Geometrization of three-manifolds.

Joan Porti (UAB)

RIMS Seminar

Representation spaces, twisted topological invariants and

geometric structures of 3-manifolds.

May 28, 2012

- Poincaré, H. **Analysis situs.** J. de l'Éc. Pol. (2) I. 1-123 (1895)
- Poincaré, H. **Complément à l'analysis situs.** Palermo Rend. 13, 285-343 (1899)
- Poincaré, H. **Second complément à l'analysis situs** Lond. M. S. Proc. 32, 277-308 (1900).
- Poincaré, H. **Sur certaines surfaces algébriques. III$^{\text{ième}}$ complément à l'analysis situs.** S. M. F. Bull. 30, 49-70 (1902).
- Poincaré, H. **Sur l'analysis situs.** C. R. 133, 707-709 (1902).
- Poincaré, H. **Cinquième complément à l'analysis situs.** Palermo Rend. 18, 45-110 (1904)

In “Cinquième complément à l’Analysis Situs" (1904):

Let M^3 be a closed 3-manifold.

Assume that M^3 is simply connected ($\pi_1(M^3) = 0$),

is M^3 homeomorphic to S^3?

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

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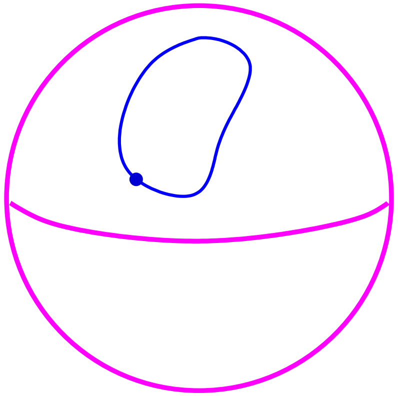

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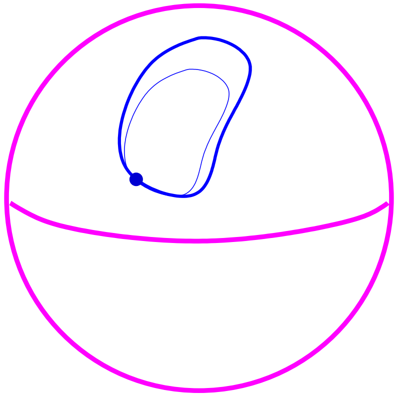

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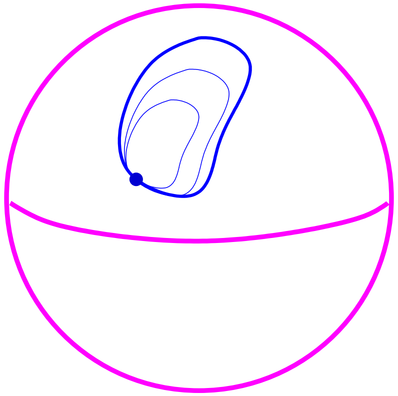

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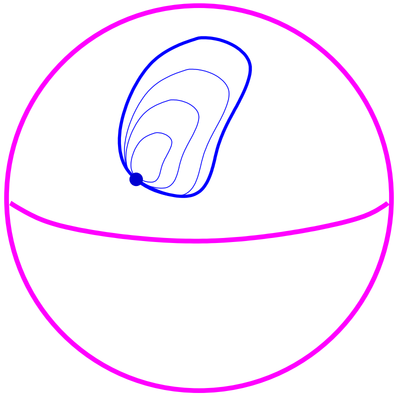

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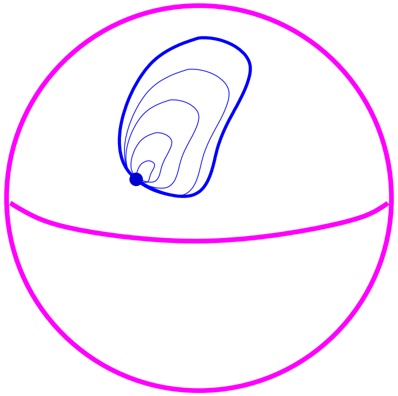

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In dim 2, $\pi_1(F^2) = 0$ characterizes the sphere among all surfaces.

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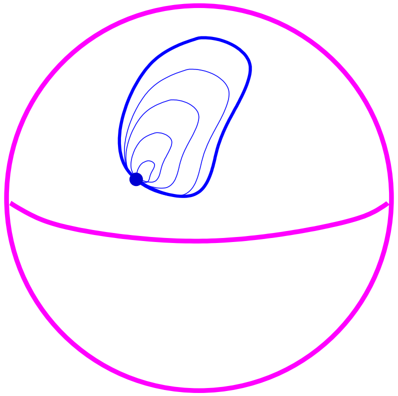
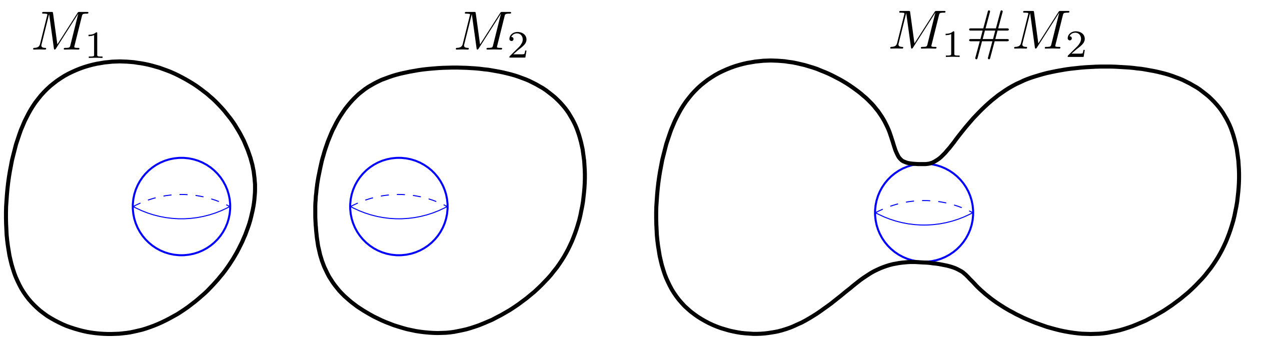


...mais cette question nous entrainerait trop loin.

Kneser and connected sum (1929)



$$M_1 \# M_2 = (M_1 - B^3) \cup_\partial (M_2 - B^3)$$

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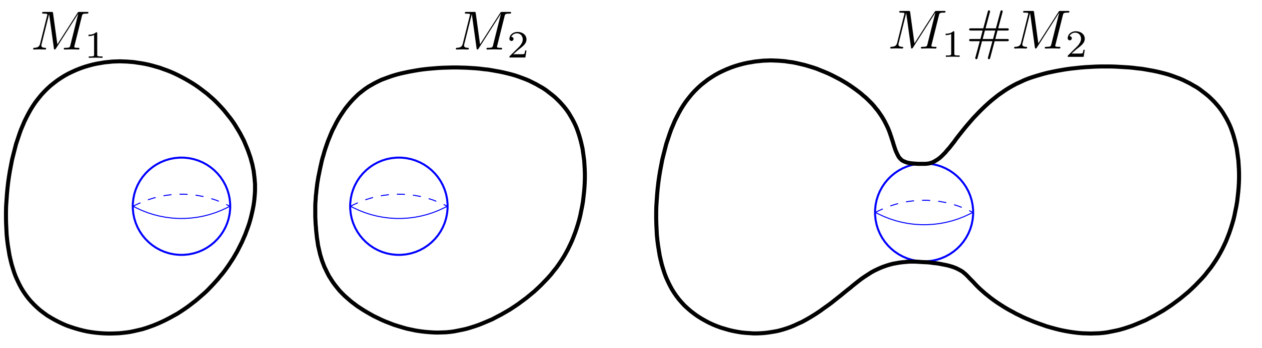

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Kneser's Theorem (1929) M^3 closed and orientable

$$\Longrightarrow M^3 \cong M_1^3 \# \cdots \# M_k^3 \ .$$

$M_1^3, \ldots, M_k^3$ unique (up to homeomorphism) and prime.

