# The notion of commensurability in group theory and geometry

Luisa Paoluzzi

(LATP Marseilles – France)

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## Definition:

Let $H_1$ and $H_2$ be subgroups of a group G. We say that $H_1$ and $H_2$ are commensurable if $[H_1:K]$, $[H_2:K]<\infty$, where $K=H_1\cap H_2$.

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## Remark:

“Being commensurable” is an equivalence relation.

## Examples:

G finite: any two $H_1$ and $H_2$ are commensurable.

$G=\mathbf{Z}$: $H_1$ and $H_2$ are commensurable iff they are isomorphic.

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6. Commensurability and geometric group theory

## Where's the geometry/topology in this?

Look at $H_1$ and $H_2$ as fundamental groups.

For instance, let G = PSL(2,**C**) acting on $\mathbf{H}^3$ and let $H_1$ and $H_2$ be lattices which are fundamental groups of hyperbolic manifolds (or, more generally, orbifolds). Obviously, if $H_1$ and $H_2$ are commensurable $X/H_1$ and $X/H_2$ have a common finite cover.

Since (orbifold) fundamental groups are defined as subgroups of G only up to conjugacy, it is natural to allow subgroups to have a finite index intersection only up to conjugacy.

Definition:

Let $H_1$ and $H_2$ be subgroups of a group G. We say that $H_1$ and $H_2$ are weakly commensurable if there is a g in G such that $[H_1:K], [H_2:K]<\infty$, where $K=H_1 \cap gH_2g^{-1}$.

Remark:

“Weak commensurability” is also an equivalence relation.

## Definition:

Let H be a subgroup of a group G. The commensurator of H in G is defined as

$Comm_G(H)=\{g \in G \mid [H:K], [gHg^{-1}:K]<\infty \text{ where } K=H \cap gHg^{-1}\}$.

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## Remark:

$Comm_G(H)$ is a subgroup of G satisfying:

$$Z(G) \subset C_G(H) \subset N_G(H) \subset Comm_G(H)$$

and, of course,

$$H \subset N_G(H) \subset Comm_G(H)$$

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## Remark:

Note that commensurable subgroups have the same commensurator, while weakly commensurable subgroups have conjugate commensurators.

**Geometry = Topology in dimension 3**

Let us consider two 3-dimensional hyperbolic manifolds (closed or finite volume). Assume they have a common finite cover.

Let $H_1$ and $H_2$ images of faithful irreducible discrete representations of their fundamental groups inside G=PSL(2,**C**). By Mostow's rigidity, $H_1$ and $H_2$ are (weakly) commensurable.

The algebraic and “geometric” notion of commensurability coincide in this setting.

## Geometry vs Topology in dimension 2

Let $S_g$ denote the fundamental group of the genus g close orientable surface.

Of course $S_g \subset S_2$ for all $g \geq 2$.

On the other hand one can find discrete surface groups $H_1$ and $H_2$ inside $G = PSL(2,\mathbf{R})$ which are not (weakly) commensurable.

## Definition:

Let $H_1$ and $H_2$ be groups. We say that $H_1$ and $H_2$ are abstractly commensurable if there are subgroups $K_i$ of $H_i$, $i=1,2$, such that $[H_1:K_1]$, $[H_2:K_2]<\infty$ and $K_1 \cong K_2$.

## Examples:

All finite groups are abstractly commensurable.
All finite rank ≥ 2 free groups are abstractly commensurable.
All hyperbolic surface groups are abstractly commensurable.
Two free abelian groups are commensurable iff they are isomorphic.

Remark:

The observations made about surfaces show that abstract commensurability is a concept better adapted to topology or geometry in a coarse sense, while weak commensurability can capture finer geometric aspects.

## Definition:

Let H be a group. The abstract commensurator of H is

$$\mathrm{Comm}(H)=\{f : K_1 \to K_2 \text{ isomorphism} \mid [H:K_1], [H:K_2]<\infty\}/\sim$$

where ~ is the equivalence relation defined by $f \sim f'$ whenever there is a $K \subset H$, $[H:K]<\infty$, such that $f|_K = f'|_K$.

## Remarks:

“Partial composition” gives Comm(H) a group structure.
There is a natural morphism $\mathrm{Aut}(H) \to \mathrm{Comm}(H)$.

## Proposition:

Let H be a group with the unique root property. Then Aut(H) injects into Comm(H).

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Proof:

Assume $g \in \mathrm{Aut}(H)$ induces a trivial commensurator, i.e. there is $K \lhd H$, $[H:K]<\infty$, such that $g|_K=Id_K$.

