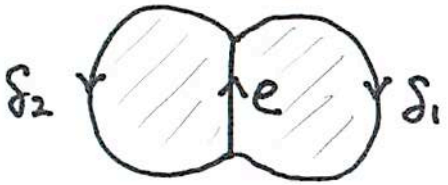
Moreover, every subword of the form

$$v_{1e}\, v_2\ ,\quad v_2\, v_{3b}\ ,\quad v_{3e}\, v_4\ ,\quad v_4\, v_{1b}$$

are pieces.

Convention for a van-Kampen diagram M

(0) M is reduced, ie for every inner edge e of M,



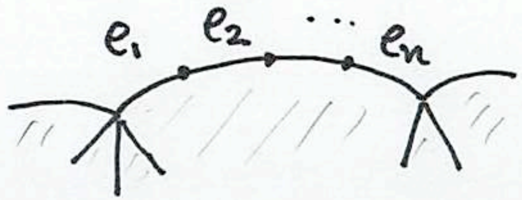
$\phi(\delta_1) \neq \phi(\delta_2)$. (Because, otherwise we can simplify M.)

(1) Every inner vertex has degree ≥ 3.

(∵)

(2) For every boundary edge e, $\phi(e)$ is a piece.

(3)



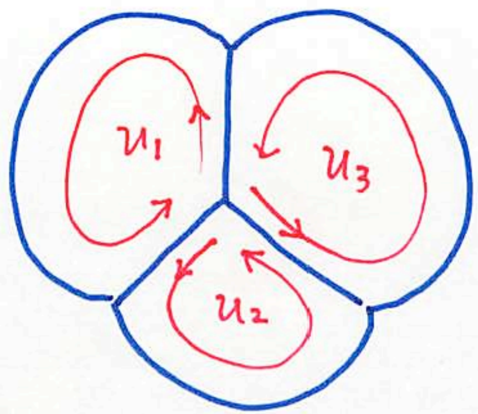
$\phi(e_1)\phi(e_2) \cdots \phi(e_n)$ cannot be expressed as a product of less than n pieces.

Key Lemma 1 R satisfies the conditions $C(4)$ and $T(4)$.

$C(4)$: Any $u \in R$ is not a product of $3 \ (= 4-1)$ pieces.

$T(4)$: If $u_1, u_2, u_3 \in R$ and if $u_{i+1} \neq u_i^{-1}$ $(1 \leq i \leq 3)$, then at least one of $u_1 u_2$, $u_2 u_3$, $u_3 u_1$ is reduced.

ie the following situation does not occur in van Kampen diagram.



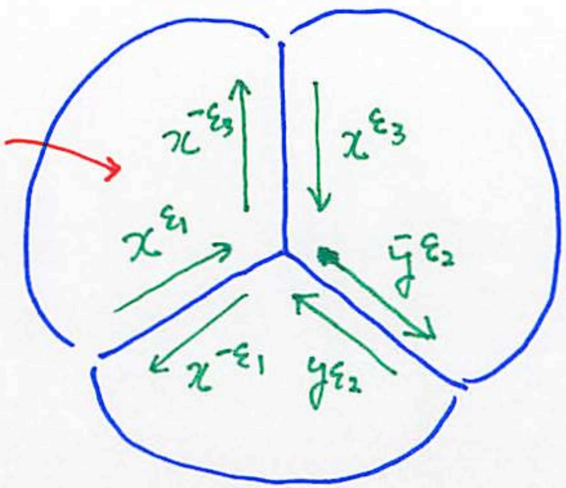
(Proof of Key Lemma)

- $C(4)$ follows from the characterization of the pieces.

(A shortest decomposition of $u = v_1 v_2 v_3 v_4$ into pieces is $v_{1b} \cdot v_{1e} v_2 \cdot v_{3b} \cdot v_{3e} v_4$, which has length $4 > 3$.)

- $T(4)$ follows from the fact that u is cyclically alternating, i.e. $u = x^{\varepsilon_1} y^{\varepsilon_2} x^{\varepsilon_3} \cdots x^{\varepsilon_{2p-1}} y^{\varepsilon_{2p}}$ $(\varepsilon_i = \pm 1)$

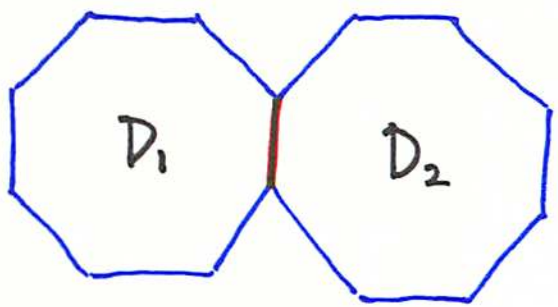
non-alternating



Heckoid group $H(r;d) = \langle x, y \mid u_r^d \rangle \quad (d \in \mathbb{N}_{\geq 2})$

Corollary $R^{(d)} := \{ u_r^{\pm d} \}$ satisfies $C(4d)$ and $T(4)$.

