Reidemeister torsion and volume for hyperbolic knots

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RIMS Seminar

Representation spaces, twisted topological invariants and geometric structures of 3-manifolds.

June 1st, 2012

- $S^3 \setminus K$ hyperbolic knot exterior.
  $hol : \pi_1(S^3 \setminus K) \to \mathrm{Isom}^+(\mathbf{H}^3) = PSL_2(\mathbf{C})$
  $\sigma_N : SL_2(\mathbf{C}) \to SL_N(\mathbf{C})$ irreducible rep.

$$\rho_N = \sigma_N \circ \widetilde{hol} : \pi_1(S^3 \setminus K) \to SL_2(\mathbf{C}) \to SL_N(\mathbf{C})$$

- The Reidemeister torsion $\tau(S^3 \setminus K, \rho_{2N}) \in \mathbf{C}$ is well defined.
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MAIN THEOREM: (Menal-Ferrer, P.)

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- Based on a theorem of Müller for closed manifolds.
- Start by recalling Reidemeister torsions

Lens spaces (Tietze 1908)

$$S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\},$$

$$t \cdot (z_1, z_2) = (e^{\frac{2\pi}{p} i} z_1, e^{\frac{2\pi q}{p} i} z_2), \qquad p, q \in \mathbf{N} \text{ coprime.}$$

$$L(p, q) = S^3 \Big/ (z_1, z_2) \sim t \cdot (z_1, z_2)$$

$$\pi_1(L(p, q)) = \mathbf{Z}/p\mathbf{Z}$$

- Question: For which q and q', $L(p, q) \cong L(p, q')$?

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Stereographic projection

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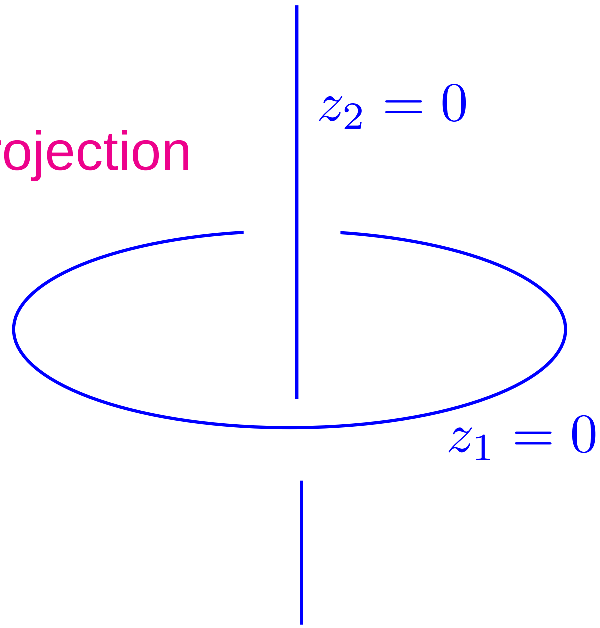

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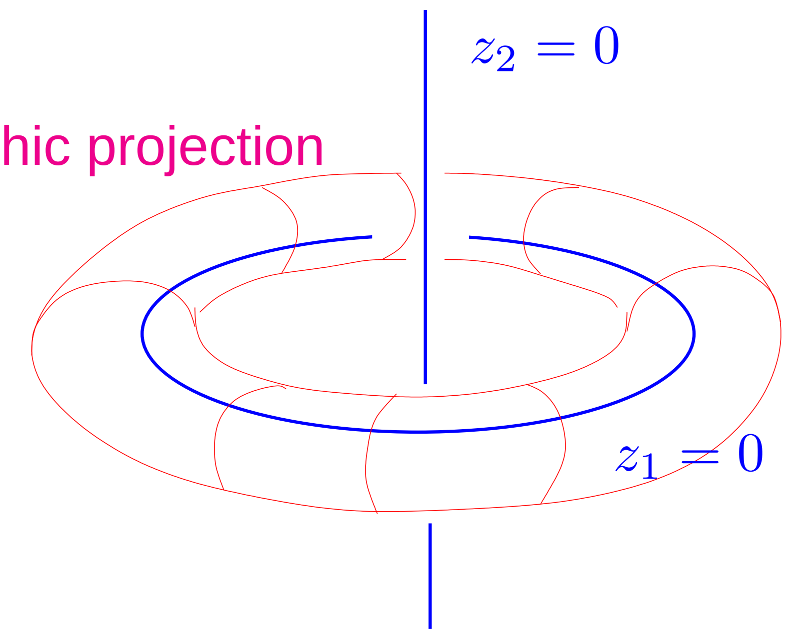