There is an integer n such that, for all $h \in H$, $h^n \in K$ and so $h^n=g(h^n)=g(h)^n$. It follows $h=g(h)$ for all $h \in H$.

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Example:

$GL(n,\mathbf{Z}) \cong \mathrm{Aut}(\mathbf{Z}^n) \subset \mathrm{Comm}(\mathbf{Z}^n) \cong GL(n,\mathbf{Q})$.

## Theorem (Bartholdi-Bogopolski):

Let H be a group with the unique root property. Assume that for infinitely many primes p there are $K \triangleleft H$ and $f_K \in \mathrm{Aut}(K)$ such that

$$1 \to K \to H \to \mathbf{Z}/p\mathbf{Z} \to 1$$

and

for all $g \in \mathrm{Aut}(H)$ $f_K$ is not the restriction of g to K.

Under these hypothesis Comm(H) is not finitely generated.

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## Corollary:

The commensurator of the free group $F_n$ is not finitely generated.

Remarks:

Obviously commensurators and abstract commensurators don't have much in common:

Theorem (Rolfsen)

$\mathrm{Comm}_{B_m}(B_n)=N_{B_m}(B_n)=\langle B_n, C_{B_m}(B_n)\rangle \cong B_n \times (B_{m-n+1})_1$

where $(B_{m-n+1})_1$ denotes the stabiliser of the “first” strand.

Theorem (Leininger-Margalit)

For $n\geq 4$, $\mathrm{Comm}(B_n)\cong \mathrm{Mod}(S^2,n+1)(\mathbf{Q}^*(\mathbf{Q}^\infty))$.

Remarks:

Abstract commensurators are not necessary “complicated”

Theorem (Farb-Handel)

$\mathrm{Comm}(\mathrm{Out}(F_n)) \cong \mathrm{Out}(F_n)$ if $n>3$.

NB

$B_n \hookrightarrow \mathrm{Aut}(F_n)$ and $B_n/Z(B_n) \hookrightarrow \mathrm{Out}(F_n)$

Remarks:

Theorem (Arzhanteva-Lafont-Minasyan)

There are

(i) a class of finite presentations of groups in which the isomorphism problem is solvable but the abstract commensurability problem is not

and, conversely,

(ii) a class of finite presentation of groups in which the abstract commensurability problem is solvable but the isomorphism problem is not.

Theorem (Margulis):

Let G=PSL(2,**C**) and H be a discrete subgroup of G of finite covolume. Then either H is of finite index in $Comm_G(H)$ or $Comm_G(H)$ is dense in G. This second situation happens iff H is *arithmetic*.

NB

Here G can be any connected semi-simple Lie group with trivial centre and H an irreducible lattice.

Definition:

A lattice H of PSL(2,**C**) is arithmetic if the trace field of H is a number field with exactly one complex place.

Example:

*Bianchi groups* PSL(2,*O*(d)) are arithmetic. Here *O*(d) is the ring of integers of the number field **Q**(√-d).

Remark (geometric consequence of Margulis' result):

Consider a commensurability class of hyperbolic 3-orbifolds.
If the orbifolds are not arithmetic, the class contains a unique minimal element, i.e. a common quotient of all elements of the class.
Its orbifold fundamental group is the commensurator of the groups of the class.

Theorem (Borel):

The commensurability class of an arithmetic 3-orbifold contains infinitely many minimal elements.

Remarks (commensurability invariants):

Commensurable orbifolds have commensurable volumes although the converse is not true in general.

(Reid) The *invariant trace field*, i.e. the smallest field containing the squares of the traces of the elements of the group.

(For cusped orbifolds) The *cusp field*, generated by the cusp parameter or cusp shape.

## What about the other 3D geometries?

Proposition:

All compact orientable manifolds admitting a geometry which is neither hyperbolic nor Sol, belong to one (or two) commensurability classes.

The commensurability classes of compact Sol manifolds are in one-to-one correspondence with real quadratic number fields (related to the eigenvalues of the monodromy of the bundle).

## Theorem (Reid):

There is a unique arithmetic knot, i.e. the figure-eight.

It follows that the figure-eight is the only knot in its commensurability class.

## Remark:

On the other hand there are infinitely many 2-component arithmetic links.

### Definition:

By an abuse of language, two knots are commensurable if their complements are.

Examples:

Other knots are known to be the alone in their commensurability classes:

All 2-bridge knots (Reid-Walsh);

Pretzel knots of type (-2,3,n) for $n \neq 7$ (Macasieb-Mattman).

Conjecture (Reid-Walsh):

A commensurability class contains at most three knot complements.

