The character varieties of Montesinos knots of Kinoshita-Terasaka type

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The computational complexity involved requires that theoretical ways to study and describe them are explored.
1. The 1-dimensional component of abelian characters
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2. The distinguished curve containing the holonomy character
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3. The Teichmüller components (for Montesinos knots)
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Standard components for a hyperbolic (Montesinos) knot:

1. The 1-dimensional component of abelian characters

2. The distinguished curve containing the holonomy character

3. The Teichmüller components (for Montesinos knots)
Examples of non standard components:
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2. For certain pretzel knots: they are due to “symmetry” *(Mattman's r-components)*
Riley studied this Montesinos knot and found that it admits several parabolic representations into $\PSL(2,p)$
2. Montesinos knots of Kinoshita-Terasaka type: arbitrary Montesinos links

A Montesinos link (left) and a rational tangle (right)

Orientations of the two vertical arcs on the far left can be the same or opposite.
Definition:

A Montesinos knot with $n+1$ tangles is of Kinoshita-Terasaka type if precisely one amongst the $\alpha_i$s is even. Up to cyclic reordering we can assume this is the $n+1^{st}$ tangle.
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Remark:

These Montesinos knots are “closely related” to composite knots whose summands are 2-bridge knots.
2. Montesinos knots of Kinoshita-Terasaka type: definition and properties
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By quotienting the two knot groups by the relations that make the meridians $\mu_1$, $\mu'_1$, $\mu_{n+1}$, and $\mu'_{n+1}$ commute one obtains the same group $\Gamma$ (commuting trick).
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Composite knots have lots of representations obtained by bending.
Theorem (P-Porti):

Let $K$ be a Montesinos knot of Kinoshita-Terasaka type with $n+1>3$ tangles. Its character variety contains the following non standard components:
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1. At least one irreducible component of dimension (at least) $n-2$ of parabolic irreducible characters;

2. At least one irreducible component of dimension (at least) $n-2$ of irreducible characters on which the trace of the meridian is not constant.
3. Main result: statement

Remarks:

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3. There are non standard irreducible components of dimension d, for each d=1,...,n-2 (uses surjections between knot groups) and again their number can be arbitrarily large.
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3. There are non standard irreducible components of dimension $d$, for each $d=1,...,n-2$ (uses surjections between knot groups) and again their number can be arbitrarily large.

4. There might be other non standard components.
Sketch of proof (parabolic component):

1. Construct irreducible parabolic representations for the associated composite knot $K' = B_1 \# \ldots \# B_n$.

Choose $\rho_i : \pi_1(B_i) \to \text{SL}(2, \mathbb{C})$ parabolic irrep (e.g. holonomy) \\

Build $\rho = \rho_1 \ast a_1 \rho_2 a_1^{-1} \ast \ldots \ast a_{n-1} \rho_n a_{n-1}^{-1}$ a parabolic irrep for $K'$ (here $a_i$ is any element belonging to the centraliser of $\rho_i(\mu_i)$ in $\text{PSL}(2, \mathbb{C})$) \\

This gives an $(n-1)$ parameter family of parabolic irreps for $K'$ up to conjugacy.
Sketch of proof (parabolic component):

2. Show that the family of representations just constructed meets the variety of representations of \( \Gamma \), thus giving representations of the Montesinos knot \( K \).

It is sufficient to choose \( a_{n-1} \) so that the fixed point on \( \mathbb{C}P^1 \) of \( \rho_1(\mu_1) \) and \( a_{n-1} \rho_n a_{n-1}^{-1}(\mu_{n+1}) \) are the same: two parabolic elements commute if and only if they have the same fixed point on \( \mathbb{C}P^1 \).

This way one loses one degree of freedom at the most.
Sketch of proof (second component):

Same strategy as before but

1. In principle one more degree of freedom (the trace of the meridian is non constant);

2. Two non parabolic elements commute if and only if they have the same axis (i.e. they fix the same two points of \( \mathbb{CP}^1 \)): one needs to conjugate twice to make sure that this is indeed the case (the argument uses cross-ratios).
4. Characteristic $p$: more motivations

The defining relations for the character variety of a group are the same in every characteristic ($\neq 2$).
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For almost every prime p the character variety in characteristic p “looks exactly like” the one in characteristic 0.
Why bother?

One can use the character variety to detect essential surfaces (**Culler-Shalen**) but not all of them can be detected in characteristic 0 (**Schanuel-Zhang**).
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One can use the character variety to detect essential surfaces (Culler-Shalen) but not all of them can be detected in characteristic 0 (Schanuel-Zhang).

Hope: (extremely hard)

Can one detect them by looking at character varieties in characteristic $p$, where $p$ is a ramified prime?
Proposition (P-Porti):

Consider $\Gamma_p = \Gamma/\langle \mu^p \rangle$. For almost every prime $p > 2$ the character variety $X(\Gamma_p)$ of $\Gamma_p$ ramifies at $p$, in the sense that its dimension is at least $n-2$ while its expected dimension is at most $n-3$. 
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Remark:

The extra ideal points should correspond to Conway spheres.
4. Characteristic $p$: a toy example

Proof:

1. $X(\Gamma_p) = X(\Gamma) \cap \left( \bigcup_k \{ \text{trace of meridian} = 2\cos(k\pi/p) \} \right)$

So, for almost all prime characteristics $>2$, $\dim X(\Gamma_p) \leq n-3$. 
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   So, for almost all prime characteristics $>2$, $\dim X(\Gamma_p) \leq n-3$.

2. In characteristic $p>2$ we have $X(\Gamma_p) = X_{\text{par}}(\Gamma)$ and we know that in characteristic $0$ $\dim X_{\text{par}}(\Gamma) \geq n-2$. 