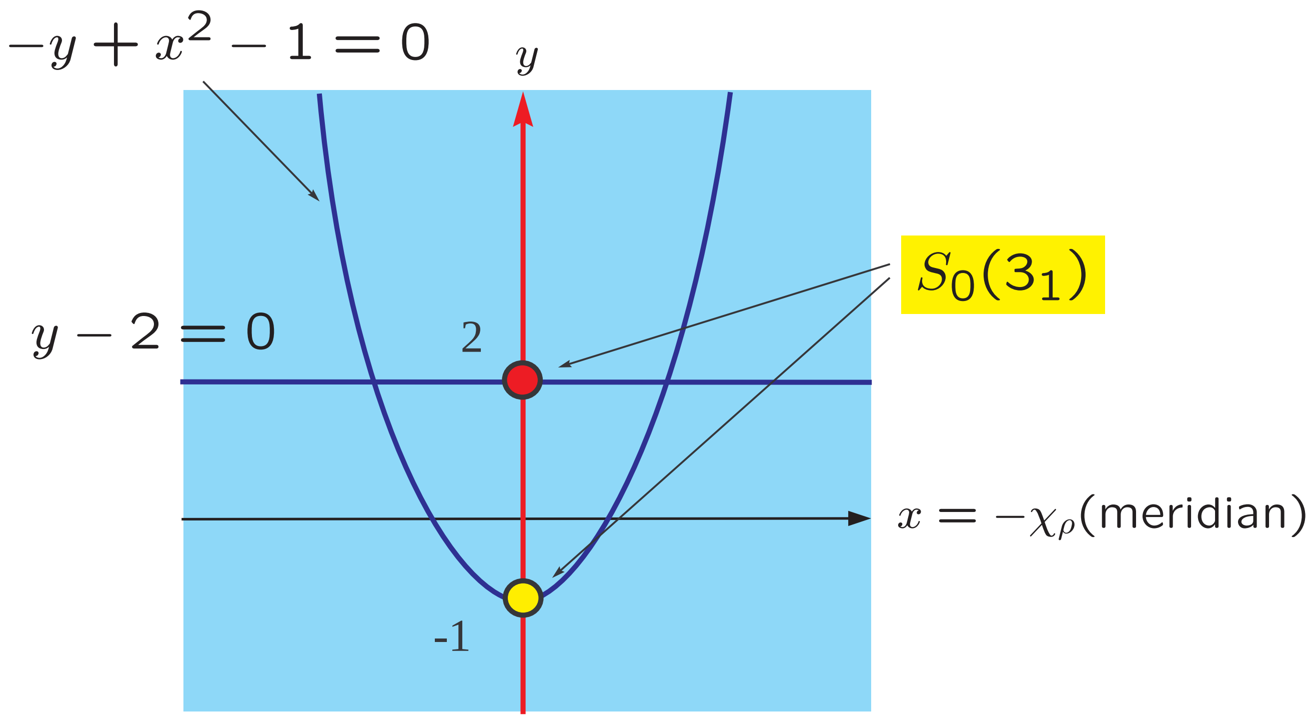
On trace-free characters and abelian knot contact homology



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31/05/2012 RIMS Seminar @ 強羅静雲荘

Notations

- K: a knot in $\mathbf{S}^3$
- $\mathbf{E}_K := \mathbf{S}^3 - \mathbf{N}(K)$
- $G(K) := \pi_1(\mathbf{E}_K)$, knot group
- $R(K) := \{\rho : G(K) \to \mathrm{SL}_2(\mathbb{C}),\ \text{representation}\}$
- $X(K) := \{\chi_\rho : G(K) \to \mathbb{C},\ \text{characters of } \rho \in R(K)\}$
- $S_0(K) := X(K) \cap \left\{ \chi_\rho(\text{meridian}) = 0 \right\}$ **trace-free characters**
- $HC_0^{\mathrm{ab}}(K)$: degree 0 abelian knot contact homology of K introduced by **Lenhard Ng**

  (**[Ng]** *Knot and braid invariants from contact homology I and II*, Geom. Topol. **9** (2005))

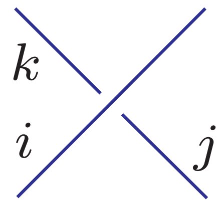
Main theorem

Let $G(K) = \langle m_1, \cdots, m_n \mid r_1 = 1, \cdots, r_n = 1 \rangle$ be a Wirtinger presentation. Then $S_0(K)$ is realized as the algebraic set

$$\left\{ (x_{ab}; x_{pqr}) \in \mathbb{C}^{\binom{n}{2}+\binom{n}{3}} \;\middle|\; \begin{array}{l} (1 \le a < b \le n) \\ (1 \le p < q < r \le n) \end{array} \right.$$

(1) $x_{ka} = x_{ij}x_{ia} - x_{ja}$, $x_{kab} = x_{ij}x_{iab} - x_{jab}$

$\left(a, b \in \{1, \cdots, n\}, \quad \forall$ **Wirtinger triple** (i, j, k) $\right)$



(2) $$x_{i_1 i_2 i_3} \cdot x_{j_1 j_2 j_3} = \frac{1}{2} \begin{vmatrix} x_{i_1 j_1} & x_{i_1 j_2} & x_{i_1 j_3} \\ x_{i_2 j_1} & x_{i_2 j_2} & x_{i_2 j_3} \\ x_{i_3 j_1} & x_{i_3 j_2} & x_{i_3 j_3} \end{vmatrix}$$

$(1 \le i_1 < i_2 < i_3 \le n,\ 1 \le j_1 < j_2 < j_3 \le n)$

(3) $$\begin{vmatrix} 2 & x_{12} & x_{1a} & x_{1b} \\ x_{21} & 2 & x_{2a} & x_{2b} \\ x_{a1} & x_{a2} & 2 & x_{ab} \\ x_{b1} & x_{b2} & x_{ba} & 2 \end{vmatrix} = 0 \quad (3 \le a < b \le n)$$

$\}$

(1) $x_{ka} = x_{ij}x_{ia} - x_{ja}$ **(F2)**, $x_{kab} = x_{ij}x_{iab} - x_{jab}$ **(F3)**

the fundamental relations (F)

(2) $$x_{i_1i_2i_3} \cdot x_{j_1j_2j_3} = \frac{1}{2}\begin{vmatrix} x_{i_1j_1} & x_{i_1j_2} & x_{i_1j_3} \\ x_{i_2j_1} & x_{i_2j_2} & x_{i_2j_3} \\ x_{i_3j_1} & x_{i_3j_2} & x_{i_3j_3} \end{vmatrix}$$

the hexagon relations (H)

(3) $$\begin{vmatrix} 2 & x_{12} & x_{1a} & x_{1b} \\ x_{21} & 2 & x_{2a} & x_{2b} \\ x_{a1} & x_{a2} & 2 & x_{ab} \\ x_{b1} & x_{b2} & x_{ba} & 2 \end{vmatrix} = 0$$

the rectangle relations (R)

NOTE.

- $x_{aa} := 2$, $x_{ba} := x_{ab}$ (symmetric)
- $x_{i_{\sigma(1)}i_{\sigma(2)}i_{\sigma(3)}} := \mathrm{sign}(\sigma)x_{i_1i_2i_3}$ $(\sigma \in \mathcal{S}_3)$ (anti-sym)
  - ➡ $x_{aab} = 0$

Plan of this talk

Section 1: An observation of the main theorem

Section 2: A rought sketch of proof of the main theorem

Section 3: An application to abelian knot contact homology
(Ng's conjecture on abelian knot contact homology)

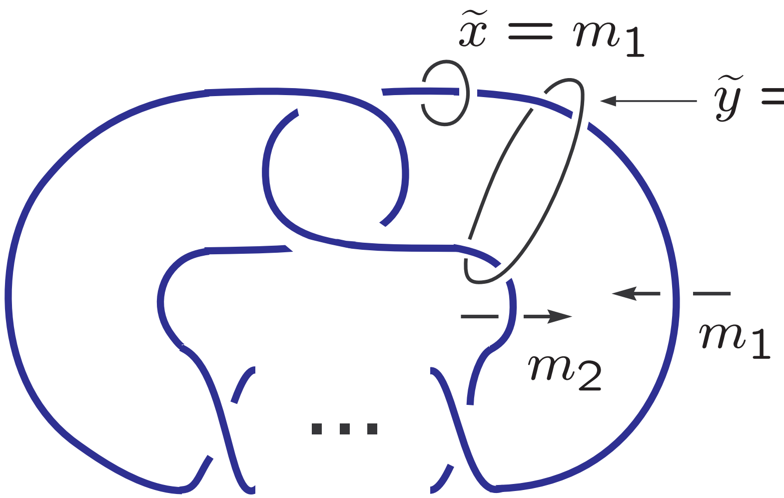
Section 1:

An observation of the main theorem

How to get the fundamental relations **(F)**

Realizing $X(K)$ as an algebraic set (a short review)

$K_m =$



$x := -\mathrm{tr}(\rho(\tilde{x})) = -\mathrm{tr}(m_1)$

$y := -\mathrm{tr}(\rho(\tilde{y})) = -\mathrm{tr}(m_1 m_2^{-1})$

Theorem [Gelca-N (JKTR)], [N (Bull. Korean Math.)]

$$X(K_m) = \left\{ (x, y) \in \mathbb{C}^2 \,\middle|\, (y-2)R_m(x, -y) = 0 \right\}, \text{ where}$$

$$R_m(x, y) := S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)$$

$$S_{n+2}(z) = zS_{n+1}(z) - S_n(z), \; S_1(z) = z, \; S_0(z) = 1.$$

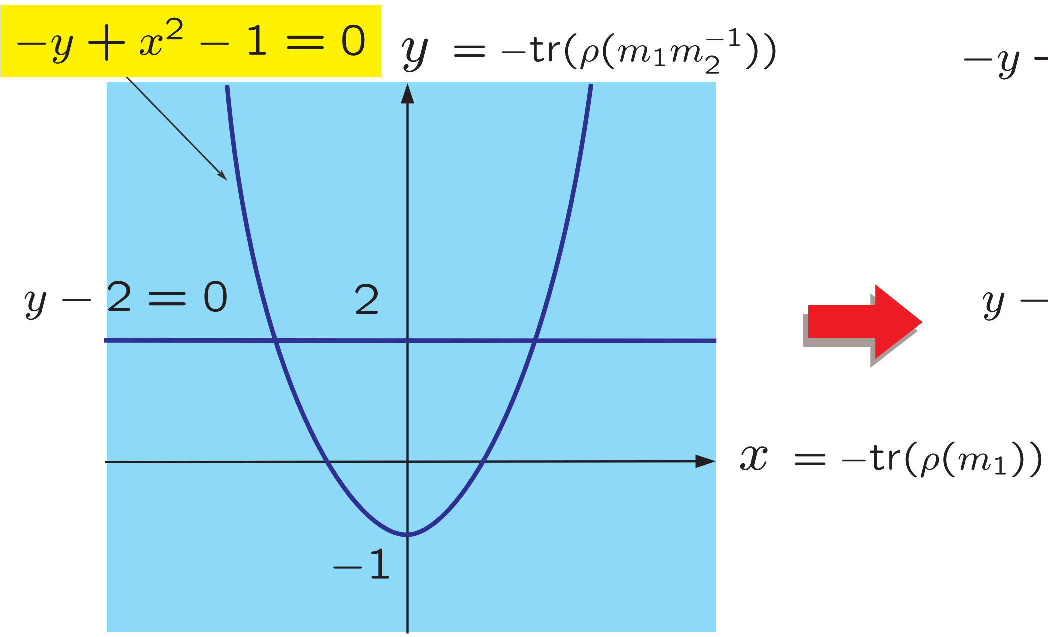
EX.