$|z_1| = c \quad |z_2| = \sqrt{1 - c^2}$

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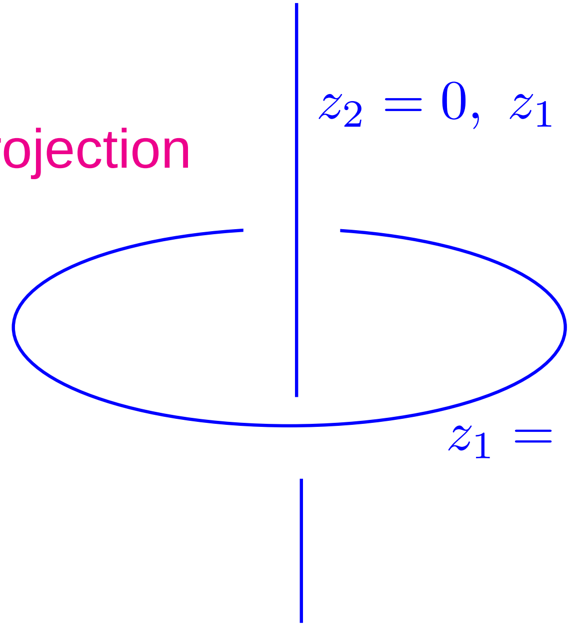

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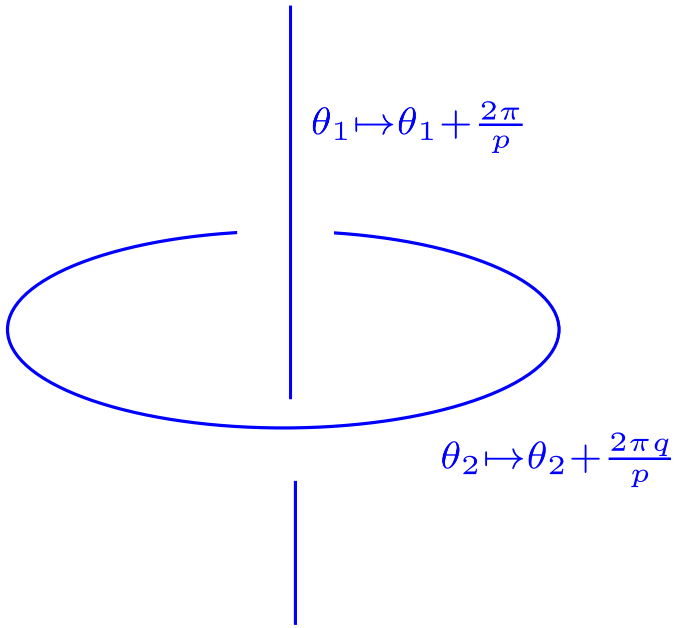
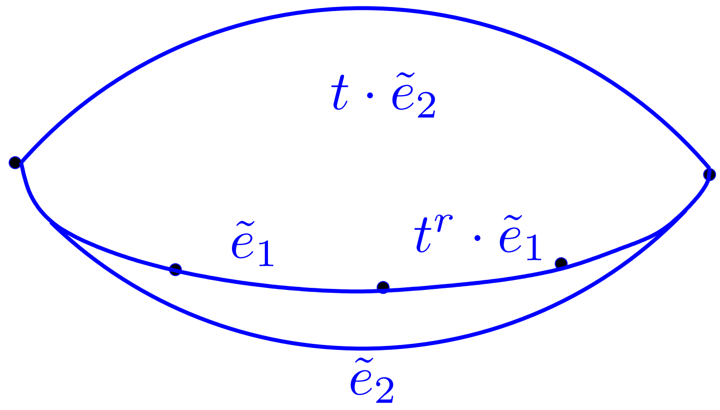