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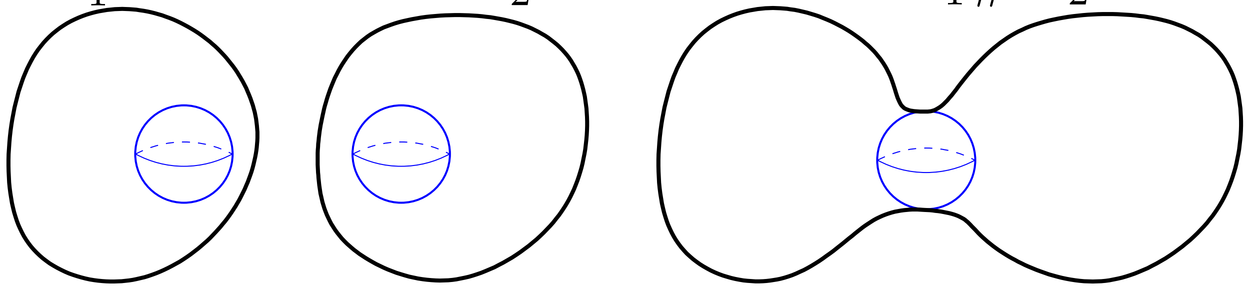

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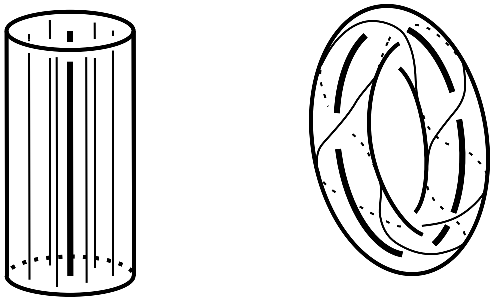
- M^3 orientable and closed, then
  M^3 is prime iff M^3 is irreducible or $M^3 \cong S^2 \times S^1$

irreducible: every embedded 2-sphere in M^3 bounds a ball in M^3

H. Seifert: fibered manifolds (1933)

Manifolds with a partition by circles with local models:



glue top and bottom of the cylinder

by a $2\pi\frac{p}{q}$-rotation, $\frac{p}{q} \in \mathbb{Q}$

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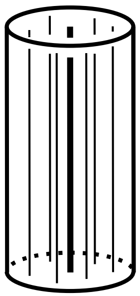

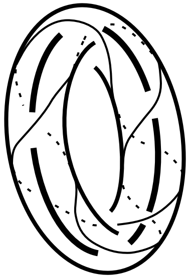

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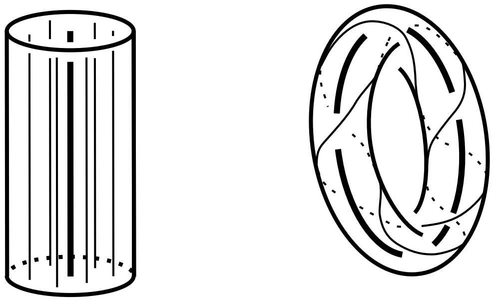

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Examples:

- $T^3 = S^1 \times S^1 \times S^1$
- $S^3 = \{z \in \mathbb{C}^2 \mid |z| = 1\}$ Hopf fibration: $S^1 \to S^3 \to \mathbb{CP}^1 \cong S^2$
- Lens Spaces: $L(p,q) = S^3/\sim, \qquad (z_1, z_2) \sim (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i\, q}{p}} z_2)$

  for p, q coprime

  (there are singular fibers when $q \neq 1$)

Jaco-Shalen and Johannson (1979)

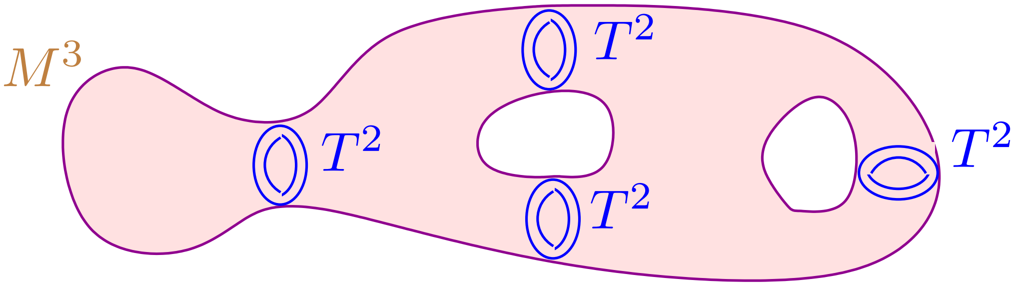
Characteristic Submanifod Theorem (JSJ 1979).

Let M^3 be irreducible, closed and orientable.

There is a canonical and minimal family of tori $T^2 \cong S^1 \times S^1 \subset M^3$

that are π_1-injective and that cut M^3

in pieces that are either Seifert fibered or simple.



N simple: not Seifert and every $\mathbb{Z} \times \mathbb{Z} \subset \pi_1(N^3)$ comes from $\pi_1(\partial N^3)$.

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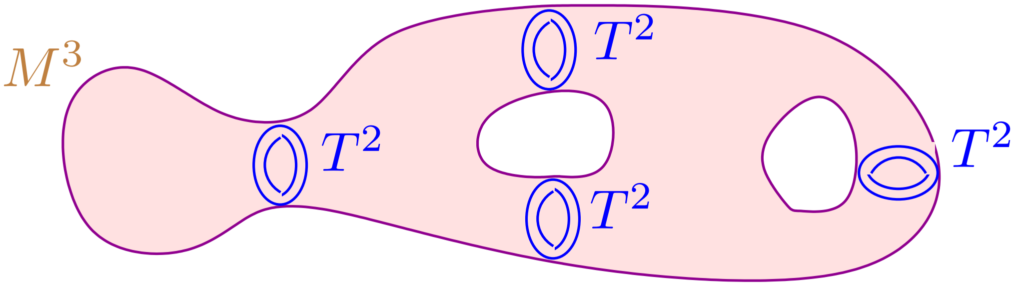

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Thurston's conjecture: simple $\Rightarrow$ hyperbolic.

Hyperbolic: int(M^3) complete Riemannian metric of curvature $\equiv -1$

Thurston's geometrization conjecture (1982)

M^3 closed admits a canonical decomposition into geometric pieces

- Canonical decomposition: connected sum and JSJ tori
- Geometric manifold: locally homogeneous metric.
  (any two points have isometric neighbourhoods)

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- L. Bianchi (1897): local classification of locally homogeneous metrics in dimension three.
- Geometric $\Leftrightarrow$ Seifert fibered , hyperbolic or $T^2 \to M^3 \to S^1$.

  Ex: S^3, $L(p,q) = S^3/\sim$, $T^3 = S^1 \times S^1 \times S^1$
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- It implies Poincaré.

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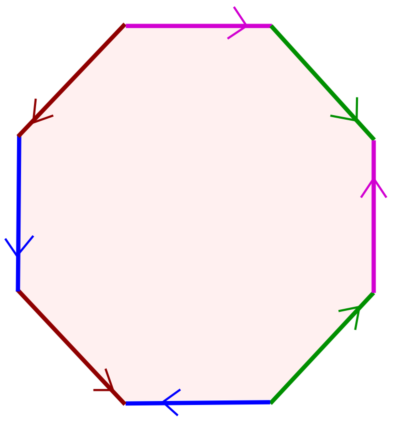
- Proved by Perelman in 2003.

Example: genus 2 surface F_2



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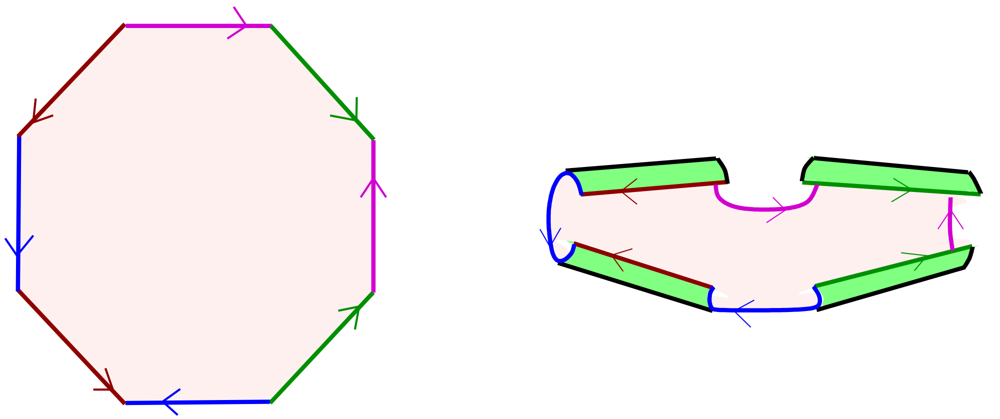

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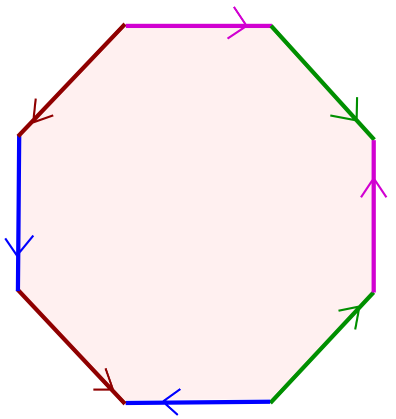