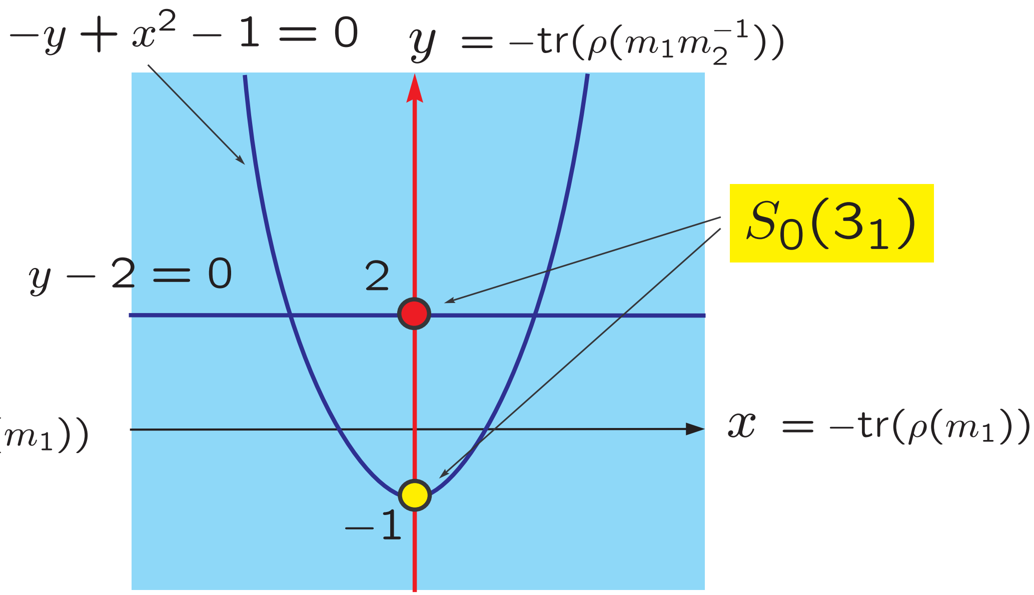
$$R_1(x, -y) = -y + x^2 - 1$$

$$R_2(x, -y) = y^2 - x^2 y + y + x^2 - 1$$

Realizing $S_0(K)$ as an algebraic set (a short review)

$$X(3_1) = \left\{(x,y) \in \mathbb{C}^2 \,\middle|\, (y-2)\left(-y+x^2-1\right) = 0\right\}$$





$X(3_1) \subset \mathbb{C}^2$

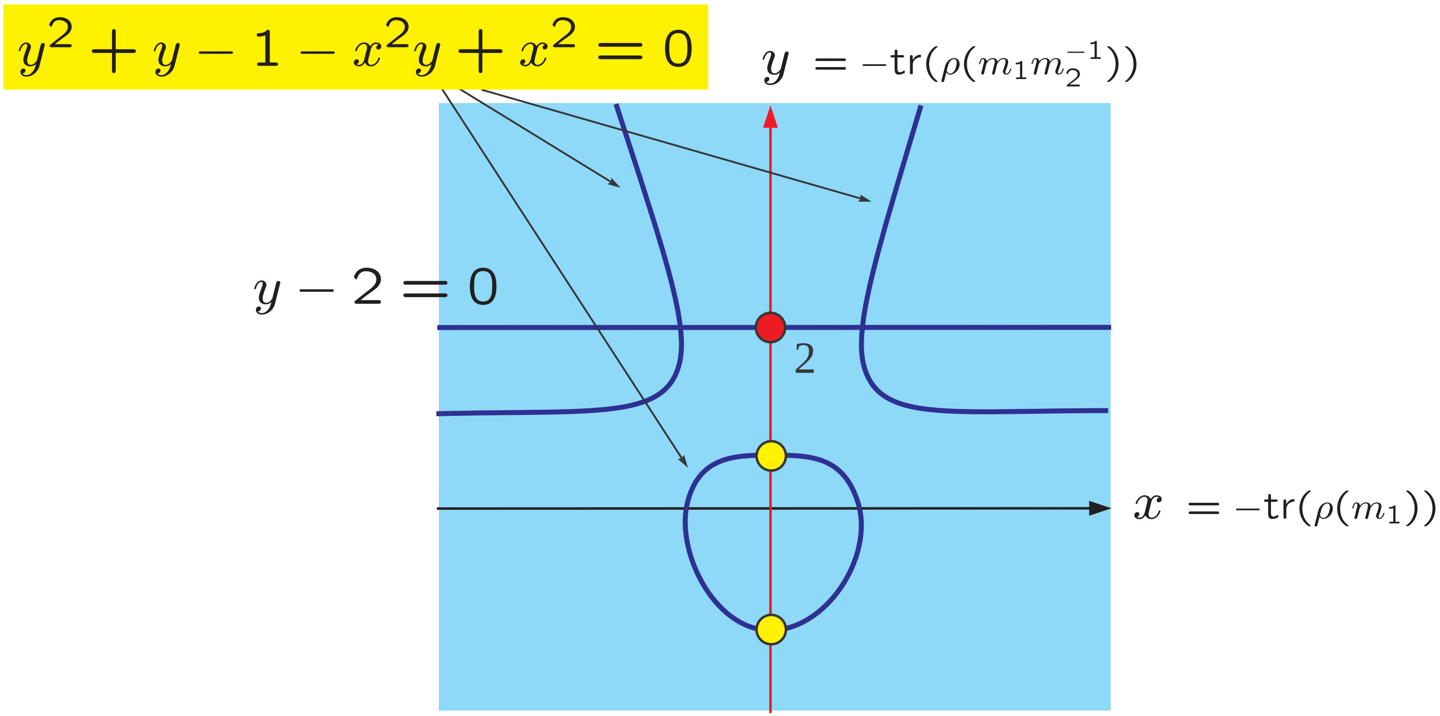
$S_0(3_1) = X(3_1) \cap \{\mathrm{tr}(\rho(m_1)) = 0\}$

Definition ($S_0(K)$ as an algebraic set)

$$S_0(K) := X(K) \cap \{\mathrm{tr}(\rho(\mu)) = 0\}$$ **(w/o multiplicity)**

Realizing $S_0(K)$ as an algebraic set (a short review)

$$X(4_1) = \left\{(x,y) \in \mathbb{C}^2 \,\middle|\, (y-2)\left(y^2+y-1-x^2y+x^2\right)=0\right\}$$

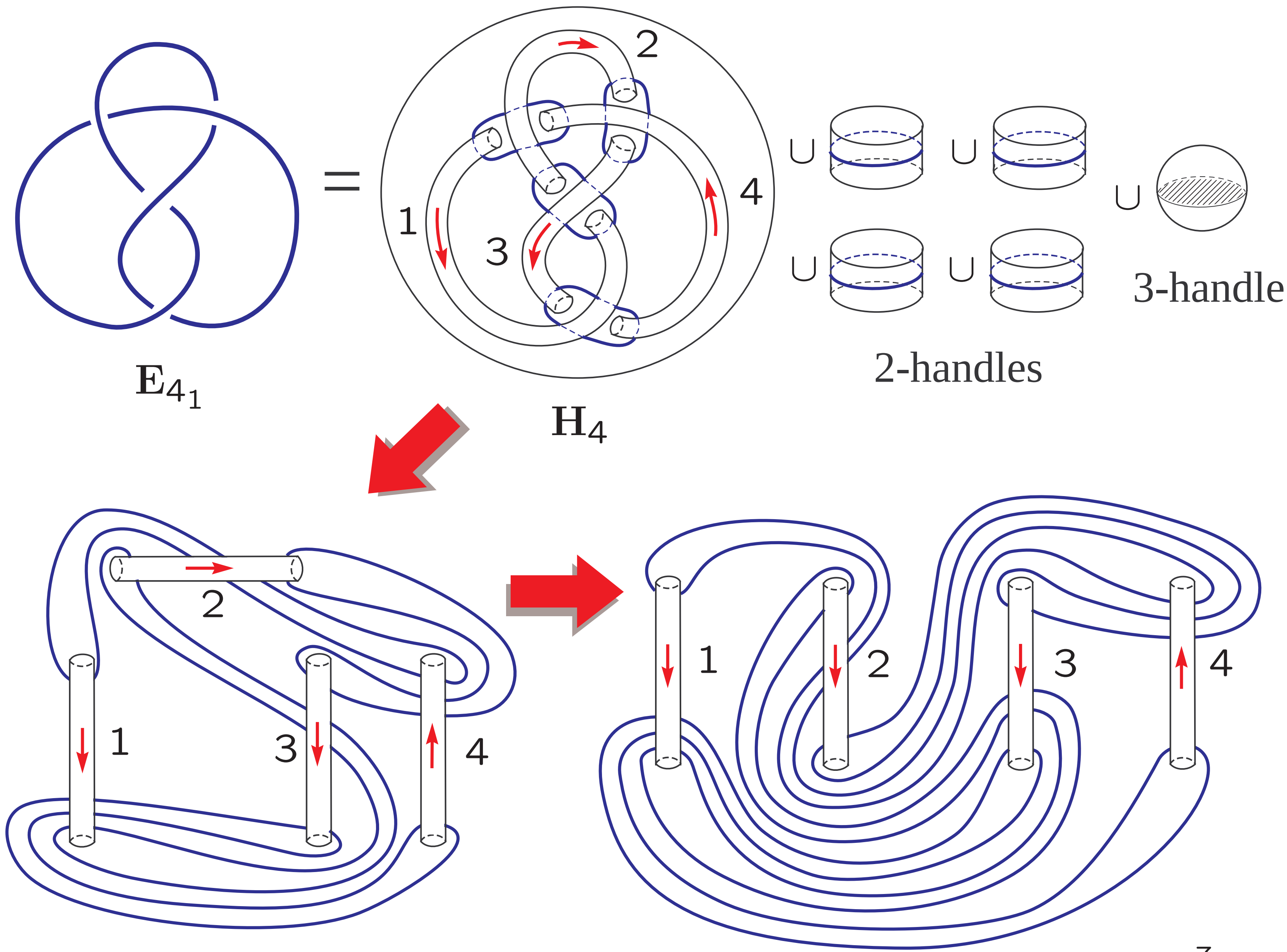


$$S_0(4_1) = X(4_1) \cap \{\mathrm{tr}(\rho(m_1))=0\} = \left\{2, \frac{-1 \pm \sqrt{5}}{2}\right\}$$