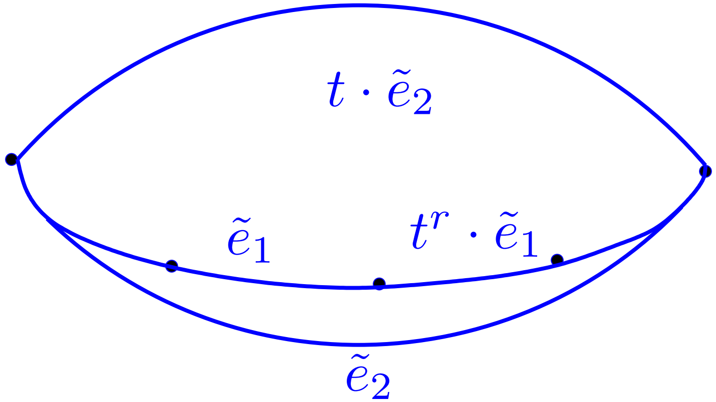
The lens is a fundamental domain for t

$$rq = 1 \mod p$$

Torsion of a lens space (1935)

$$S^3 = \{z \in \mathbf{C}^2 \mid |z| = 1\},\ t \cdot (z_1, z_2) = (e^{\frac{2\pi}{p} i} z_1, e^{\frac{2\pi q}{p} i} z_2),$$

$$L(p,q) = S^3/(z_1, z_2) \sim t \cdot (z_1, z_2)$$



$$\begin{cases} \partial \tilde{e}_3 = (1 - t)\tilde{e}_2 \\ \partial \tilde{e}_2 = (1 + t + \cdots + t^{p-1})\tilde{e}_1 \\ \partial \tilde{e}_1 = (t^r - 1)\tilde{e}_0 \end{cases}$$

$$t \to 1, \quad \begin{cases} \partial e_3 = 0 \\ \partial e_2 = p e_1 \\ \partial e_1 = 0 \end{cases} \qquad H_i^{CW}(L(p,q), \mathbf{Z}) = \begin{cases} \mathbf{Z} & i = 3 \\ \mathbf{0} & i = 2 \\ \mathbf{Z}/p\mathbf{Z} & i = 1 \\ \mathbf{Z} & i = 0 \end{cases}$$

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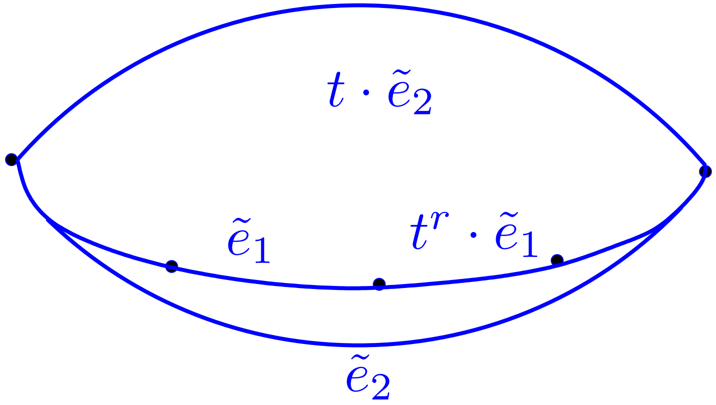