Remarks:

The commensurability class of the pretzel knot (-2,3,7) contains three knot complements (Reid-Walsh);

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The commensurability class of the pretzel knot (-2,3,7) contains three knot complements (Reid-Walsh);

There are infinitely many commensurability classes containing three knot complements (Hoffman).

Theorem (Boileau-Boyer-Cebanu-Walsh):

If a knot has no *hidden symmetries* then its commensurability class contains at most two other knots.

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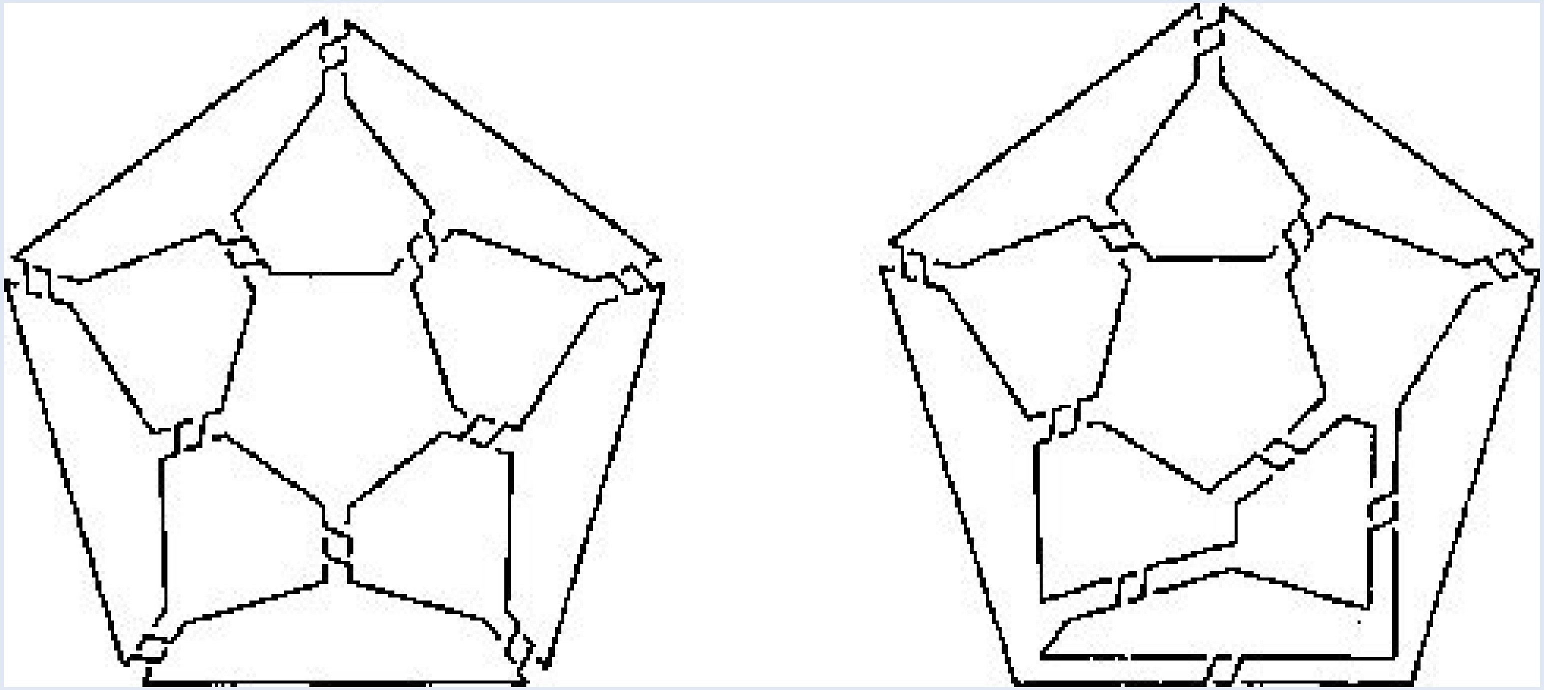
Definition:

A knot has hidden symmetries if its fundamental group is not normal in its commensurator.

## 4. Commensurability and hyperbolic knots

# Remarks:

Only three knots with hidden symmetries are known: the figure-eight knot and the two dodecahedral knots described by Aitchison and Rubinstein:



## Remarks:

The cusp shape of a knot with hidden symmetries must belong either to **Q**[√-3] or **Q**[i]; moreover if the knot is not arithmetic the cusp is rigid. (Neumann-Reid)

Also just one knot with cusp field **Q**[i] is known and it has no hidden symmetries.

## Conjecture (Neumann-Reid):

The only non arithmetic knots with hidden symmetries are the two dodecahedral knots (which belong to the same commensurability class).