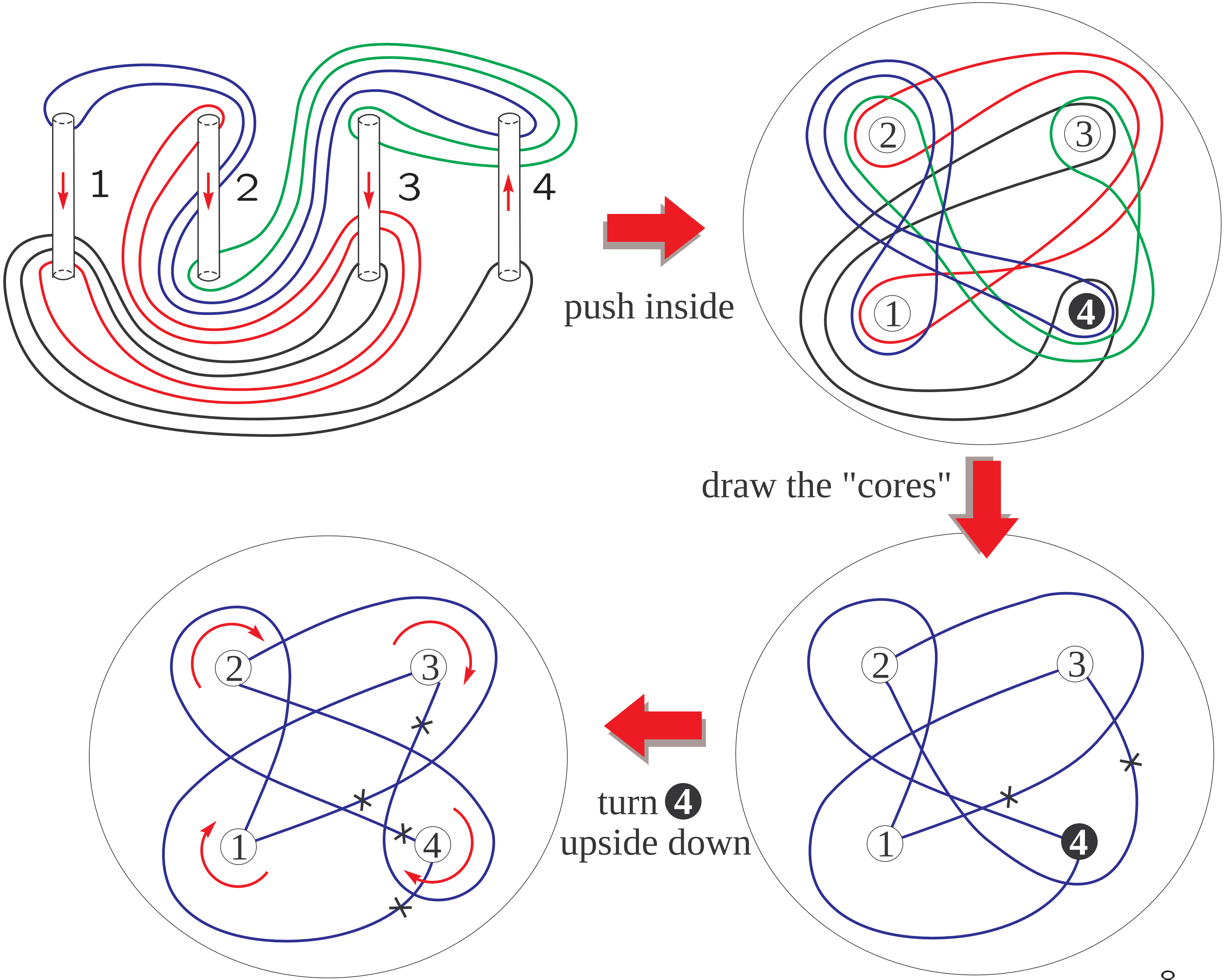
This can be done because we have the defining poly. of $X(K_m)$.

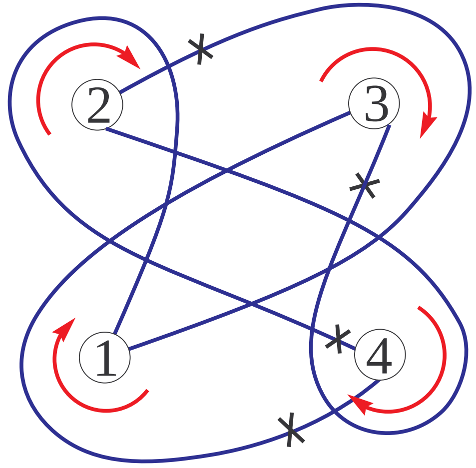
- We want to calculate $S_0(K)$ directly w/o the calculation of $X(K)$.

A decomposition of E_K into H_n and handles



Presenting attaching curves on a punctured disk



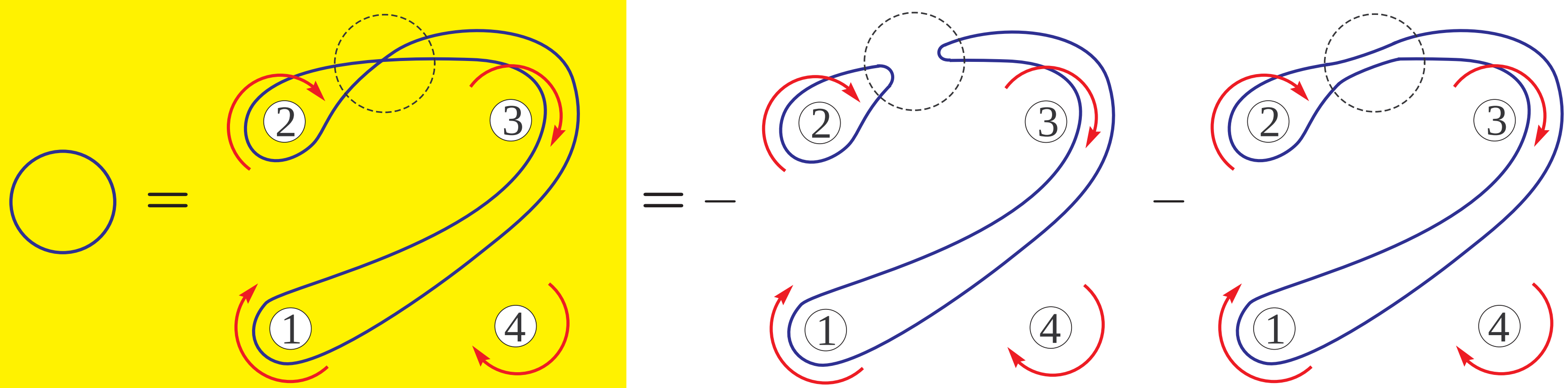


$$\begin{cases} x_{13}x_{23} - x_{12} = 2, \quad x_{13} = x_{23} \\ x_{12}x_{24} - x_{14} = 2, \quad x_{12} = x_{24} \\ x_{13}x_{14} - x_{34} = 2, \quad x_{13} = x_{14} \\ x_{24}x_{34} - x_{23} = 2, \quad x_{23} = x_{24} \end{cases}$$

a part of **(F2)**

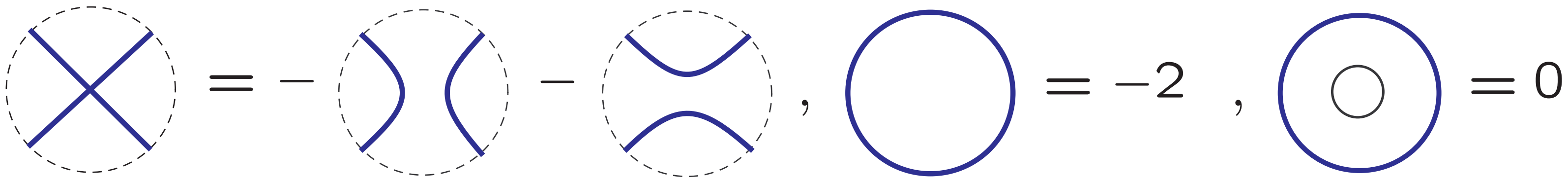
EX. For an arbitrary trace-free character χ_ρ :