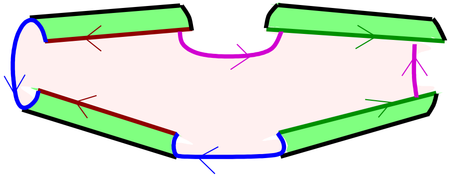

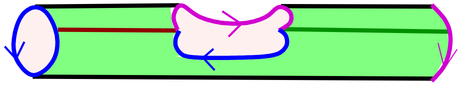

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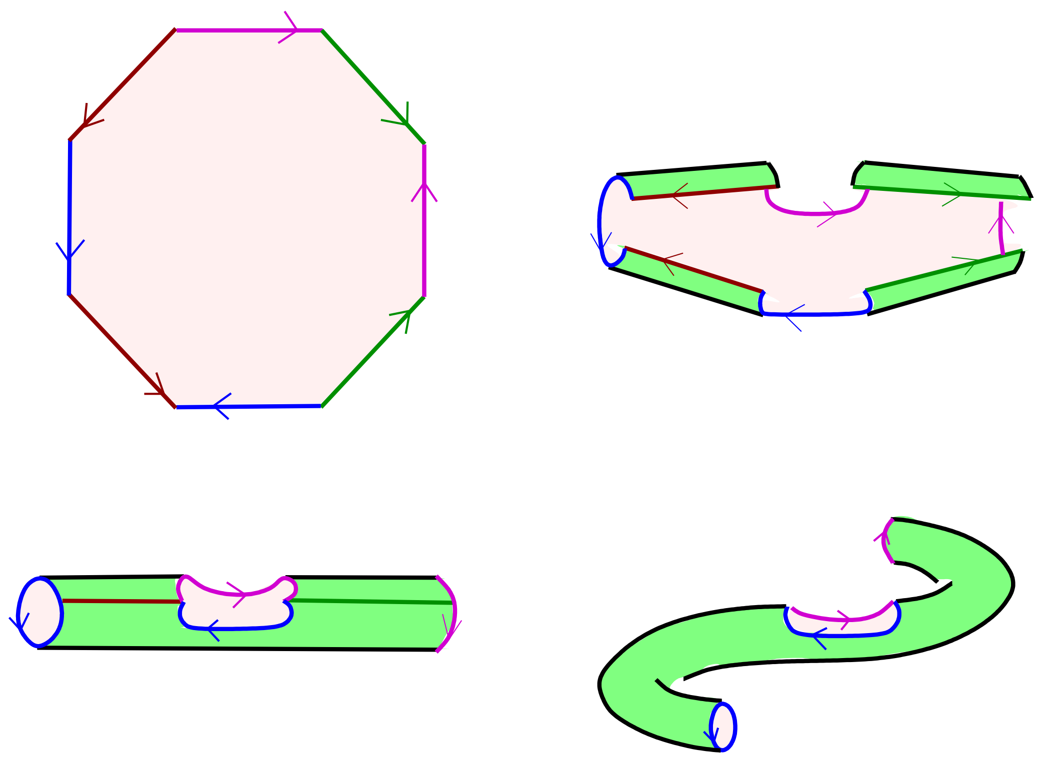

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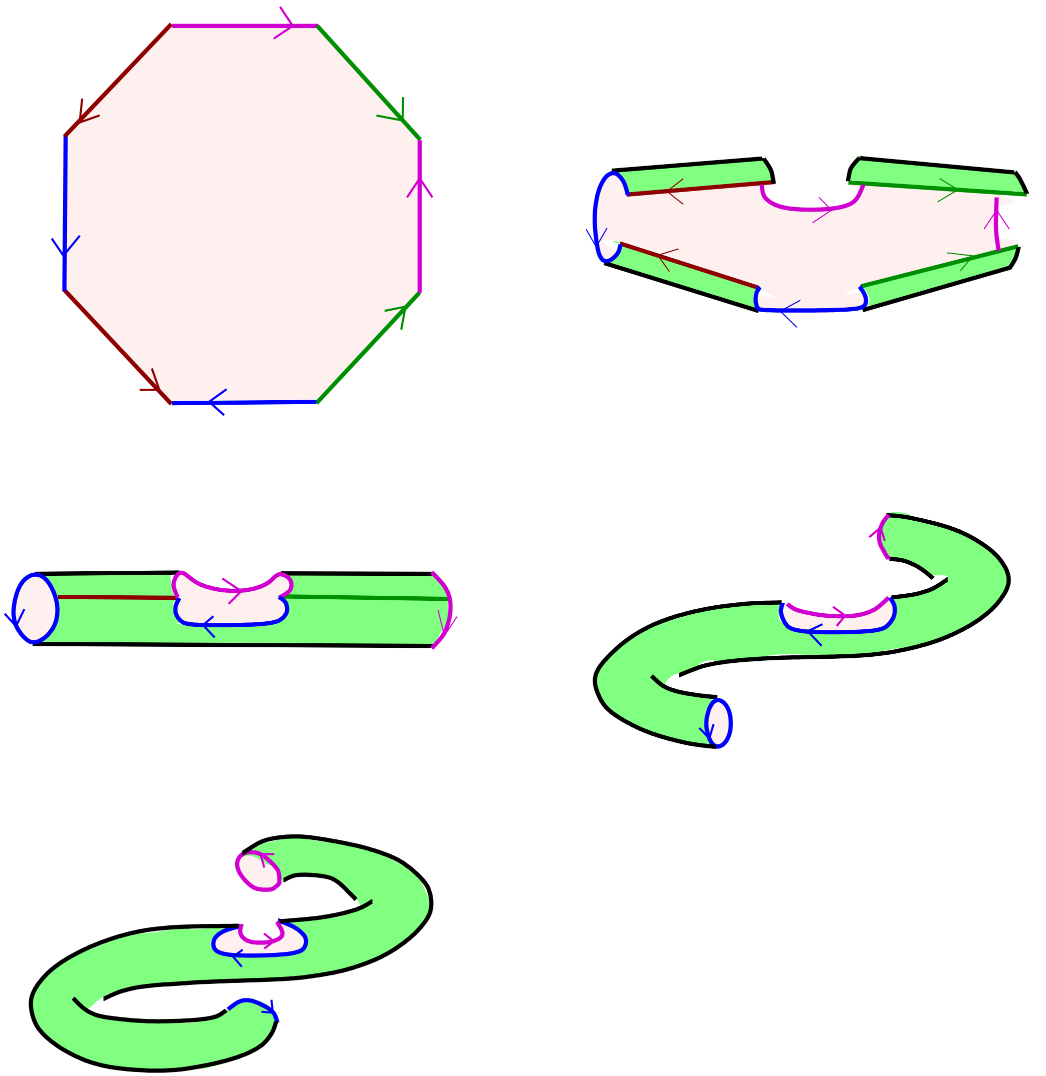

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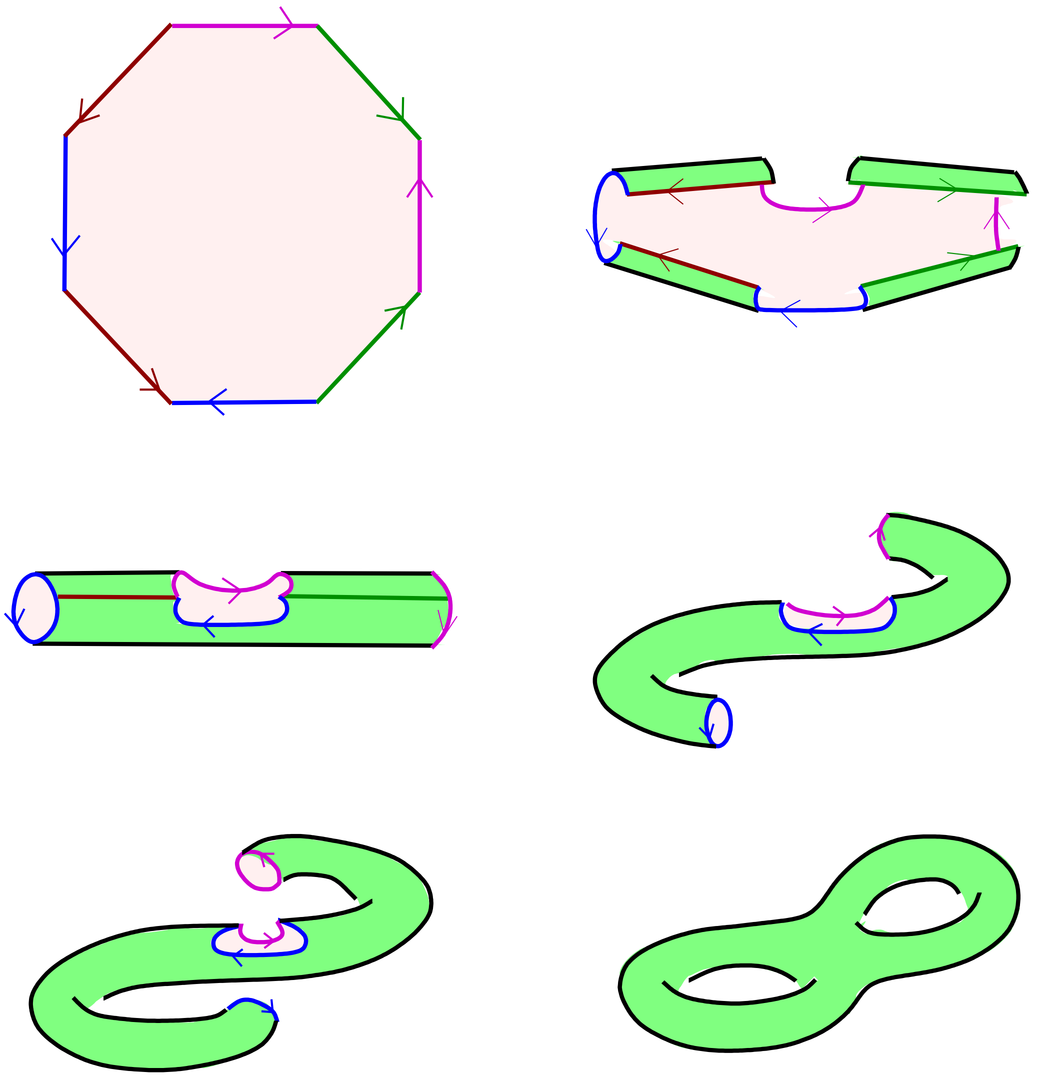