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$$t \mapsto \xi,\ \xi^p = 1,\ \xi \neq 1, \qquad \begin{cases} \partial \tilde{e}_3 = (1 - \xi)\tilde{e}_2 \\ \partial \tilde{e}_2 = 0 \\ \partial \tilde{e}_1 = (\xi^r - 1)\tilde{e}_0 \end{cases} \qquad H_*(L(p,q), \xi) = 0$$

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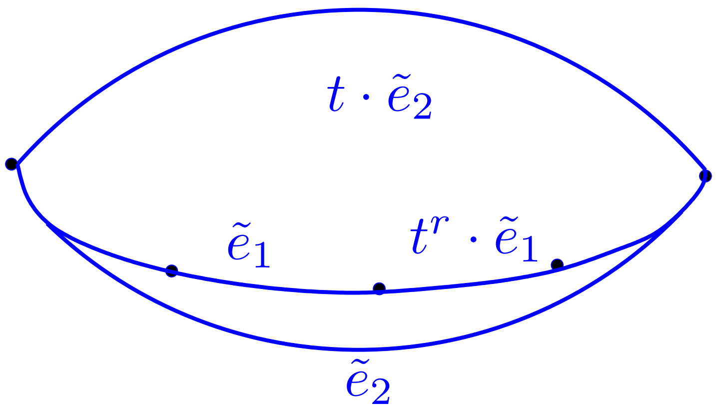

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Def. (Reidemeister, 1935): $\tau(L(p,q),\xi) := |(1-\xi)(1-\xi^r)|$

$$\{\tau(L(p,q),\xi)\}_{\xi^p=1,\xi\neq 1} = \{\tau(L(p,q'),\xi)\}_{\xi^p=1,\xi\neq 1} \Leftrightarrow q' = \pm q^{\pm 1} \mod p$$

K compact CW-cplx, $\rho\colon\pi_1 K \to GL(V)$, V=fin dim F-vect.space

Def: $C_*(K,\rho) := V \otimes_\rho C_*^{CW}(\tilde{K},\mathbf{Z})$

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- $B_i := Im(\partial : C_{i+1}(K,\rho) \to C_i(K,\rho))$

If $H_*(K,\rho) = 0$ (ρ is acyclic), then

$$0 \to B_i \to C_i(K,\rho) \xrightarrow{\partial} B_{i-1} \to 0 \text{ is exact.}$$

b_i F-basis for B_i, $\Longrightarrow b_i \sqcup \tilde{b}_{i-1}$ F-basis for $C_i(K,\rho)$.

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Def: $$\tau(K,\rho) := \prod_{i=0}^{n} \det(b_i \sqcup \tilde{b}_{i-1} \mid c_i)^{(-1)^i} \in F^*/\{\pm \det \rho(\pi_1 K)\}$$

- $\tau(K,\rho)$ combinatorial invariant (cellular homeos and subdivision) and the conjugacy class of ρ

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- If $\rho\colon \pi_1 K \to SL(V)$, then $\tau(K,\rho) \in F^*/\{\pm 1\}$

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- If $H_*(K,\rho) \neq 0$, define $\tau(K,\rho,h_*)$ for h_* = F-basis of $H_*(K,\rho)$.
  Look for a *natural* basis h_*!!

Combinatorial invariant (no topological)

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TWO COMPLEXES WHICH ARE HOMEOMORPHIC BUT COMBINATORIALLY DISTINCT

BY JOHN MILNOR[1]

(Received March 14, 1961)

Let L_q denote the 3-dimensional lens manifold of type $(7, q)$, suitably triangulated (see § 1), and let σ^n denote an n-simplex. A finite simplicial complex X_q is obtained from the product $L_q \times \sigma^n$ by adjoining a cone over the boundary $L_q \times \partial\sigma^n$. The dimension of X_q is $n + 3$.

THEOREM 1. *For $n + 3 \geqq 6$ the complex X_1 is homeomorphic to X_2.*

THEOREM 2. *No finite cell subdivision of the simplicial complex X_1 is isomorphic to a cell subdivision of X_2. In particular there is no piecewise linear homeomorphism from X_1 to X_2.*

The proof of Theorem 1 will be based on a recent result of B. Mazur. For the special case $n = 3$ (which is somewhat more difficult) the proof will make use of theorems of A. Haefliger and J. Stallings.