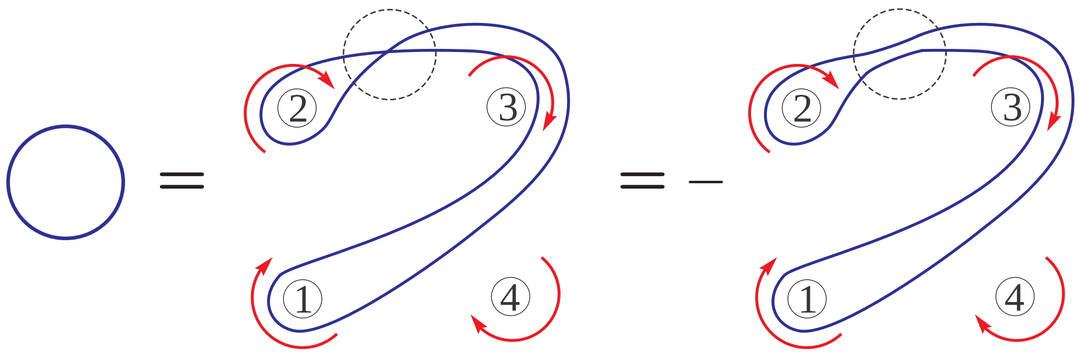
$$1 = m_3 m_1 m_3^{-1} m_2^{-1} \Rightarrow -2 = -\mathrm{tr}\left(\rho(m_3 m_1 m_3^{-1} m_2^{-1})\right)$$



$$-2 = -\mathrm{tr}\left(\rho(m_3 m_1 m_3^{-1})\right) \mathrm{tr}\left(\rho(m_2^{-1})\right) + \mathrm{tr}\left(\rho(m_3 m_1 m_3^{-1} m_2)\right)$$

trace-free Kauffman brakcet skein relation (at $t = -1$)





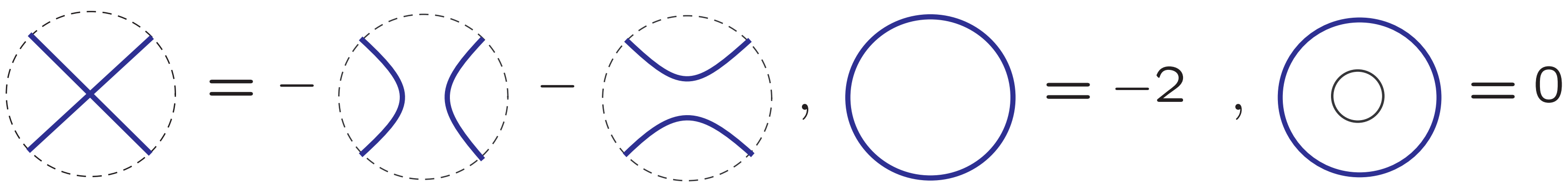
$$-2 = -\mathrm{tr}(\rho(m_3 m_1 m_3^{-1} m_2^{-1})) = \mathrm{tr}\left(\rho(m_3 m_1 m_3^{-1} m_2)\right)$$

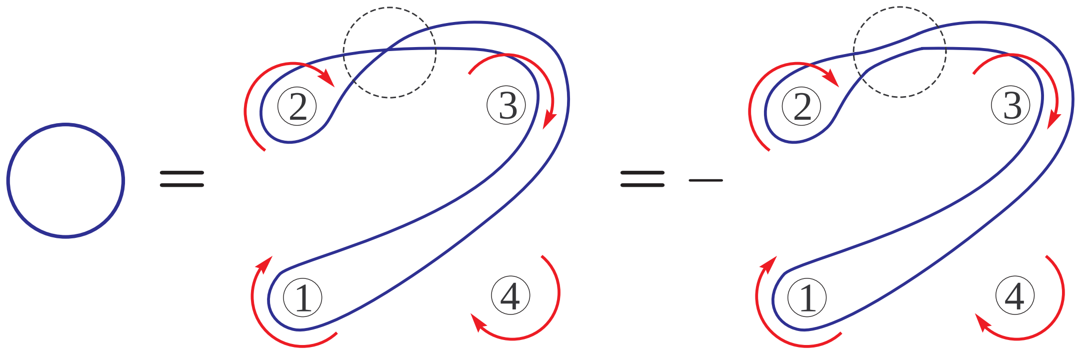
in general

a loop γ in $\mathbf{E}_K \Leftrightarrow -t_{[\gamma]}(\rho) := -\mathrm{tr}(\rho([\gamma]))$ $(\chi_\rho \in S_0(K))$

trace-free KBSR $\Leftrightarrow$ trace-identity

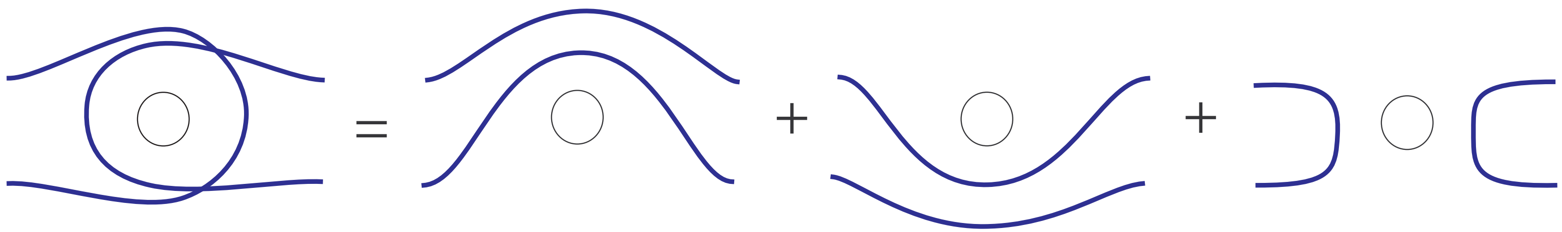
trace-free Kauffman brakcet skein relation (at $t=-1$)



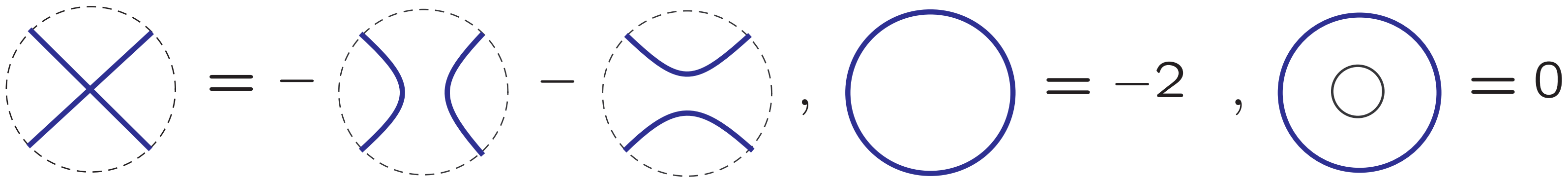


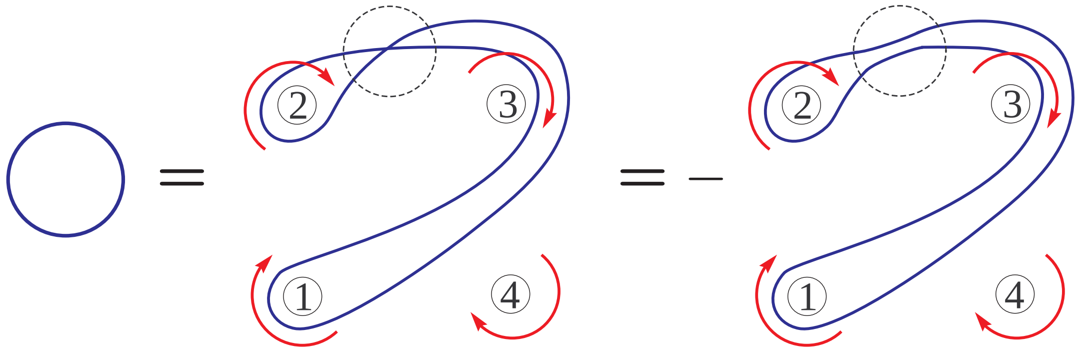
$$-2 = -\mathrm{tr}(\rho(m_3 m_1 m_3^{-1} m_2^{-1})) = \mathrm{tr}\left(\rho(m_3 m_1 m_3^{-1} m_2)\right)$$

Here, by skein relation, we obtain

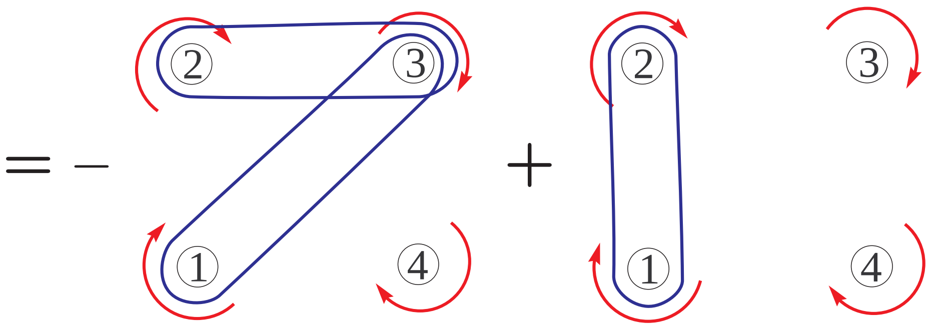


trace-free Kauffman brakcet skein relation (at $t = -1$)



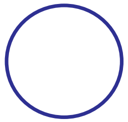
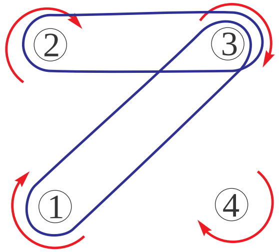


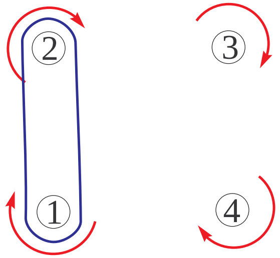
$$-2 = -\mathrm{tr}(\rho(m_3 m_1 m_3^{-1} m_2^{-1})) = \mathrm{tr}\left(\rho(m_3 m_1 m_3^{-1} m_2)\right)$$



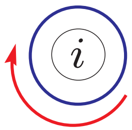
$$= \mathrm{tr}\left(\rho(m_1 m_3)\right) \mathrm{tr}\left(\rho(m_2 m_3^{-1})\right) + \mathrm{tr}\left(\rho(m_1 m_3^2 m_2^{-1})\right) \cdots$$ continued