i.e. In any reduced van-Kampen diagram (M, φ) over $\langle x, y \mid u_r^d \rangle$, any two 2-cells can share only a "short word",



$$(\text{the "length" of the common edge}) \leq \frac{1}{4d} (\text{"length" of } \partial D_i)$$

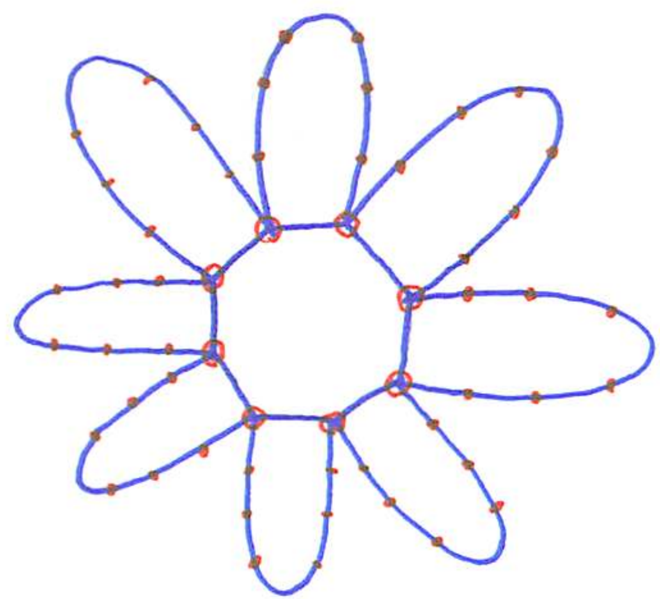
Prop (M, Φ) : reduced van-Kampen diagram over $H(r;d) = \langle x, y \mid u_r^d \rangle$,
st the boundary word $\Phi(\partial M)$ is alternating.

Set $A := \#\{ v \in \partial M^{(0)} \mid \deg_M(v) = 2 \}$

$B := \#\{ v \in \partial M^{(0)} \mid \deg_M(v) > 2 \}$

Then $A \gg B$, to be precise, $A \geqq (4d-3)B + 4d$

(proof) Use the curvature formula.

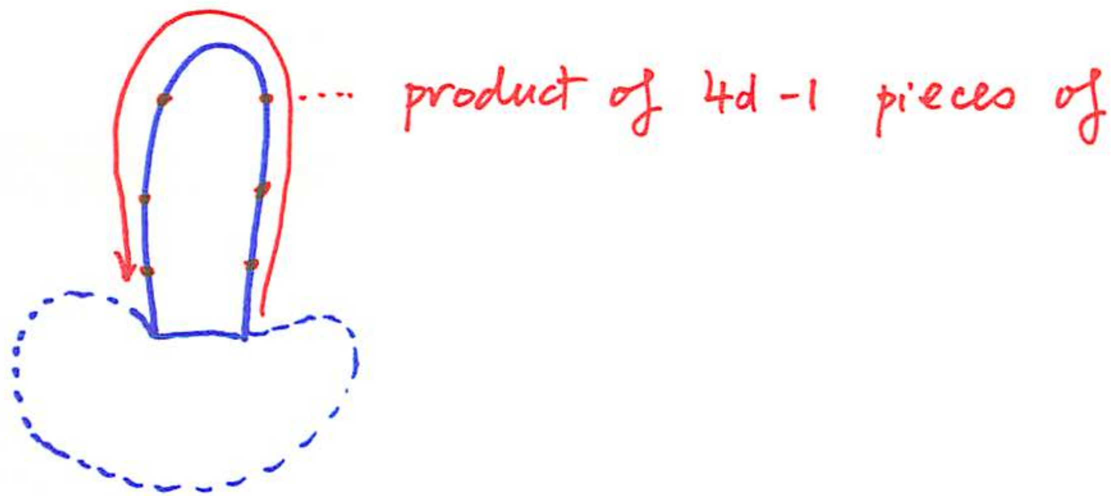


$A = \#\{ \bullet \}$

$\gg$

$B = \#\{ \circ \}$

Cor ∂M contains at least $4d-2$ consecutive vertices of degree 2.
So $\Phi(\partial M)$ contains a subword which is a product of $4d-1$ pieces of $\{u_r^{\pm d}\}$.



Cor If an alternating word w in $\{x, y\}$ represents the trivial element in $H(r:d) = \langle x, y \mid u_r^d \rangle$, then w shares a long subword with $u_r^{\pm d}$.

ie w contains a subword which is a product of $4d-1$ pieces.

Proof of Main Theorem (1)

Suppose $\alpha_s = 1$ in $H(r:d) = \langle x, y \mid u_r^d \rangle$ for some $s \in I_1 \cup I_2^{(d)}$.

Then α_s shares a long subword with $u_r^{\pm d}$.

To be precise, the cyclic S-sequence $CS(\alpha_s)$ contains a subword w, st

$$S(w) = \langle \underbrace{S_1, S_2}, \underbrace{S_1, S_2}, \cdots, \underbrace{S_1, S_2} \rangle$$

(2d-1) pairs.