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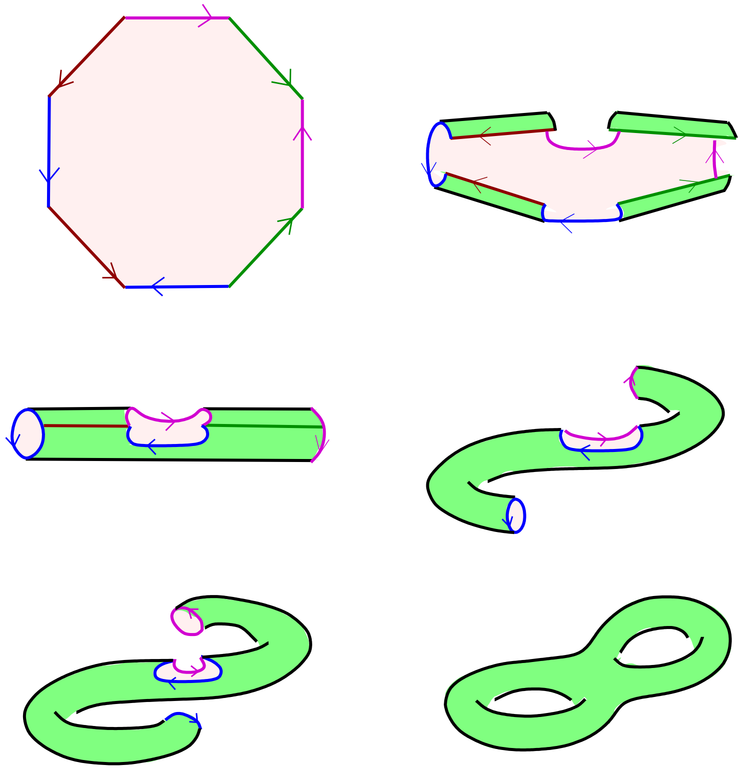
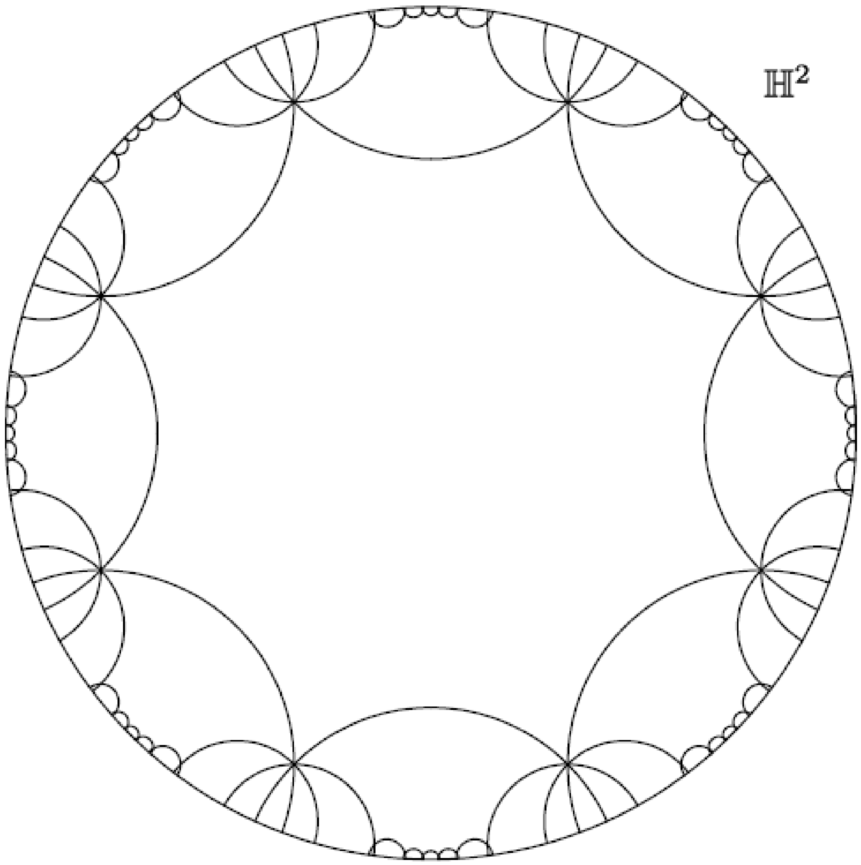


$$F_2 = \mathbf{H}^2/\Gamma$$



$$\frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$$

- M^3 compact, irreducible, or., with $\partial M^3 = \emptyset$ or $\partial M^3 = T^2 \sqcup \cdots \sqcup T^2$.
  $\pi = \pi_1(M^3)$

Some consequences of geometrization

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- If π is infinite $\Rightarrow \pi$ determines M^3 ($\pi_1(M) \cong \pi_1(M') \Leftrightarrow M \cong M'$)
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- In π the word and conjugacy problems can be solved (Sela, Préaux)
- $\tilde{M} \to M$ covering of order $[M : \tilde{M}] < \infty$, $b_1(\tilde{M}) = \dim_{\mathbb{Q}} H_1(\tilde{M}; \mathbb{Q})$.

$$\lim_{\tilde{M}} \frac{b_1(\tilde{M})}{[M : \tilde{M}]} = 0$$ (Lück)

- For π infinite and non-solvable,

$$\limsup_{\tilde{M}} b_1(\tilde{M}) = \infty$$ (Agol, Kahn-Markovic, Wise)

Status on Thurston's conjecture in 1981:
Thurston's conjecture equivalent to Conj 1 + Conj 2:

- Conj 1: If $|\pi_1 M^3| < \infty$ then M^3 spherical ($M \cong \Gamma \backslash S^3$, $\Gamma < SO(4)$).
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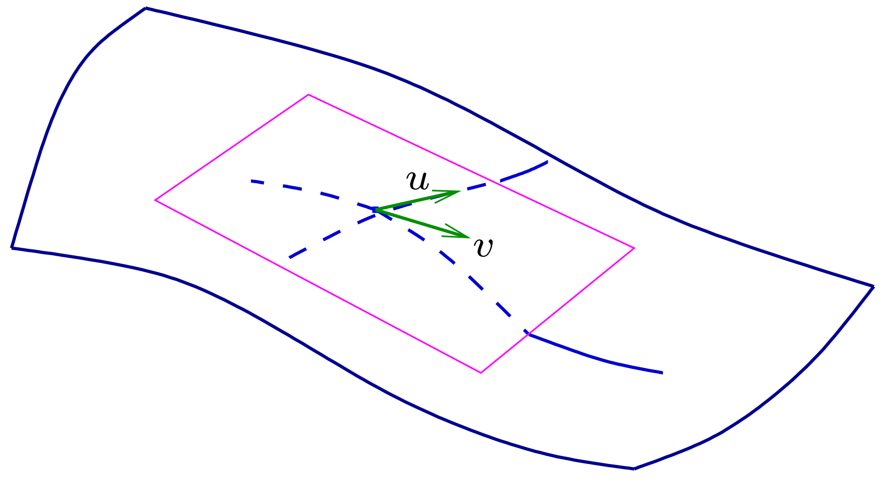
- Thurston knew how to prove it for Haken manifolds
  - M^3 is Haken if irreducible and $\exists F^2 \subset M^3$, $\pi_1(F^2) \hookrightarrow \pi_1(M^3)$
  - If M^3 irreducible and $H_1(M^3; \mathbf{Q}) \neq 0$ then M^3 Haken
  - If M^3 irreducible and $\partial M^3 \neq \emptyset$ then M^3 Haken

  - M^3 Haken iff it has a hierarchy ("nice" decomposition into balls).

Riemannian geometry (Riemann 1854)

At the tangent space at each point, there is a scalar product.



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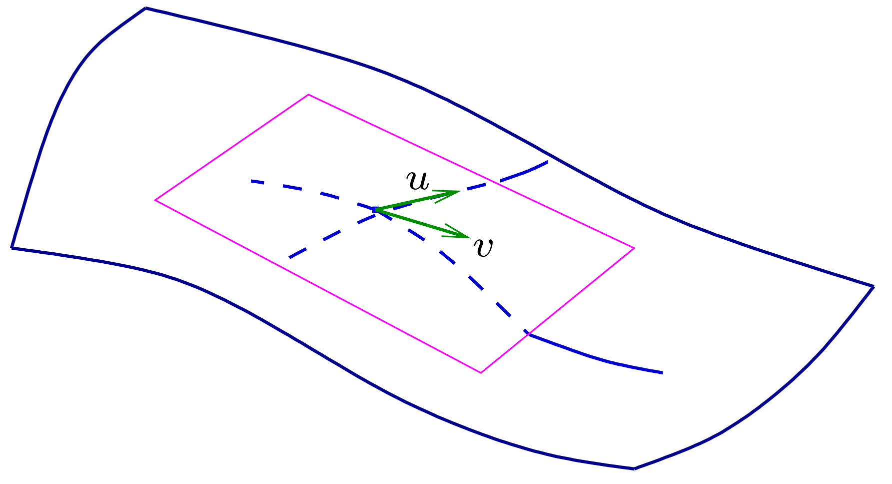

$\langle u, v \rangle$

In coordinates $(x^1, \ldots, x^n)$, $\quad g_{ij}(x) = \langle \partial_i, \partial_j \rangle \qquad \partial_i = \frac{\partial}{\partial x^i}$

$$\left.\begin{aligned} u &= u^i \partial_i \\ v &= v^j \partial_j \end{aligned}\right\} \langle u, v \rangle = \sum u^i g_{ij}(x) v^j = (u^1 \cdots u^n) \begin{pmatrix} g_{11}(x) & \cdots & g_{1n}(x) \\ \vdots & & \vdots \\ g_{n1}(x) & \cdots & g_{nn}(x) \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$$