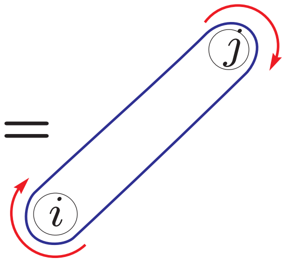
$$
\begin{aligned}
-2 = \cdots &= \mathrm{tr}(\rho(m_1m_3))\mathrm{tr}\left(\rho(m_2m_3^{-1})\right) + \mathrm{tr}(\rho(m_2^{-1}m_1m_3^2)) \\
&= \mathrm{tr}\,(\rho(m_1m_3))\,\{\mathrm{tr}(\rho(m_2))\mathrm{tr}(\rho(m_3)) - \mathrm{tr}\,(\rho(m_2m_3))\} \\
&\quad + \mathrm{tr}(\rho(m_2^{-1}m_1m_3))\mathrm{tr}(\rho(m_3)) - \mathrm{tr}\left(\rho(m_2^{-1}m_1)\right) \\
&= -\mathrm{tr}(\rho(m_1m_3))\mathrm{tr}(\rho(m_2m_3)) + \mathrm{tr}(\rho(m_1m_2))
\end{aligned}
$$

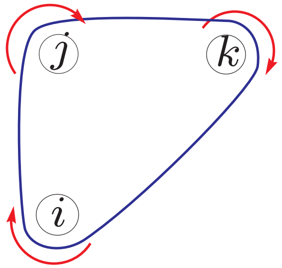
 $= \cdots = -$ 

$+$ 

Set the followings:

$x_i :=$ 

$= 0$, $x_{ij} :=$ 

, $x_{ijk} :=$ 

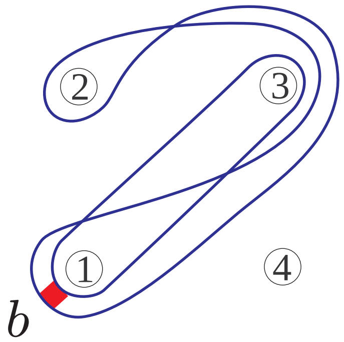
$\boxed{-\mathrm{tr}(\rho(m_i))} = 0$ $\quad \boxed{-\mathrm{tr}(\rho(m_im_j))}$ $\quad \boxed{-\mathrm{tr}(\rho(m_1m_jm_k))}$

➡ $-2 = -x_{13}x_{23} + x_{12}$ (the desired equation)

Also we can get: $x_{13} = x_{23}$, $x_{12} = x_{24}$, $x_{13} = x_{14}$, $x_{23} = x_{24}$

EX.

$x_{13} = sl_b(x_{13}) :=$



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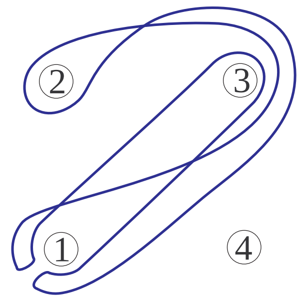
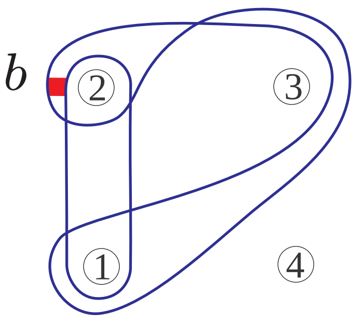


$= x_{23}$

Other relations: $x_{12} = x_{13}^2 - 2$, $x_{24} = x_{13}^2 - 2$, $x_{34} = x_{13}^2 - 2$

EX.

$x_{12} = sl_b(x_{12}) =$



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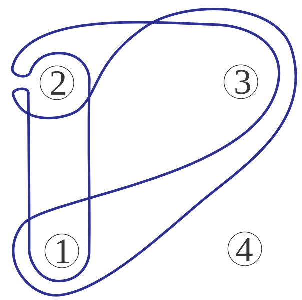

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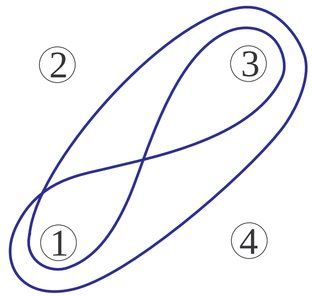

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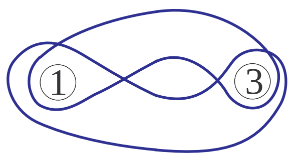

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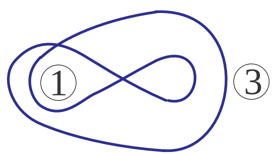

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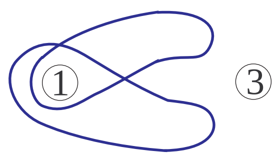


$= x_{13}^2 - 2$

Continuing this work, we obtain all the **(F2):**

$$\left\{\begin{array}{c} x_{13}x_{23}-x_{12}=2,\ x_{12}x_{24}-x_{14}=2,\ x_{13}x_{14}-x_{34}=2,\ x_{24}x_{34}-x_{23}=2 \\ x_{13}=x_{23},\ \boxed{x_{12}=x_{24}},\ \boxed{x_{13}=x_{14}},\ x_{23}=x_{24} \\ \boxed{x_{12}=x_{13}^2-2},\ x_{24}=x_{13}^2-2,\ x_{34}=x_{13}^2-2 \\ \boxed{x_{13}=x_{14}x_{24}-x_{12}},\ x_{14}=x_{23}x_{34}-x_{24} \\ x_{13}=x_{23}x_{24}-x_{34},\ x_{23}=x_{12}x_{14}-x_{24} \end{array}\right\}$$

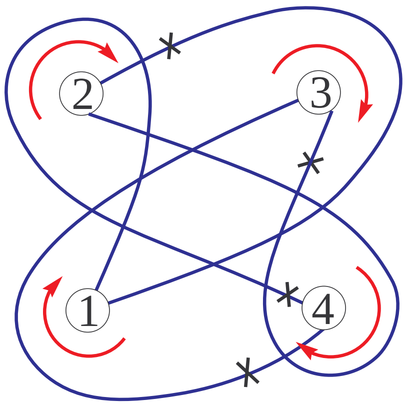
$$F_2(4_1) := \left\{ (x_{12},\cdots,x_{45}) \in \mathbb{C}^{10} \;\middle|\; \begin{array}{l} x_{ka}=x_{ik}x_{ia}-x_{ja} \quad \textbf{(F2)} \\ \text{for any Wirtinger triple } (i,j,k) \\ a \in \{1,\cdots,4\} \ (x_{aa}=2) \end{array} \right\}$$

So $F_2(4_1)$ is parametrized by x_{13} and

$$x_{13}=x_{14}x_{24}-x_{12} \Rightarrow x_{13}=x_{13}(x_{13}^2-2)-(x_{13}^2-2)$$

$$\Rightarrow (x_{13}-2)(x_{13}^2+x_{13}-1)=0$$

Hence we get $F_2(4_1) = \left\{2, \frac{-1\pm\sqrt{5}}{2}\right\} = S_0(4_1)$.



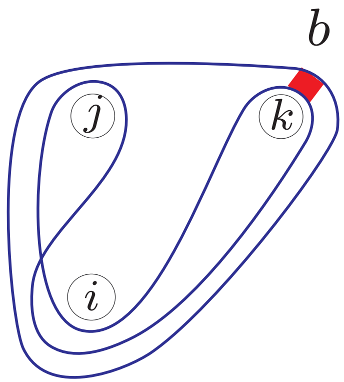
$$\begin{cases} x_{123} = 0,\ x_{124} = 0 \\ x_{134} = 0,\ x_{234} = 0 \end{cases}$$

(F3) becomes trivial !

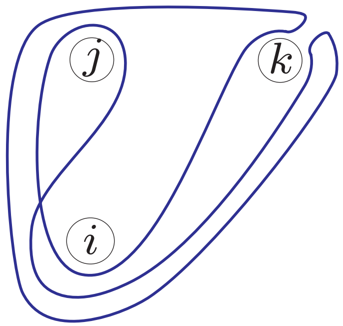
Indeed, we can check this like...

EX.

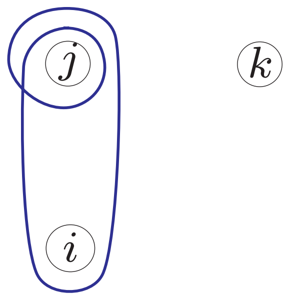
$$x_{ijk} = sl_b(x_{ijk}) =$$



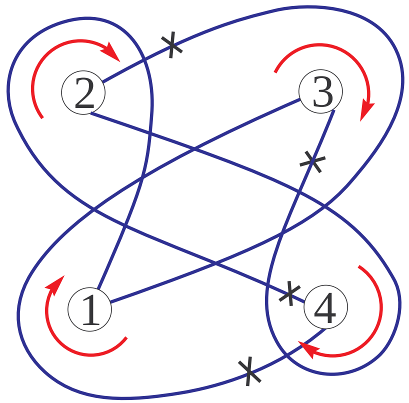
$$=$$



$$=$$



$$= -x_i x_{ij} - x_i = 0$$

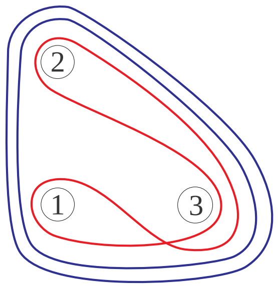


$$\begin{cases} x_{123} = 0,\ x_{124} = 0 \\ x_{134} = 0,\ x_{234} = 0 \\ \textbf{(F3) becomes trivial !} \end{cases}$$

Or we can check this by **the hexagon relation**:

(H) $$x_{i_1 i_2 i_3} \cdot x_{j_1 j_2 j_3} = \frac{1}{2} \begin{vmatrix} x_{i_1 j_1} & x_{i_1 j_2} & x_{i_1 j_3} \\ x_{i_2 j_1} & x_{i_2 j_2} & x_{i_2 j_3} \\ x_{i_3 j_1} & x_{i_3 j_2} & x_{i_3 j_3} \end{vmatrix} \qquad \begin{matrix} (1 \le i_1 < i_2 < i_3 \le 4) \\ (1 \le j_1 < j_2 < j_3 \le 4) \end{matrix}$$

EX.



$$= x_{123}^2 = \frac{1}{2} \begin{vmatrix} 2 & x_{12} & x_{13} \\ x_{21} & 2 & x_{23} \\ x_{31} & x_{32} & 2 \end{vmatrix} = x_{12}x_{13}x_{23} - x_{12}^2 - x_{13}^2 - x_{23}^2 + 4$$