The proof of Theorem 2 will be based on the concept of "torsion" as defined by Reidemeister, Franz, and de Rham.

- $|K| = M^n$ closed & smooth, $\rho : \pi_1 M^n \to SO(m, \mathbf{R})$, $H^*(M^n, \rho) = 0$

  - Ray-Singer (1971): Definition of $\tau(M, \rho)$
    that works for differential forms
    (de Rham cohomology)

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$\delta_i : C^i(K;\rho) \to C^{i+1}(K,\rho)$,
$\delta^t_{i-1} : C^i(K;\rho) \to C^{i-1}(K,\rho)$ (use the $\mathbf{R}$-basis for transpose)
$\Delta_i^{comb}(\rho) := \delta_i^t \circ \delta_i + \delta_{i-1} \circ \delta_{i-1}^t$
$\tau(K,\rho)^2 = \prod_{k=0}^n (\det \Delta_k^{comb}(\rho))^{k(-1)^{k+1}}$

- $\log|\tau(K,\rho)| = \frac{1}{2} \sum_{k=0}^{n} (-1)^{k+1} k \log(\det(\Delta_k^{comb}(\rho)))$

- Idea: Define torsion from "*determinant*" of Laplacian on k-forms

$$\Delta^k : \Omega^k(M; V_\rho) \to \Omega^k(M; V_\rho)$$

- $|K| = M^n$ closed & smooth, $\rho : \pi_1 M^n \to SO(m, \mathbf{R})$, $H^*(M^n, \rho) = 0$
  $\Delta_i^{comb}(\rho) := \delta_i^t \circ \delta_i + \delta_{i-1} \circ \delta_{i-1}^t$
  $\log|\tau(K, \rho)| = \frac{1}{2} \sum_{k=0}^{n} (-1)^{k+1} k \, \log(\det(\Delta_k^{comb}(\rho)))$

- $\Delta^k : \Omega^k(M; V_\rho) \to \Omega^k(M; V_\rho)$ Laplacian on k-forms
- $\mathrm{Spec}(\Delta^k) = \{\lambda \in \mathbf{R} \mid \exists \omega \in \Omega^k(M; V_\rho), \Delta^k \omega = \lambda \omega\}$
  $\mathrm{Spec}(\Delta^k) > 0$ is discrete.

$$\zeta_k(s) = \sum_{\lambda \in \mathrm{Spec}(\Delta^k)} \lambda^{-s} \qquad \text{for } s \in \mathbf{C},\ Re(s) \gg 0$$

- $\zeta_k(s)$ extends meromorphically to $s = 0$.

$\zeta_k'(s) = \sum -\lambda^{-s} \log \lambda$, $\Rightarrow \zeta_k'(0) = -\sum \log \lambda = -\log \det \Delta^k$ (FORMAL!)

Ray-Singer's analytic torsion

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$$\zeta_k(s) = \sum_{\lambda \in \mathrm{Spec}(\Delta^k)} \lambda^{-s} \qquad \text{for } s \in \mathbf{C},\ Re(s) \gg 0$$

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Def (Ray-Singer 1971)

Analytic torsion: $\log RS(M,\rho) := \frac{1}{2} \sum_k (-1)^k k \zeta_k'(0)$

- $K = M^n$ closed & smooth, $\rho : \pi_1 M^n \to SO(m, \mathbf{R})$, $H^*(M, \rho) = 0$
  $\Delta^k : \Omega^k(M; V_\rho) \to \Omega^k(M; V_\rho)$ Laplacian on k-form
  $\log RS(M, \rho) := \frac{1}{2} \sum_k (-1)^{k+1} k \zeta_k'(0)$

Theorem (Cheeger-Müller)

Analytic and combinatorial torsions are the same:

$$RS(M, \rho) = |\tau(M, \rho)|$$

- Proved by Jeff Cheeger & Werner Müller
  for $\rho : \pi_1 M^n \to SO(m, \mathbf{R})$ (1978)
- Proved by W. Müller for $SL(m, \mathbf{R})$ (1993).