This is an example of tensor

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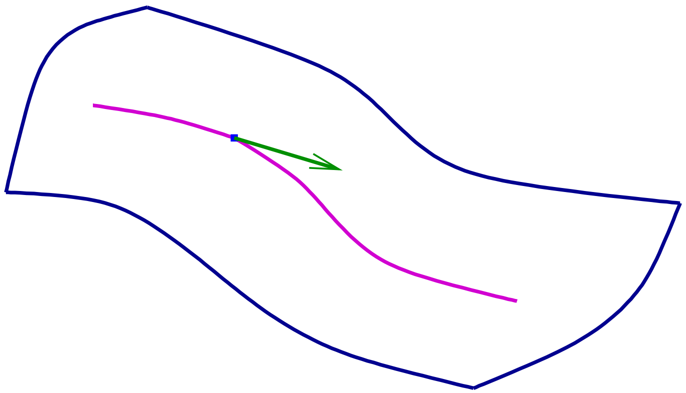
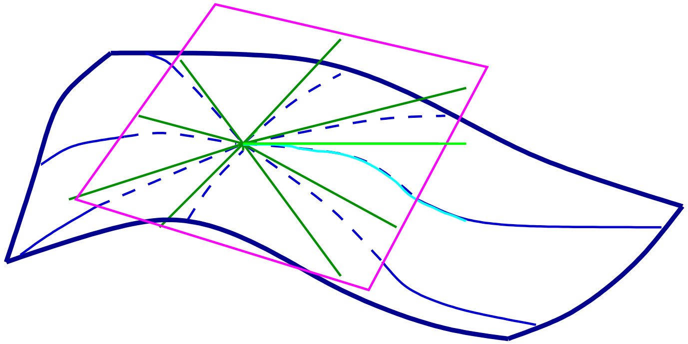
Length of curves $\gamma(t) = (x_1(t), \ldots, x_n(t))$, $a \leq t \leq b$

$$L = \int_a^b |\gamma'(t)| dt = \int_a^b \sqrt{\sum_{ij} x_i'(t) g_{ij}(\gamma(t)) x_j'(t) dt}$$



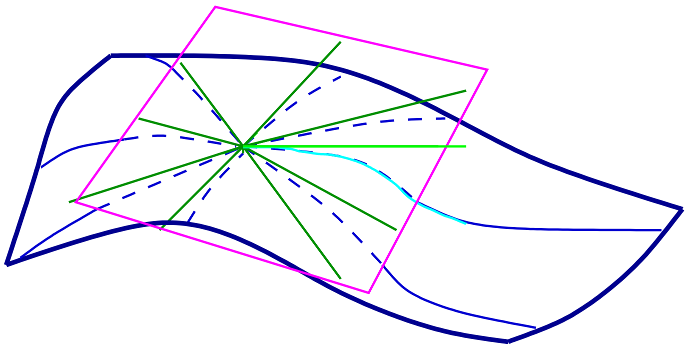


The geodesic exponential map identifies

— radial straight lines starting at 0, in the tangent space

— with minimizing geodesics starting at the point, in the manifold.

Geodesic or normal coordinates



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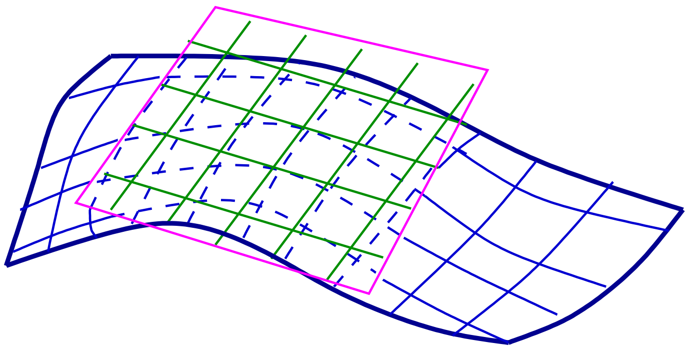

Normal coordinates $\longleftrightarrow$ "squared-gird" coordinates in the tangent

In normal coordinates, Riemann proved in his habilitation (1854):

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}\sum_{\alpha,\beta} R_{i\alpha\beta j} x^\alpha x^\beta + O(|x|^3)$$

- $R_{i\alpha\beta j} = -R_{i\alpha j\beta} = -R_{\alpha i\beta j} = R_{\beta j i\alpha}$
- $R_{i\alpha\beta j} + R_{i\beta j\alpha} + R_{ij\alpha\beta} = 0.$

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$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}\sum_{\alpha,\beta} R_{i\alpha\beta j} x^\alpha x^\beta + O(|x|^3)$$

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- $R_{i\alpha\beta j} + R_{i\beta j\alpha} + R_{ij\alpha\beta} = 0.$

- $R_{i\alpha\beta j}$ is the Riemannian curvature tensor.
  Currently defined with covariant derivatives.
- Riemann finds the Gauss curvature K for surfaces:

$$K = R_{1212} = -R_{1221} = -R_{2112} = R_{2121}$$

Ricci, scalar, and sectional curvatures

In geodesic coordinates

- Ricci curvature $R_{ij} = \sum_\alpha R_{i\alpha\alpha j}$
- Scalar curvature $R = \sum_i R_{ii}$
- Sectional curvature of the plane $x_3 = \cdots = x_n = 0$, $K = R_{1212}$.

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- "Ricci is $-\frac{1}{3}$ of Hessian matrix of volume"

$$d\,vol = \sqrt{det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n$$

$$d\,vol(x) = \left(1 - \frac{1}{6}\sum_{ij} R_{ij}x^i x^j + O(|x|^3)\right) dx^1 \wedge \cdots \wedge dx^n.$$

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Einstein’s equation: $R_{ij} - \frac{1}{2} R g_{ij} = T_{ij}$

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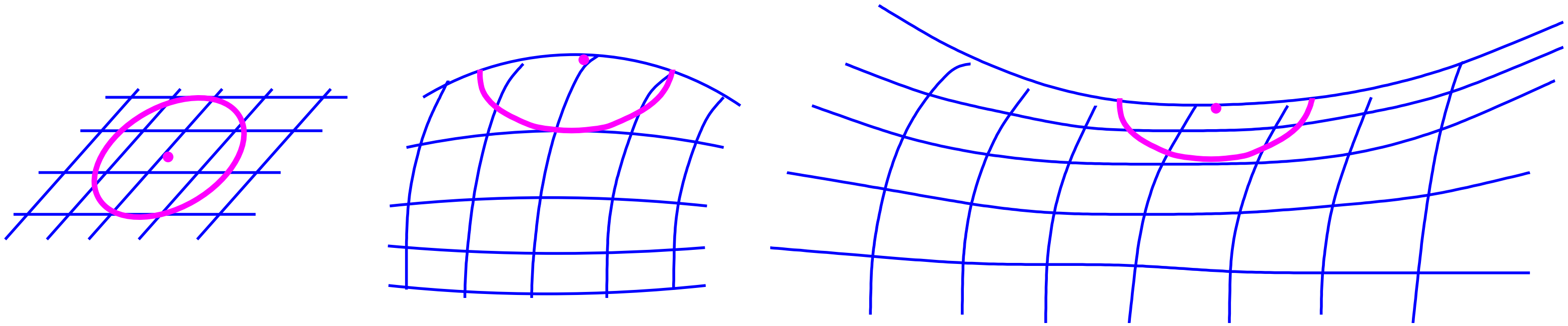
$(R_{ij}) = 0$ $(R_{ij}) > 0$ $(R_{ij}) < 0$



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where
$$\begin{cases} \Delta(g_{ij}) = \text{Laplacian of the scalar function } g_{ij} \\ Q_{ij} = \text{quadratic expression} \end{cases}$$

It is a **reaction-diffusion** equation

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- Hamilton/DeTurck:

Short time existence and uniqueness

When M^n is compact there is a unique solution defined for $t \in [0, T)$, $T > 0$.

Example

- Assume that $g(0)$ has constant sectional curvature K.

  $\Rightarrow R_{ij} = (n-1)Kg_{ij}(0)$

  Set $g_{ij}(t) = f(t)g_{ij}(0)$,

  then $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$ is equivalent to the ODE

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$$\boxed{g(t) = (1 - 2K(n-1)t)g(0)}$$

$$\begin{cases} \text{if } K < 0 \text{ it expands forever} \\ \text{if } K = 0 \text{ it keeps constant} \\ \text{if } K > 0 \text{ it collapses at time } T = \frac{1}{2K(n-1)} \end{cases}$$

Example: Solitons

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

A solution g_t is a soliton if $\boxed{g_t = \lambda(t)\Phi_t^* g_0}$.
Shrinking if $\lambda < 1$, steady if $\lambda = 1$ and expanding if $\lambda > 1$.

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A gradient soliton if $\boxed{\frac{\partial}{\partial t}\Phi_t = \nabla f}$

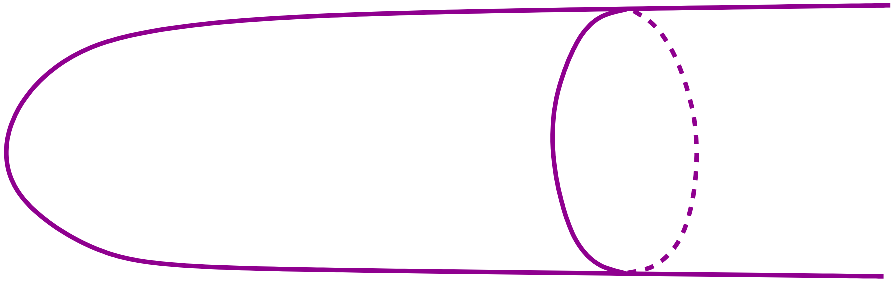
Equivalently:

$$\boxed{R_{ij} + Hess_{ij}(f) + c\, g_{ij} = 0}$$

- Gradient solitons of curvature ≥ 0 appear after blowing up singularities.