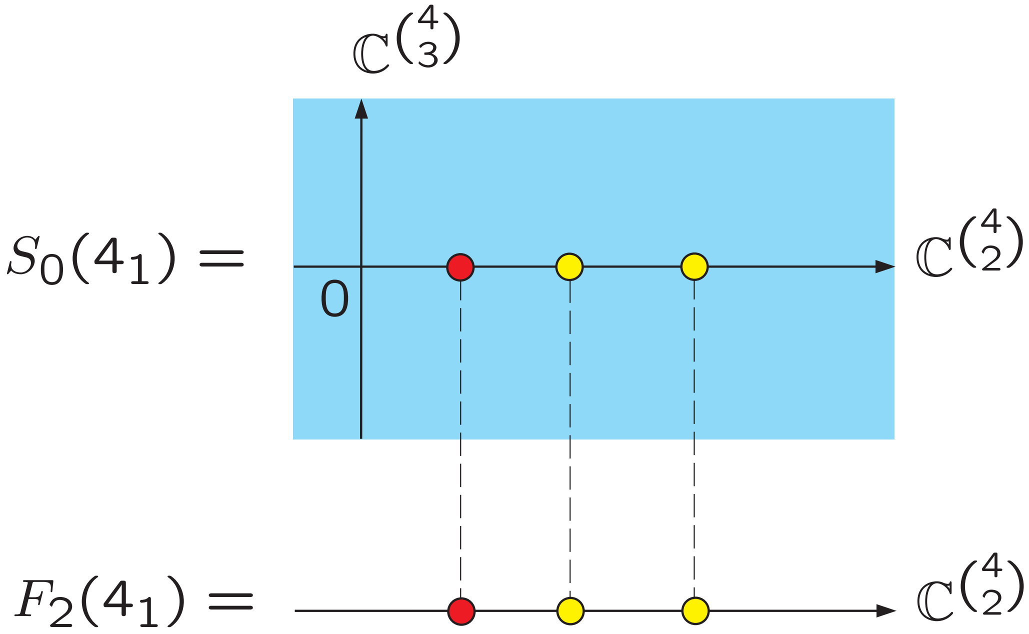
$$= (x_{13}^2 - 2)x_{13}^2 - (x_{13}^2 - 2)^2 - x_{13}^2 - x_{13}^2 + 4 = 0$$

Then actually all the point in $F_2(4_1)$ satisfy **(H)** and

$$\textbf{(R)} \quad \begin{vmatrix} 2 & x_{12} & x_{1a} & x_{1b} \\ x_{21} & 2 & x_{2a} & x_{2b} \\ x_{a1} & x_{a2} & 2 & x_{ab} \\ x_{b1} & x_{b2} & x_{ba} & 2 \end{vmatrix} = 0 \qquad (3 \le a < b \le 4)$$

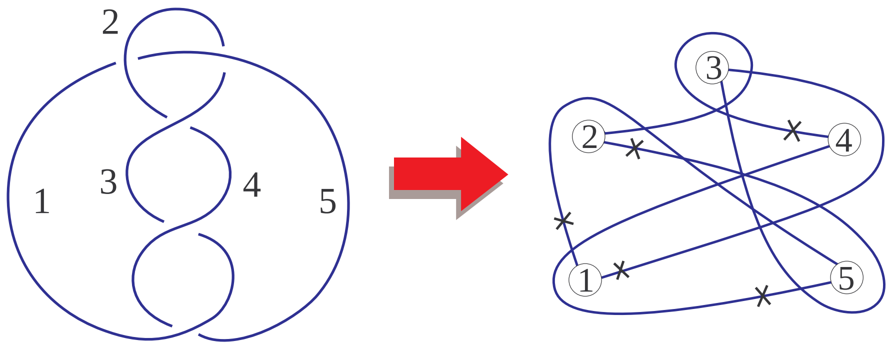
Hence all points in $F_2(4_1)$ **lift** to $S_0(4_1)$.

$S_0(4_1) = F_2(4_1)$ and thus the main theorem holds for 4_1.



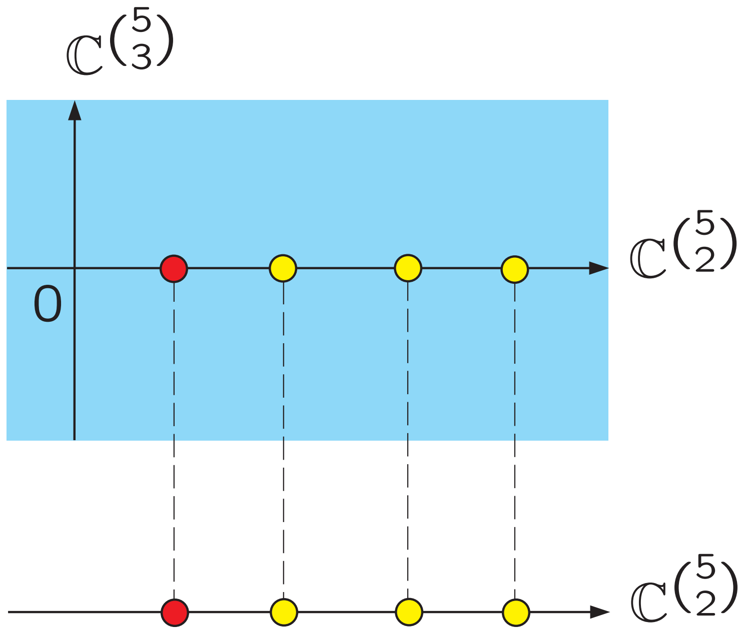
Calculate $F_2(4_1)$ first, check the liftability second.

The case of $K = 5_2$



$$S_0(5_2) = F_2(5_2) = \left\{ x_{14} \in \mathbb{C} \,\middle|\, (x_{14} - 2)\left(x_{14}^3 + x_{14}^2 - 2x_{14} - 1\right) = 0 \right\}$$

$S_0(5_2) =$



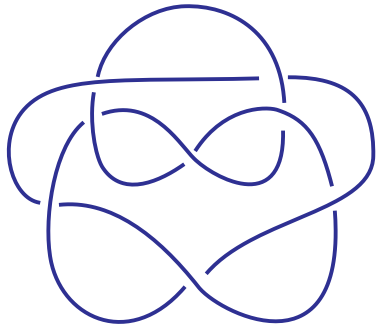
$F_2(5_2) =$

All points in $F_2(5_2)$

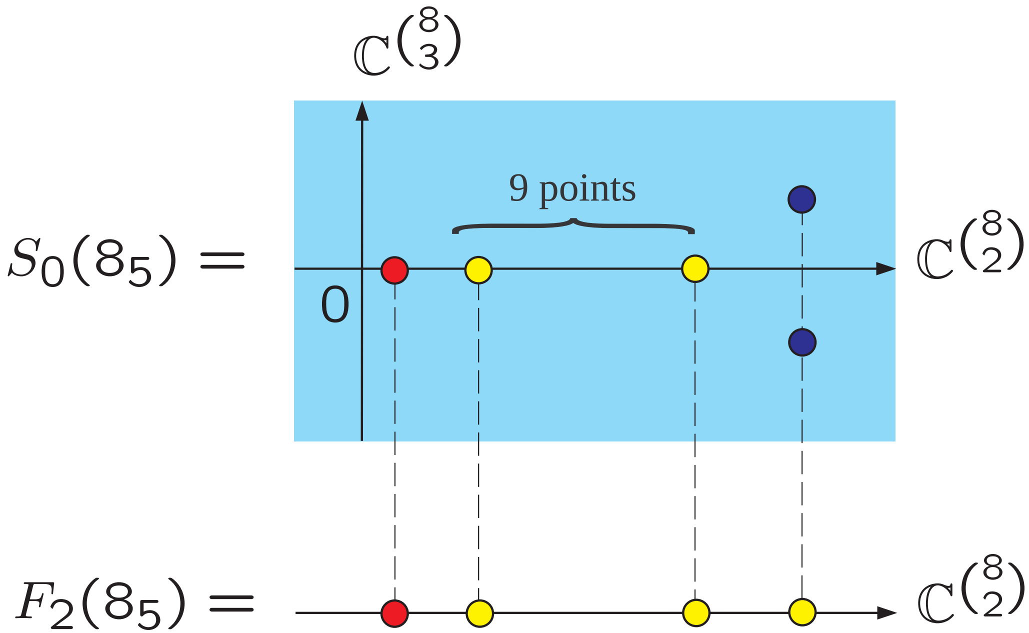
also **lift** to $S_0(5_2)$

Calculate $F_2(5_2)$ first, check the liftability second.

The case of $K = 8_5$



$F_2(8_5) = \{11 \text{ points}\}$, $S_0(8_5) = \{12 \text{ points}\}$



All points in $F_2(8_5)$ also **lift** to $S_0(8_5)$

Calculate $F_2(8_5)$ first, check the liftability second.

- Can any point of $F_2(K)$ lift to $S_0(K)$?

We look into this property.

Section 2:

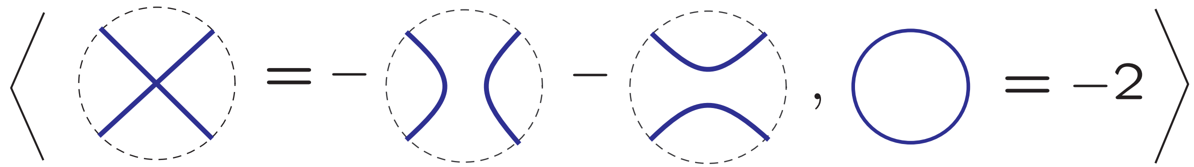
A rough sketch of proof of the main theorem

On liftability problem of $F_2(K)$ to $S_0(K)$

In a general setting, the following holds:

- $\mathbf{M}$: a compact orientable 3-manifold

- $\mathcal{K}_{-1}(\mathrm{M}) := \dfrac{\mathbb{C}\,[\text{loops in } \mathrm{M}]}{\langle \ \cdots \ \rangle}$



Kauffman bracket skein algebra (KBSA) at $t = -1$ of M

- $\chi(\mathrm{M}) := \mathbb{C}[x_1, \cdots, x_N] / \sqrt{\langle \text{polynomials vanishing on } X(\mathrm{M}) \rangle}$

coordinate ring of the character variety $X(\mathrm{M})$

Theorem [Bullock], [Przytycki-Sikora]

$\exists$ a surjective homomrphism $\varphi : \mathcal{K}_{-1}(\mathrm{M}) \to \chi(\mathrm{M})$ defined by $\varphi(\gamma) := -t_{[\gamma]}$ (a loop $\gamma \in M$). Ker$(\varphi) = \sqrt{0}$ (the nilradical).