- $K = M^n$ closed & smooth, $\rho : \pi_1 M^n \to SO(m, \mathbf{R})$, $H^*(M, \rho) = 0$
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- $SL(m, \mathbf{C}) \subset SL(2m, \mathbf{R})$ $\qquad a + b\,i \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

- Closed hyperbolic M^n, $n \geq 3$, $\rho : \pi_1(M^n) \to SO(m)$ acyclic.
- Ruelle Zeta Function:

$$R_\rho(s) = \prod_{\gamma \in PCG(M^n)} \det(I - \rho(\gamma) e^{-sl(\gamma)}), \qquad s \in \mathbf{C},\ \mathrm{Re}(s) > n-1.$$

$PCG(M^n) = \{$oriented primitive closed geodesics $\gamma \subset M^n\}$.

Theorem (D. Fried 1986):

$R_\rho(s)$ extends meromorphically to $\mathbf{C}$ and

$$|R_\rho(0)^{(-1)^{n+1}}| = \tau(M^n, \rho)^2,$$

- $S^3 \setminus K$ orientable hyperbolic, $S^3 \setminus K \cong \mathbf{H}^3/\Gamma$
  $\Gamma < \mathrm{Isom}^+(\mathbf{H}^3) = PSL_2(\mathbf{C}) = SL_2(\mathbf{C})/\{\pm Id\}$
  $hol : \pi_1(S^3 \setminus K) \cong \Gamma < PSL_2(\mathbf{C})$ lifts to $\widetilde{hol} : \pi_1(S^3 \setminus K) \to SL_2(\mathbf{C})$.

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  $hol : \pi_1(S^3 \setminus K) \cong \Gamma < PSL_2(\mathbf{C})$ lifts to $\widetilde{hol} : \pi_1(S^3 \setminus K) \to SL_2(\mathbf{C})$.
- $\sigma_N : SL_2(\mathbf{C}) \to SL_N(\mathbf{C})$ irreducible
  $\mathbf{C}^N = \mathrm{Sym}^{N-1}(\mathbf{C}^2)$= homogeneous polynomials on $\mathbf{C}^2$ of deg $N-1$
  If $\mathbf{C}^2 = \langle v_1, v_2 \rangle$, then $\mathrm{Sym}^{N-1}(\mathbf{C}^2) = \langle v_1^{N-1}, v_1^{N-2} v_2, \cdots, v_2^{N-1} \rangle$.

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Theorem (P. Menal Ferrer - J. P. 2010)

$K \subset S^3$ hyperbolic knot, $N > 1$, $H^1(S^3 \setminus K, \rho_N) = \begin{cases} 0 & \text{for } N \text{ even} \\ \mathbf{C} & \text{for } N \text{ odd} \end{cases}$

Cohomology for hyperbolic knots

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  (For N odd one has to chose a basis in cohomology.)
- For N odd this is related to deformations
  $X(\pi_1(S^3 \setminus K), SL_m(\mathbf{C})) = \hom(\pi_1(S^3 \setminus K), SL_m(\mathbf{C}))//SL_m(\mathbf{C})$
  $[\rho_m]$ is a smooth point of local dimension $m-1$.
- $T_{[\rho_m]}X(\pi_1(S^3 \setminus K), SL_m(\mathbf{C})) \cong H^1(S^3 \setminus K, \mathfrak{sl}_m(\mathbf{C})_{Ad\rho_m}) \cong$
  $\cong H^1(S^3 \setminus K, \rho_3) \oplus H^1(S^3 \setminus K, \rho_5) \oplus \cdots \oplus H^1(S^3 \setminus K, \rho_{2m-1}) \cong$
  $\cong \mathbf{C}^{m-1}$

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Theorem (W. Müller 2010) M^3 closed.