Theorem (Boileau-Boyer-Cebanu-Walsh):

Let K be a hyperbolic knot whose *cyclic* commensurability class contains another knot K'. Then

(i) K and K' are fibred (follows from work of Ni);
(ii) K and K' have the same genus;
(iii) K and K' have different volume; in particular K and K' are not mutants;
(iv) K and K' are chiral and not commensurable with their mirror images.

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(iv) K and K' are chiral and not commensurable with their mirror images.

Note that the two dodecahedral knots (i) are one fibred and the other not; (ii) have different genera; (iii) have the same volume; (iv) are both amphicheiral.

## The case of links

Recall that mutation preserves volume (trace field and Bloch invariant) but can give rise to non isometric hyperbolic links.

Theorem (Chesebro-Deblois):

Non isometric mutant links may be commensurable but this is not necessarily the case.

Theorem (Boileau-Boyer-Cebanu-Walsh):

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The conclusion follows from the cyclic surgery theorem: at most three slopes give lens spaces.

Abstractly commensurable (finitely presented) groups are easily seen to be quasi-isometric, however abstract commensurability is in general a stronger notion.

Example:

All hyperbolic 3-orbifold groups are quasi-isometric, for they are quasi-isometric to $\mathbf{H}^3$, but they are not all commensurable.

Sometimes the two notions coincide, though.

Theorem (Schwartz):

Non uniform lattices of PSL(2,**C**) are commensurable iff they are quasi isometric.

Theorem (Behrstock-Januszkiewicz-Neumann + Papasoglu-Whyte):

Let $H=H_1*...*H_n$ and $H'=H'_1*...*H'_m$ two non trivial free products of finitely generated non trivial abelian groups. Assume H, H'≠ $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$. The following are equivalent:

(i) H and H' are commensurable;
(ii) H and H' are quasi-isometric;
(iii) {rk(H1),...,rk(Hn)}={rk(H'1),...,rk(H'm)}.

## Remarks:

The groups in the previous theorem are (virtually) right-angled Artin groups of a special type (with associated graph a disjoint union of complete graphs).

For right-angled Artin groups with associated graph a tree see Behrstock-Januszkiewicz-Neumann.
For a classifification up to quasi-isometry of higher dimensional right-angled Artin groups see Behrstock-Neumann.
For commensurators of “generic” Artin groups see Crisp.
For commensurability between right-angled Artin and right-angled Coxeter groups see Davis-Januszkiewicz (this implies that right-angled Artin groups are linear).

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The classification of virtual surface groups up to abstract commensurability coincides with their classification up to quasi-isometry.

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Moreover, a group which is quasi-isometric to a hyperbolic surface group is virtually a cocompact Fuchsian group (Casson-Jungreis, Gabai).

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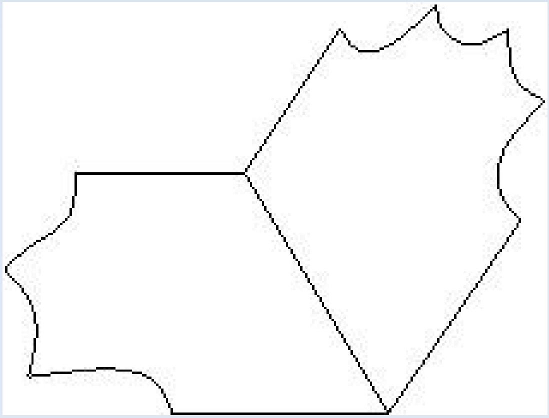
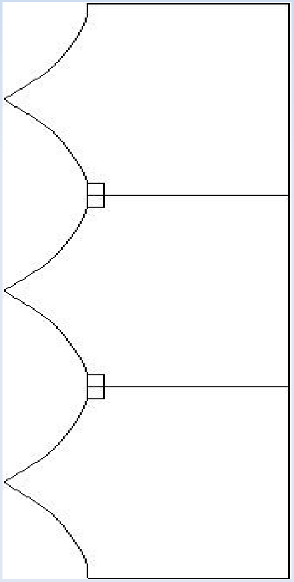
Moreover, a group which is quasi-isometric to a hyperbolic surface group is virtually a cocompact Fuchsian group (Casson-Jungreis, Gabai).

The classification up to abstract commensurability or quasi-isometry of right-angled polygon Coxeter groups follows.

Consider the 2-complex obtained by gluing together two right-angled hyperbolic polygons along an edge (the polygons do not lie in the same plane):





Note that a right-angled hyperbolic n-gon, n>4, can be obtained by gluing together n-4 copies of a right-angled regular hyperbolic pentagon.