The sliding ideal S_K and the fundamental ideal F_K

"van-Kampen's theorem-like theorem" [Przytycki]

$$\mathcal{K}_{-1,TF}(\mathbf{E}_K) = \frac{\mathcal{K}_{-1.TF}(\mathbf{H}_n)}{\left\langle z - sl_b(z) \,\middle|\, z\text{: any loop in } \mathcal{K}_{-1,TF}(\mathbf{H}_n)\right\rangle}$$

We define two ideals in $\mathcal{K}_{-1,TF}(\mathbf{H}_n)$:

sliding ideal for K

$$S_K := \left\langle z - sl_b(z) \,\middle|\, z\text{: any loop in } \mathcal{K}_{-1,TF}(\mathbf{H}_n)\right\rangle$$

fundamental ideal of K

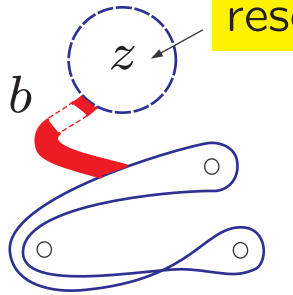
$$F_K := \left\langle \begin{matrix} x_{ka} - x_{ij}x_{ia} + x_{ja} \ \textbf{(F2)} \\ x_{kab} - x_{ij}x_{iab} + x_{jab} \ \textbf{(F3)} \end{matrix} \,\middle|\, \begin{matrix} (i,j,k)\text{: any Wirtinger triangle} \\ (a, b \in \{1, \cdots, n\}) \end{matrix} \right\rangle$$

By definition, $S_K \supset F_K$.

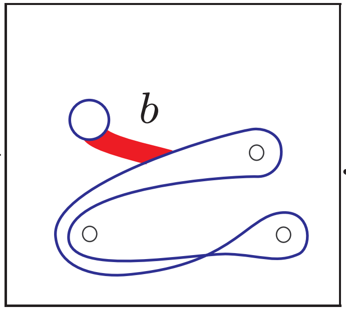
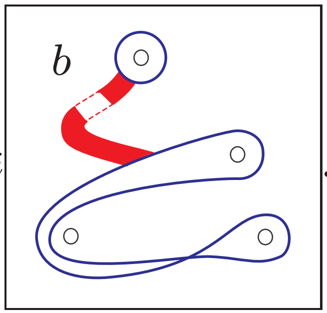
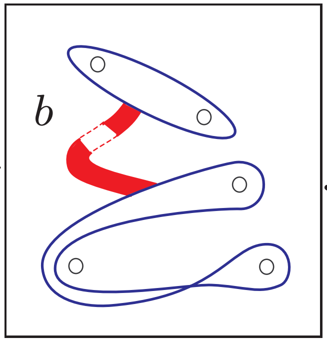
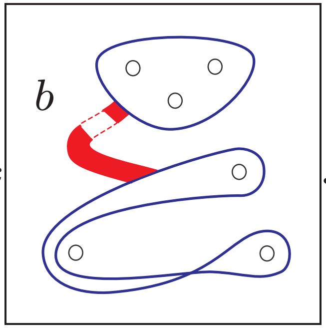
We show that they coincides, i.e., $S_K = F_K$.

Step1 Take an aribitrary band b for a handle sliding sl_b associated with b along an slope.

$$z - sl_b(z) = z -$$



$$= \boxed{\bigcirc}\, f - \Sigma_i \boxed{x_i}\, f_i - \Sigma_{i,j} \boxed{x_{ij}}\, f_{ij} - \Sigma_{i,j,k} \boxed{x_{ijk}}\, f_{ijk}$$ (resolution of z by skein relations)

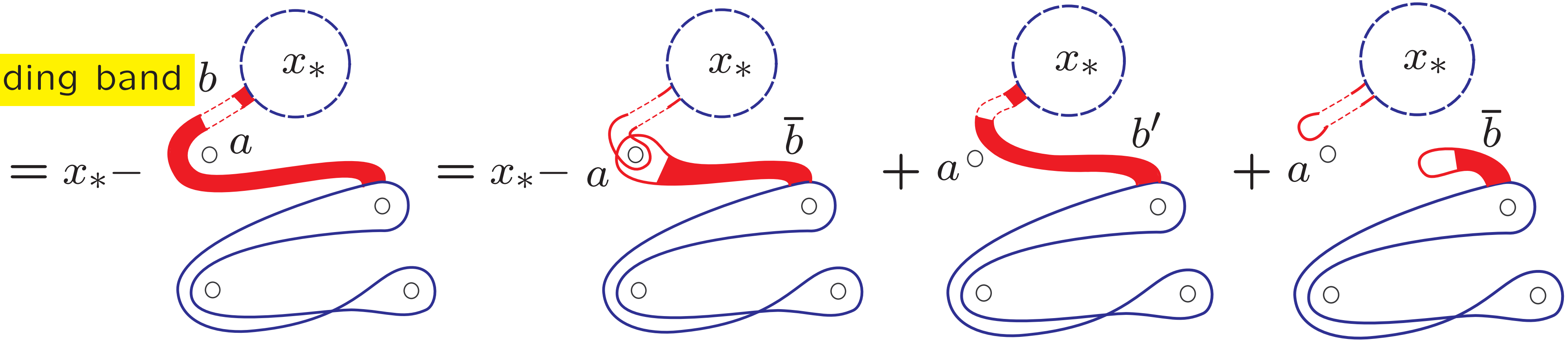
$-$  $f - \Sigma_i$  $f_i - \Sigma_{i,j}$  $f_{ij} - \Sigma_{i,j,k}$  f_{ijk}

$$= (\bigcirc - sl_b(\bigcirc))f + \Sigma_i(x_i - sl_b(x_i))f_i + \Sigma_{i,j}(x_{ij} - sl_b(x_{ij}))f_{ij} + \Sigma_{i,j,k}(x_{ijk} - sl_b(x_{ijk}))f_{ijk}$$

$\Rightarrow$ $S_K = \langle \bigcirc - sl_b(\bigcirc),\ sl_b(x_i),\ x_{ij} - sl_b(x_{ij}),\ x_{ijk} - sl_b(x_{ijk}) \mid b\text{: any band} \rangle$

Step2 Consider sl_b for $x_* \in \{\bigcirc,\ x_i(=0),\ x_{ij},\ x_{ijk}\}$.

If the band b is **"winding"**, then we can actually "straighten" the band $\widetilde{b}$ by the skein relation: $x_* - sl_b(x_*)$



$= x_* - \ldots = x_* - a \ldots + a \ldots + a \ldots$

$$= (x_* \# x_a)\left(-sl_{\bar{b}}(x_a)\right) - \left(x_* - sl_{b'}(x_*)\right) - x_*\left(-2 - sl_{\bar{b}}(\bigcirc)\right)$$

$$= \Sigma\left(-sl_{\bar{b}}(x_a)\right) f + \Sigma\left(x_* - sl_{\bar{b}}(x_*)\right) g + \Sigma\left(-2 - sl_{\bar{b}}(\bigcirc)\right) h$$

$$S_K = \left\langle \begin{matrix} \bigcirc - sl_{\bar{b}}(\bigcirc),\ sl_{\bar{b}}(x_i) \\ x_{ij} - sl_{\bar{b}}(x_{ij}),\ x_{ijk} - sl_{\bar{b}}(x_{ijk}) \end{matrix} \ \middle|\ \bar{b}\text{: a non-winding band} \right\rangle$$

the fundamental relations (F) $\Rightarrow$ $S_K = F_K$

$$\mathcal{K}_{-1,TF}(\mathbf{E}_K) = \cfrac{\mathcal{K}_{-1,TF}(\mathbf{H}_n)}{F_K = \left\langle \begin{array}{l|l} x_{ka} - x_{ik}x_{ia} + x_{ja} & (i,j,k)\text{: any Wirtinger triple} \\ x_{kab} - x_{ik}x_{iab} + x_{jab} & a, b \in \{1, \cdots, n\} \end{array} \right\rangle}$$

$$\mathcal{K}_{-1,TF}(\mathbf{H}_n)/\sqrt{0} = \cfrac{\mathbb{C}[x_{ij}; x_{ijk} \mid 1 \le i < j < k \le n]}{\sqrt{\left\langle (\mathbf{H}), (\mathbf{R}), \begin{vmatrix} 2 & x_{12} & x_{13} & x_{1a} \\ x_{21} & 2 & x_{23} & x_{2a} \\ x_{31} & x_{32} & 2 & x_{3a} \\ x_{b1} & x_{b2} & x_{b3} & x_{ab} \end{vmatrix} \ (4 \le a < b \le n) \right\rangle}}$$

[González=Acuña-Montesinos]

by taking $(1, 2, 3)$ as a Wirtinger triple, the extra relation

$$= x_{ab} \begin{vmatrix} 2 & x_{12} & x_{13} \\ x_{21} & 2 & x_{23} \\ x_{31} & x_{32} & 2 \end{vmatrix} - x_{b3} \begin{vmatrix} 2 & x_{12} & x_{1a} \\ x_{21} & 2 & x_{2a} \\ x_{31} & x_{32} & x_{3a} \end{vmatrix} + x_{b2} \begin{vmatrix} 2 & x_{13} & x_{1a} \\ x_{21} & x_{2a} & x_{2a} \\ x_{31} & 2 & x_{3a} \end{vmatrix} - x_{b1} \begin{vmatrix} x_{12} & x_{13} & x_{1a} \\ 2 & x_{23} & x_{2a} \\ x_{32} & 2 & x_{3a} \end{vmatrix}$$

$$= x_{ab}x_{123}^2 - x_{b3}x_{123}x_{12a} + x_{b2}x_{123}x_{13a} - x_{b1}x_{123}x_{23a}$$

$$= x_{123}(x_{ab}x_{123} - x_{b3}x_{12a} + x_{b2}x_{13a} - x_{b1}x_{23a}) = 0$$

trivial !