Example: Cigar soliton

$$g = \frac{dx^2+dy^2}{1+x^2+y^2} = \frac{dr^2+r^2d\theta^2}{1+r^2} = d\rho^2 + \tanh^2 \rho \, d\theta^2 \qquad \text{in } \mathbb{R}^2$$



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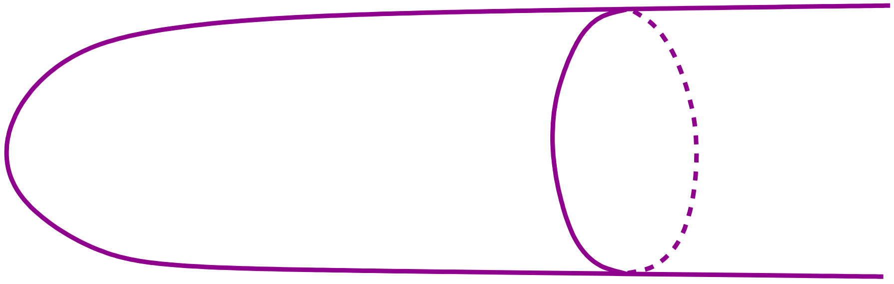

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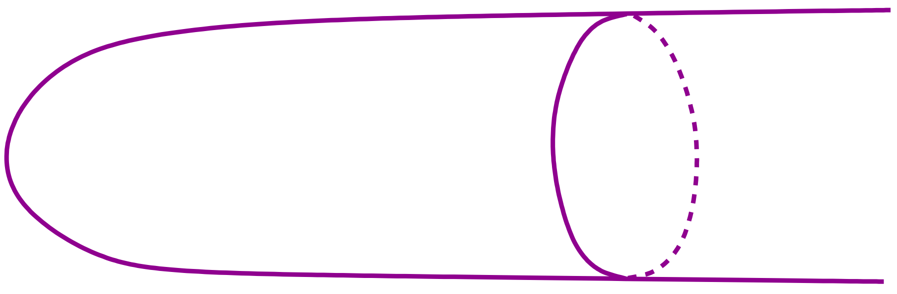

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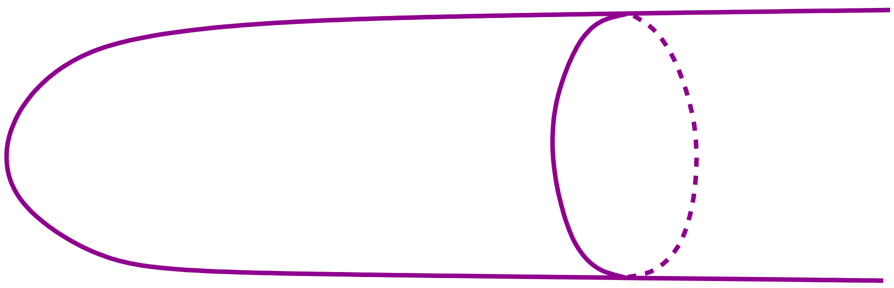
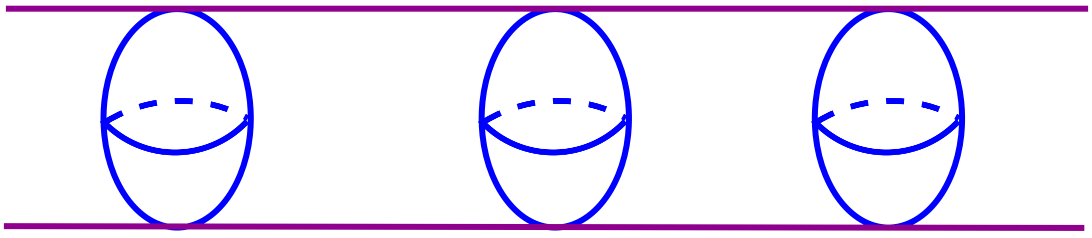


- Asymptotic to a cylinder ($\tanh\rho \to 1$ when $\rho \to \infty$)
- $sec = \frac{2}{\cosh^2\rho} > 0$ and $sec \to 0$ when $\rho \to \infty$.
- It is a steady gradient soliton:

$f = -2\log\cosh\rho$ satisfies $Hess(f) + \frac{2}{\cosh^2\rho} g = 0$

- Cylinder $S^2 \times \mathbb{R}$:



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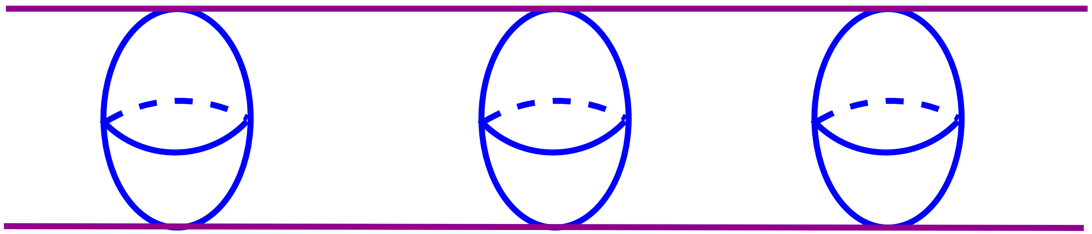

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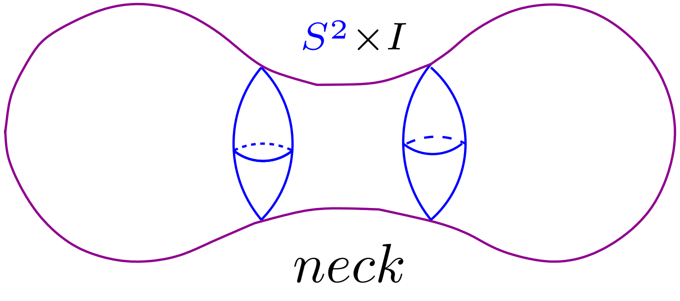
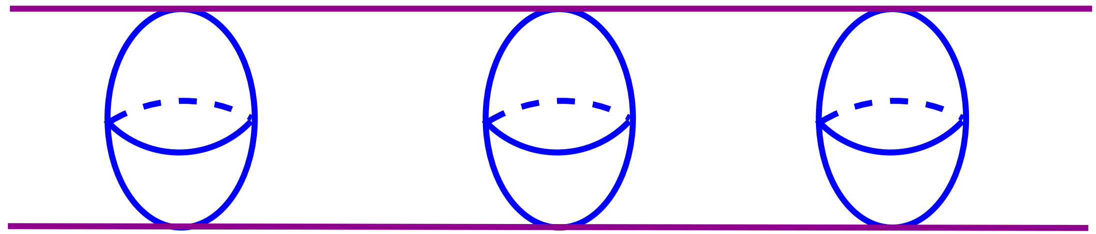


More examples

- Cylinder $S^2 \times \mathbb{R}$:



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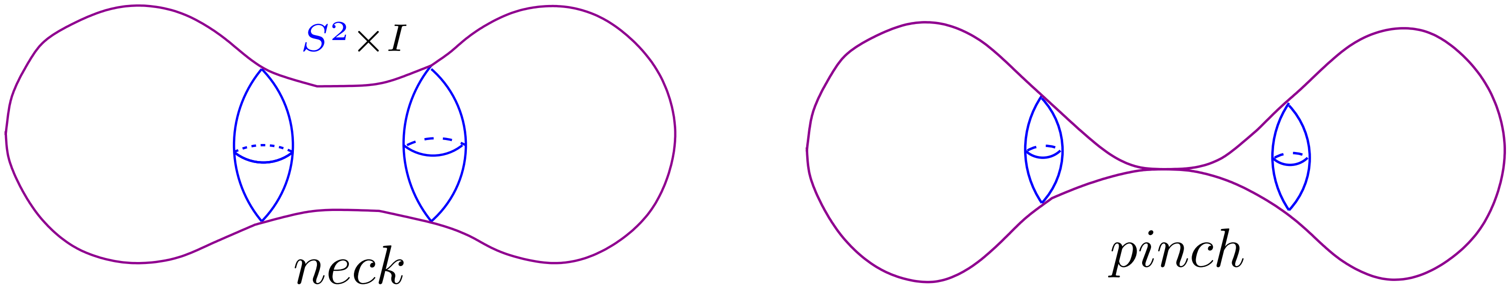
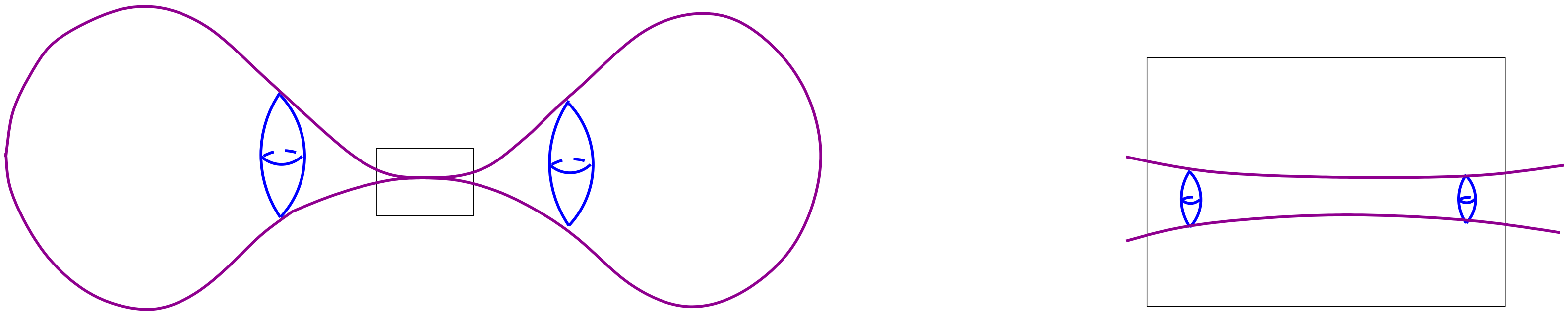