Let $n \geq m \geq 1$. Consider $W_{m,n}$ the Coxeter group of reflections in the edges of the 2-complex defined before by identifying along an edge a hyperbolic right-angled (n+4)-gon and a hyperblic right-angled (m+4)-gon.

$$W_{m,n} = C_{m+4} *_{D_\infty \times \mathbf{Z}/2\mathbf{Z}} C_{n+4}$$

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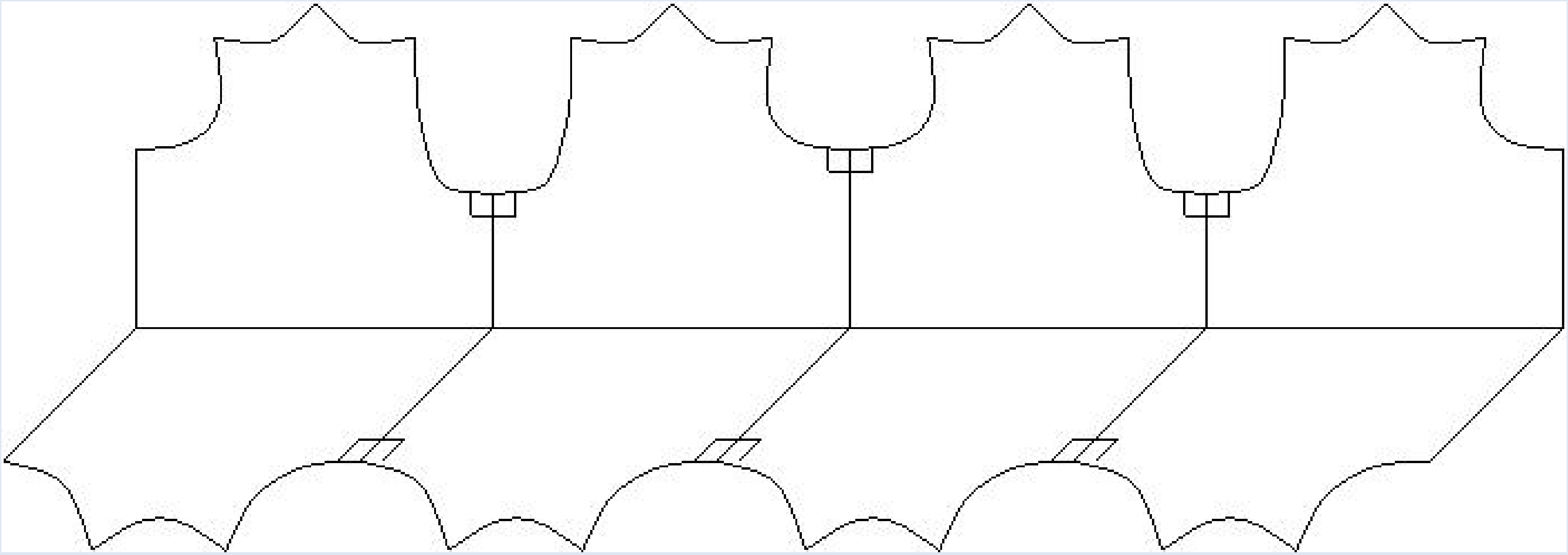
Theorem (Crisp-P.):

$W_{m,n}$ and $W_{k,l}$ are abstractly commensurable iff $m/n=k/l$.

Idea of proof (sufficiency):



Idea of proof (necessity):

Lemma (Lafont):

Any isomorphism between finite index subgroups of $W_{m,n}$ and $W_{k,l}$ is induced by a homeomorphism of the corresponding covering spaces.

One can assume that the corresponding covering spaces are obtained by gluing together “tiled surfaces” along simple closed “singular” geodesics.

## Idea of proof (necessity):

The lemma implies that we have a homeomorphism

$$h: S_1 \cup S_2 \rightarrow S'_1 \cup S'_2$$

where $S_1$ is an (m+4)-tiled surface, $S_2$ an (n+4)-tiled surface, $S'_1$ a (k+4)-tiled surface, and $S'_2$ an (l+4)-tiled surface.

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We can assume that $h^{-1}(S'_1) \cap S_1$ is not empty.

It is easy to see that the number τ of tiles in $U=h^{-1}(S'_1) \cap S_1$ equals the number of tiles in $V=h^{-1}(S'_2) \cap S_2$.

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An Euler characteristic computation gives:

$$m/n=m\tau/n\tau=\chi(U)/\chi(V) = \chi(h(U)) / \chi(h(V))=k/l$$

## Thank you for your attention!

Pssst, it's over now...