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Theorem (P. Menal Ferrer - J. P. 2011) $K \subset S^3$ hyperbolic:

$$\lim_{N\to\infty} \frac{\log|\tau(S^3 \setminus K, \rho_{2N})|}{(2N)^2} = -\frac{\mathrm{Vol}(S^3 \setminus K)}{4\pi}$$

Müller: M^3 closed. $\lim_{N\to\infty} \frac{\log|\tau(M^3,\rho_N)|}{N^2} = -\frac{\mathrm{Vol}(M^3)}{4\pi}$

- Ingredients of Müller's proof :

– $R_\rho(s) = \prod_{\gamma\in PCG(S^3\setminus K)} \det(I-\rho(\gamma)e^{-sl(\gamma)}), \qquad s\in\mathbf{C},\ \mathrm{Re}(s) > n-1.$

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$$\log \frac{|\tau(M^3,\rho_{2N})|}{|\tau(M^3,\rho_4)|} = \sum_{k=3}^{N} \log|R_{2k}(k)| - \frac{1}{\pi}\mathrm{Vol}(M^3)(N(N+1)-6)$$

where $R_{2k}(s) = \prod_{\gamma\in PCG(S^3\setminus K)} (1 - e^{ki\theta(\gamma)-sl(\gamma)})$ & $\sum_{k=3}^{N} |\log|R_{2k}(k)|| < C$

l =length, θ = Riemannian torsion and $\lambda = l + i\theta$ =complex length

$$hol(\gamma) = \pm \begin{pmatrix} e^{\lambda(\gamma)/2} & 0 \\ 0 & e^{-\lambda(\gamma)/2} \end{pmatrix}$$

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Goal: Prove this functional equation for $S^3 \setminus K$ by approximating it by closed manifolds (Dehn fillings).

Approximate by Dehn fillings

$K_{p/q} = (p,q)$- Dehn filling on $S^3 \setminus K$
$K_{p/q} = S^3 \setminus \mathcal{N}(K) \cup_\varphi D^2 \times S^1$, with $\varphi(\partial D^2 \times *) = p$ meridian $+ q$ longitude
$\gamma_{p/q}$= core geodesic of $D^2 \times S^1$

Thurston's thm: $K_{p/q}$ is hyperbolic for almost every $p/q \in \mathbf{Q} \cup \infty$
and $\lim_{p^2+q^2 \to \infty} K_{p/q} = S^3 \setminus K$ for the geometric topology
(pointed bi-Lipchitz convergence).

In particular $\mathrm{Vol}(K_{p/q}) \to \mathrm{Vol}(S^3 \setminus K)$ and length$(\gamma_{p/q}) \to 0$

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$\lambda PCG(K_{p/q})$= complex length of or.primitive closed geodesics of $K_{p/q}$

Lemma: For every compact $C \subset \{z \in \mathbf{C} \mid 0 < \mathrm{Re}(z), 0 \le \mathrm{Im}(z) \le 2\pi\}$

$$\lambda PCG(K_{p/q}) \cap C \to \lambda PCG(S^3 \setminus K) \cap C$$

Proof Thin-thick decomposition: the thick parts converge, and geodesics different from $\pm\gamma_{p/q}$ that enter the thin part become arbitrarily long

Equation when $p^2 + q^2 \to \infty$

$$K_{p/q} \to S^3 \setminus K$$

$$\log \left| \frac{\tau(K_{p/q}, \rho_{2N}^{p/q})}{\tau(K_{p/q}, \rho_4^{p/q})} \right| = \sum_{k=3}^{N} \log |R_{2k}^{p/q}(k)| - \left(\frac{1}{\pi} \operatorname{Vol}(K_{p/q})(N(N+1) - 6) \right)$$

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$\Longrightarrow$

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Final remarks and questions

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- ...and thank you for your attention!