Zoom of singularities in dimension three



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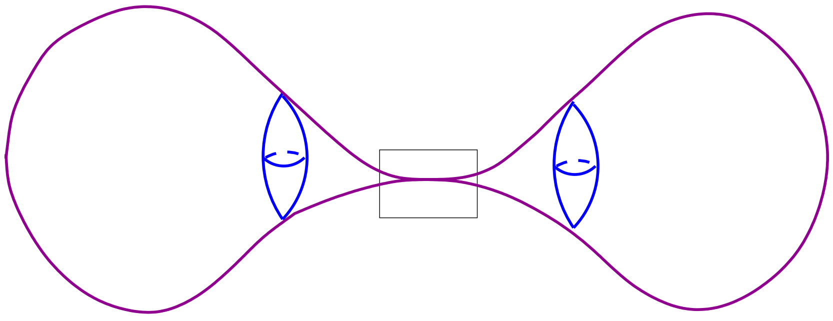
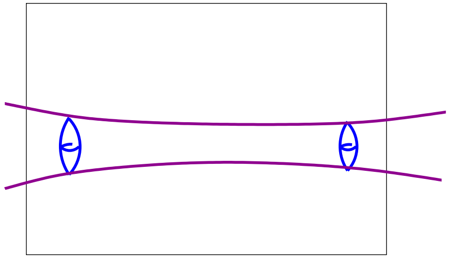
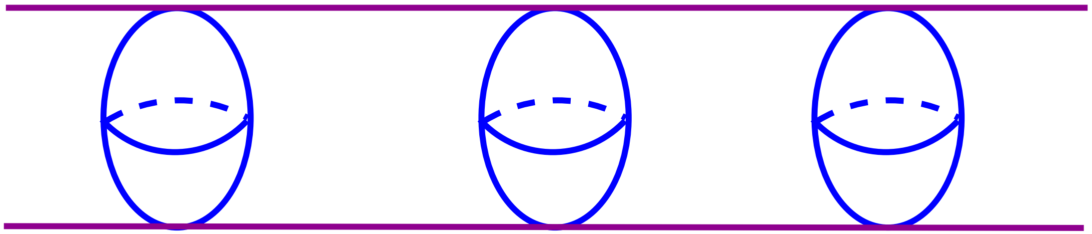




When zoom and blow up a singularity
we would like to get a cylinder $S^2 \times \mathbf{R}$



Theorem (Hamilton 1982)

If M^3 admits a metric with $(R_{ij}) > 0$

$\Rightarrow M^3$ admits a metric with curv $\equiv 1$

Idea: • $(R_{ij}) > 0$ is an invariant condition for the flow in dim 3.

• One can control the eigenvalues of R_{ij}.

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Generalization:

- If $(R_{ij}) \geq 0$, it admits a loc. homogeneous metric, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$, S^3.

(Strong maximum principle for tensors (Hamilton)).

Scalar curvature R

$$R = \sum R_{ii}$$

- Evolution of R for the Ricci flow:

$$\frac{\partial R}{\partial t} = \Delta R + 2|(R_{ij})|^2$$

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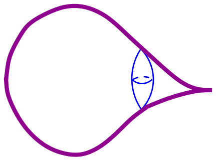
- Hamilton-Ivey: R controls singularities in dim 3:

When approaching the limit time, $R \to \infty$ at some point.

Singularities



Singularities appear at limit time T of existence of the flow.

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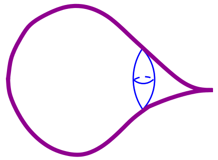

Singularities appear at limit time T of existence of the flow.

When $t \to T$, $R \to \infty$ at some point.

- Hamilton's question: How to control the injectivity radius around singularities?

- Perelman 2002: Solutions to Ricci flow are locally non-collapsed (after rescaling at $R = 1$).

Theorem: κ-non collapse

$\exists \kappa > 0$ s.t. $\forall r > 0$, $\forall x \in M$ and $\forall t \in [1, T)$,

If $\forall y \in B(x,t,r)$, $|R(y,t)| \leq r^{-2} \Rightarrow \frac{vol(B(x,t,r))}{r^3} \geq \kappa$

$\Rightarrow$ When we rescale to $|R(y,t)| = 1$,

lower bound of injectivity radius.

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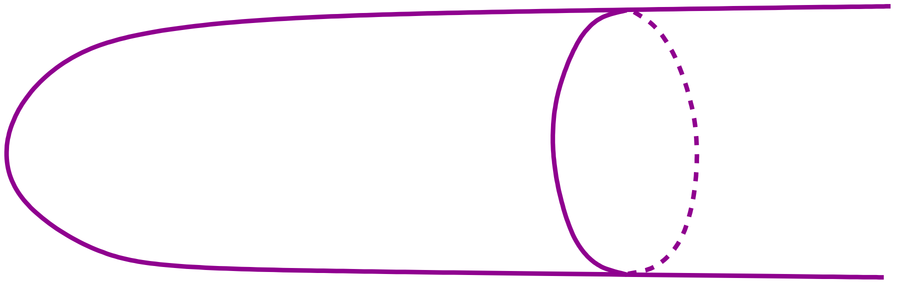
- Idea: "$\mathcal{L}$-geodesics" and "reduced volume".
- This excludes the cigar soliton as local model for singularities. Seek cylinders $S^2 \times \mathbb{R}$ as local models for singularities

The cigar soliton is κ-collapsed

$$g_{cigar} = \frac{dx^2+dy^2}{1+x^2+y^2} = d\rho^2 + \tanh^2\rho\, d\theta^2 \qquad \text{in } \mathbb{R}^2$$



Consider $g_{cigar} + dz^2$ in $\mathbb{R}^3$ or in $\mathbb{R}^2 \times S^1$.

Since $R = \frac{2}{\cosh^2\rho} \to 0$ and $inj \to 1$ when $\rho \to \infty$,
it is excluded as local model for singularities
(by the κ-non collapse)

(κ-non collapse: when rescale at $|R| = 1$, $inj > c(\kappa) > 0$)

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Theorem: canonical neighbourhoods

$\forall \varepsilon > 0$, $\exists r > 0$, s.t. $\forall x \in M$ an $\forall t \in [1, T)$,

If $R(x,t) \geq r^{-2} \Rightarrow x \in (M, g(t))$ lies in a ε-canonical neighbourhood.

ε-canonical neighbourhood:

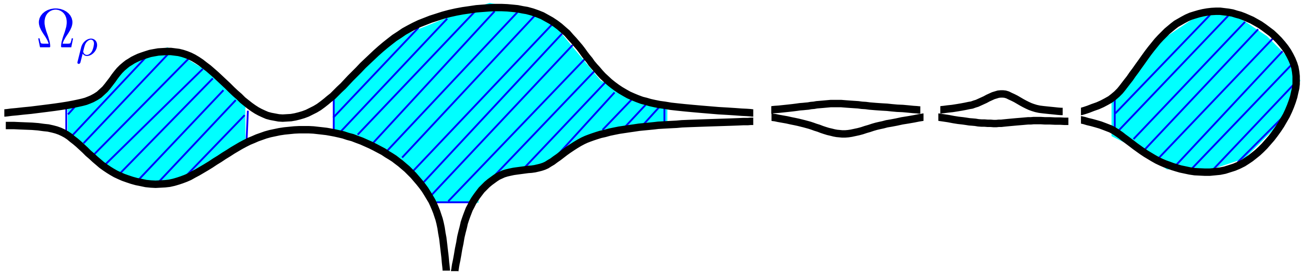
- ε-close to a cylinder $S^2 \times (0, l)$
- ε-close to B^3 open with cylindrical end
- manifold with $sec > 0$.

Ricci flow with δ-surgery

$(M^3, g(t))$ Ricci flow, $t \in [0, T)$.

$\Omega_\rho = \{x \in M \mid R(x,t) \leq \rho^{-2}, t \to T\}$ compact.

$\Omega = \bigcup_{\rho>0} \Omega_\rho$ open. $g_\infty =$ limit metric on Ω.



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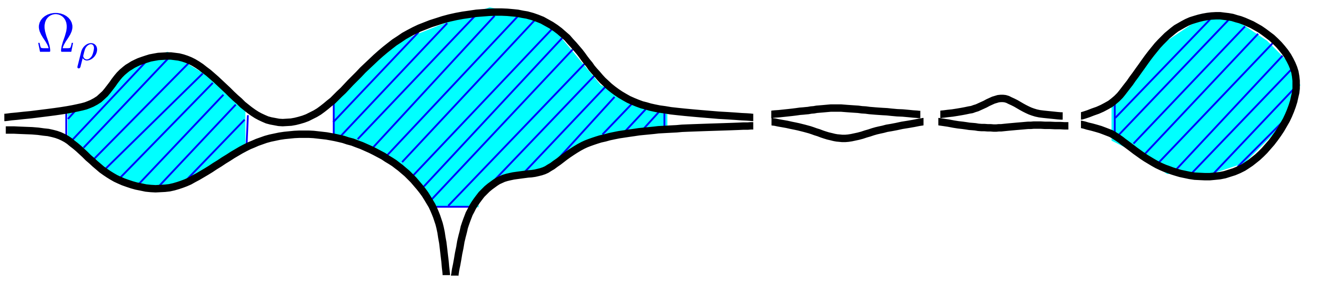

If $t \lesssim T \Rightarrow (M^3 - \Omega_r, g(t)) =$ union of ε-canonical neighbourhoods.