This contradicts the assumption that $s \in I_1 \cup I_2^{(d)}$, which says that s is "far from" r.

This last step can be proved by induction on the length of the continued fraction expansion of r.

Remark

- The key lemma implies that the cyclic word u contains a subword, w, of $(u_r)^d$ or $(u_r)^{-d}$, st

$$|w| \geqq \frac{2d-1}{2d} |(u_r)^d| = \left(1 - \frac{1}{2d}\right) |(u_r)^d|$$

  This is slightly stronger than the following consequence of Newmann's spelling theorem:

  - u contains a subword, w, of $(u_r)^d$ or $(u_r)^{-d}$, st

$$|w| \geqq \frac{d-1}{d} |(u_r)^d| = \left(1 - \frac{1}{d}\right) |(u_r)^d|$$

- The key lemma holds even when $d = 1$, where Newman's theorem cannot be applied.

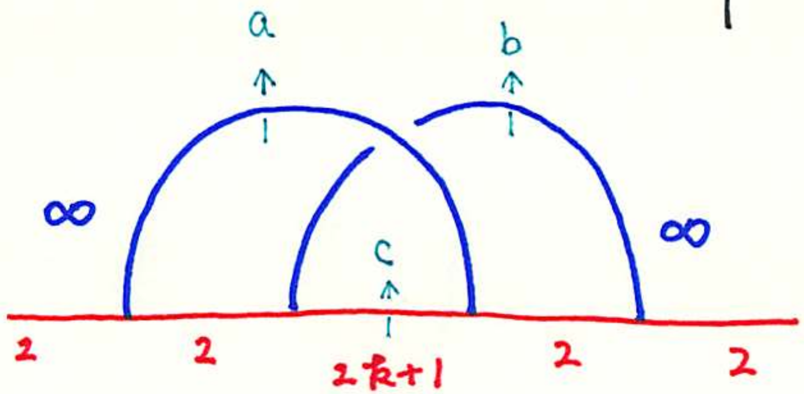
Remark

- The if part of the main theorem holds for odd Heckoid groups $H(r; \frac{2k+1}{2})$, too.
- Odd Heckoid groups are not one-relator groups

$$H(1/2, \tfrac{2k+1}{2}) \cong \left\langle a, b, c \;\middle|\; \begin{array}{l}(ca)^2 = (cab)^2 = (cab\bar{a})^2 = 1 \\ c^{2k+1} = 1\end{array} \right\rangle$$

$$\cong \pi_1^{orb}(S^2(2,2,2,2k+1))$$

$$\cong \left\langle a, b \;\middle|\; \begin{array}{l}(aba^{-1}b^{-1})^{2k+1}, ((aba^{-1}b^{-1})^k a)^2, \\ ((aba^{-1}b^{-1})^k b^{-1})^2\end{array} \right\rangle$$



Speculation following [Minsky, Geom. Top. Monograph 12]

- $(S^3, K) = (B_1^3, t_1) \underset{S}{\cup} (B_2^3, t_2)$ n-bridge decomposition
- $\mathcal{M}(S) = \pi_0 \mathrm{Diff}(S)$ where $S = \partial B_i^3 - t_i$

  $\cup$

  $\mathcal{M}_0(B_i^3, t_i) := \{ f \in \pi_0 \mathrm{Diff}(B_i^3, t_i) \mid f_* = id \in \mathrm{Out}(\pi_1(B_i^3 - t_i)) \}$

  $\Gamma := \langle \mathcal{M}_0(B_1^3, t_1), \mathcal{M}_0(B_2^3, t_2) \rangle \subset \mathcal{M}(S)$
- $\mathcal{C}^{(0)}(S) = \{$essential simple loops on $S\} /$isotopy

  $\cup$

  $\Delta_i := \{$the boundaries of essential disks in $B_i^3 - t_i\}$

  $\Delta := \Delta_1 \cup \Delta_2$

Observation If $\alpha \in \Gamma \cdot \Delta$, then $\alpha \sim 1$ in $S^3 - K$

Question Is the converse true ?

[Masur] $\mathcal{M}_0(B_i^3, t_i) \curvearrowright \mathcal{PML}(S)$ has a non-empty domain of discontinuity.

Question Suppose the bridge decomposition is "sufficiently complicated".

(1) Does $\Gamma = \langle \mathcal{M}_0(B_1^3, t_1), \mathcal{M}_0(B_2^3, t_2) \rangle \curvearrowright \mathcal{PML}(S)$ have a non-empty domain of discontinuity?

(2) $\Gamma \cong \mathcal{M}_0(B_1^3, t_1) * \mathcal{M}_0(B_2^3, t_2)$?

(3) Suppose $\alpha \in \mathcal{C}^{(0)}(S)$ is contained in the domain of discontinuity $\Omega(\Gamma) \subset \mathcal{PML}(S)$. Then can $\alpha \sim 1$ in $S^3 - K$?

(4) Does $cl\{\alpha \in \mathcal{C}^{(0)}(S) \mid \alpha \sim 1 \text{ in } S^3 - K(r)\} \subset \mathcal{PML}(S)$ have measure 0 ?