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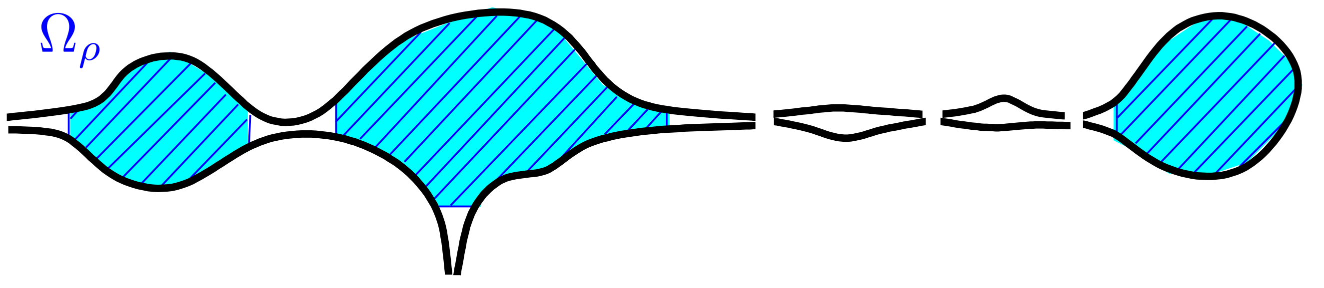

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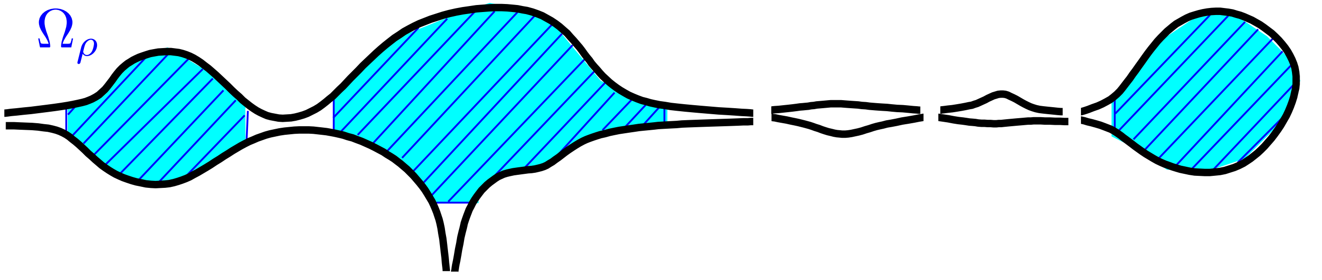

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- δ-surgery: Glue hemispheres to the boundary of (Ω_ρ, g_∞), smooth them out and continue the flow.

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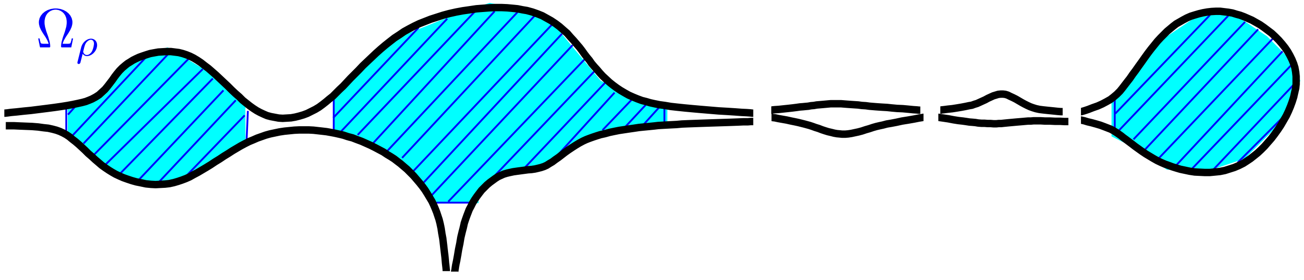

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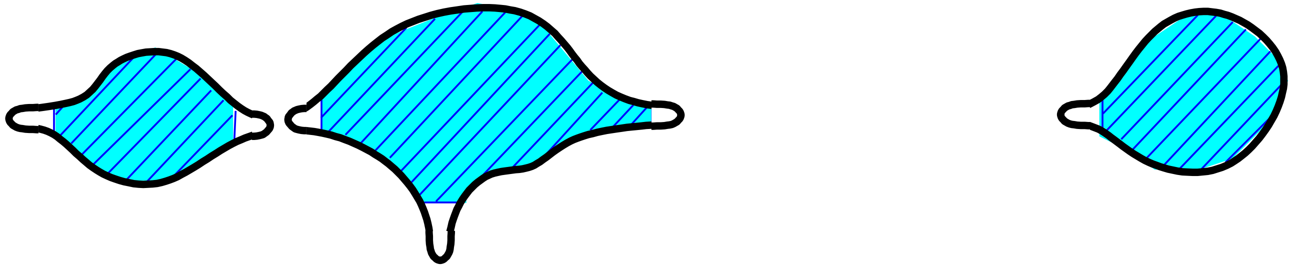

...and apply again the flow.

Evolution of Ricci flow with δ-surgery

1 There could be infinitely many surgery times.
Surgery times do not accumulate (volume estimates)

$$\frac{d}{dt}\,\mathrm{vol}(M, g(t)) = -\int_M R \leq ctnt \cdot \mathrm{vol}(M, g(t)) \qquad (\min_M R \text{ non-decreasing})$$

and every surgery decreases at least a certain amount of volume

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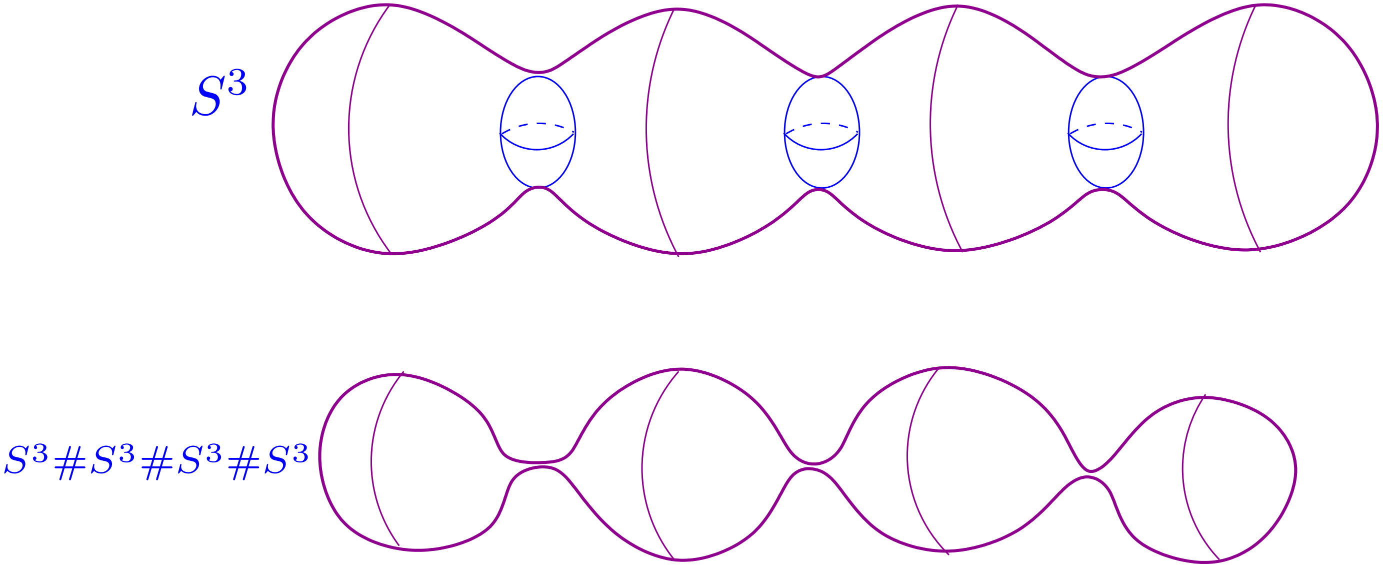

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4 By 1:
- Either ends up with a connected sum of manifolds of constant curvature $\equiv +1$ and $S^2 \times S^1$,
- or continues forever.

For sufficiently large time, M_t splits into:

$$M_t = M_t^{thin} \cup M_t^{thick}$$

thin/thick according to whether inj-rad is larger/less than $c(R, t, \delta)$.

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$$M_t = M_t^{thin} \cup M_t^{thick}$$

thin/thick according to whether inj-rad is larger/less than $c(R, t, \delta)$.

This corresponds to the JSJ splitting.

$$\begin{cases} M_t^{thick} = \text{hyperbolic (by regularization of flow)} \\ M_t^{thin} = \text{union of Seifert fibrations, called GRAPH manifold} \\ \qquad \text{(using techniques of collapsed manifolds)} \end{cases}$$

...and thank you for your attention!