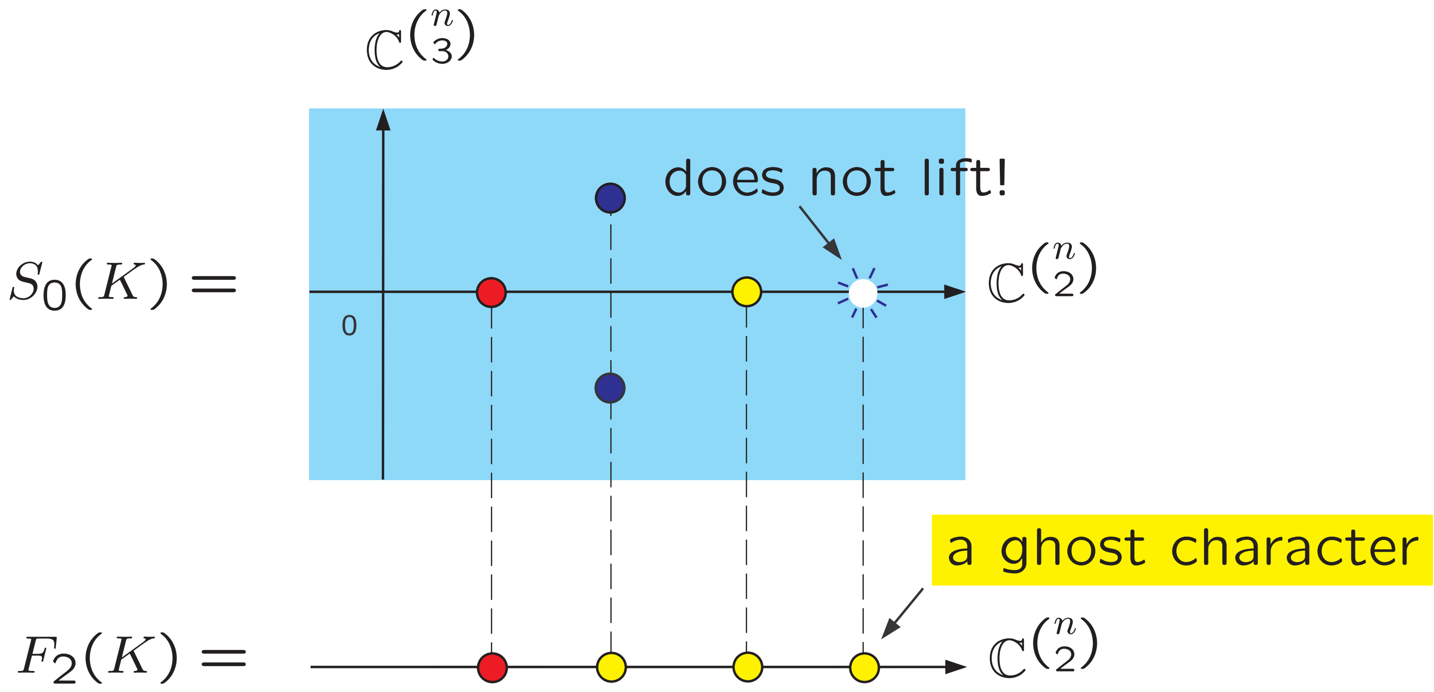
Main theorem (but $(1, 2, 3)$ should be a Wirtinger triple)

Liftability problem of $F_2(K)$: ghost characters

- Can any point of $F_2(K)$ lift to $S_0(K)$?

Definition (ghost characters)

If $\exists$ a point in $F_2(K)$ which does not lift to $S_0(K)$, then we call it **a ghost character**.



- When do the ghost characters appear?

- For $(x_{ab}) \in F_2(K)$, a point $(x_{ab}; x_{pqr})$ satisfying **(H)** in fact, satisfies **(F3)** $x_{kab} = x_{ij}x_{iab} - x_{jab}$, because...

(i) if all $x_{pqr} = 0$, then **(F3)** is trivial.

(ii) if $\exists$ a coordinate $x_{stu} \neq 0$, then

$$x_{stu}(\mathrm{LHS}) = x_{stu}x_{kab} = \frac{1}{2}\begin{vmatrix} x_{sk} & x_{sa} & x_{sb} \\ x_{tk} & x_{ta} & x_{tb} \\ x_{uk} & x_{ua} & x_{ub} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_{ij}x_{si} - x_{sj} & x_{sa} & x_{sb} \\ x_{ij}x_{ti} - x_{tj} & x_{ta} & x_{tb} \\ x_{ij}x_{ui} - x_{uj} & x_{ua} & x_{ub} \end{vmatrix}$$

$$= x_{ij}\frac{1}{2}\begin{vmatrix} x_{si} & x_{sa} & x_{sb} \\ x_{ti} & x_{ta} & x_{tb} \\ x_{ui} & x_{ua} & x_{ub} \end{vmatrix} - \frac{1}{2}\begin{vmatrix} x_{sj} & x_{sa} & x_{sb} \\ x_{tj} & x_{ta} & x_{tb} \\ x_{uj} & x_{ua} & x_{ub} \end{vmatrix}$$

$$= x_{ij}x_{stu}x_{iab} - x_{stu}x_{jab} = x_{stu}(\mathrm{RHS})$$

⇒ (LHS)=(RHS)

- Hence, actually, we do not need **(F3)** if we assume **(H)**.

▶ For $(x_{ab}) \in F_2(K)$, a point $(x_{ab}; x_{pqr})$ satisfying **(R)** lifts to $S_0(K)$:

EX. $$x_{124} = \pm\frac{1}{2}\sqrt{\begin{vmatrix} 2 & x_{12} & x_{14} \\ x_{21} & 2 & x_{24} \\ x_{41} & x_{42} & 2 \end{vmatrix}},\quad x_{125} = \pm\frac{1}{2}\sqrt{\begin{vmatrix} 2 & x_{12} & x_{15} \\ x_{21} & 2 & x_{25} \\ x_{51} & x_{52} & 2 \end{vmatrix}}$$

So **(H)** determines x_{124}, x_{125} up to $\pm$

$$x_{124}^2 x_{125}^2 = \cdots \textbf{(R)} \cdots = x_{124}x_{125} \cdot \frac{1}{4}\begin{vmatrix} x_{21} & 2 & x_{24} & 0 \\ 2 & x_{12} & x_{14} & 2 \\ x_{21} & 2 & x_{24} & 2 \\ x_{51} & x_{52} & x_{54} & x_{25} \end{vmatrix}$$

$$= x_{124}x_{125} \cdot \frac{1}{2}\begin{vmatrix} x_{11} & x_{12} & x_{14} \\ x_{21} & x_{22} & x_{24} \\ x_{51} & x_{52} & x_{54} \end{vmatrix} = (x_{124}x_{125})^2$$

So **the rectangle relations (R)** give **an obstruction of liftability** of $F_2(K)$ to $S_0(K)$.

Section 3:

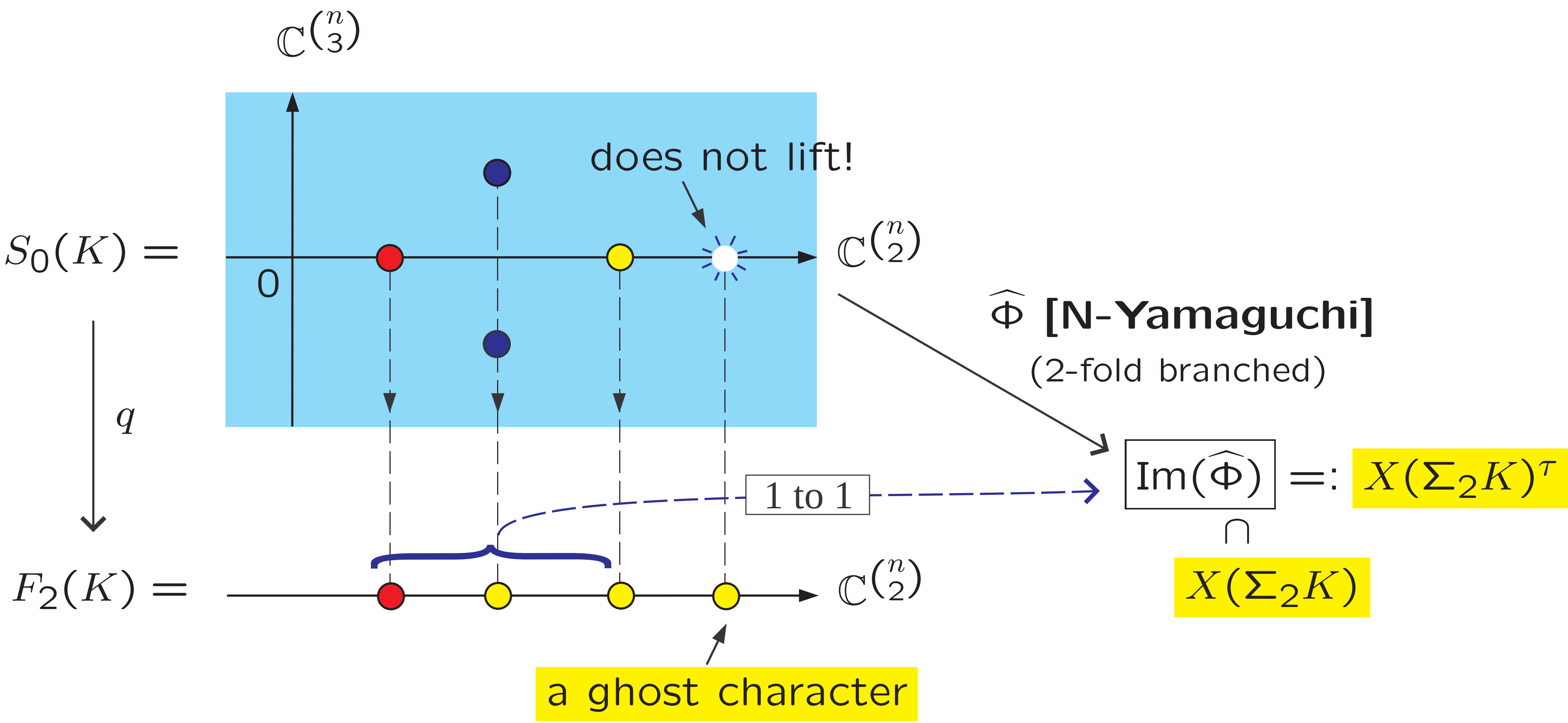
An application to abelian knot contact homology

Ng's conjecture on abelian knot contact homology

(heavily rely on **[N-Yamaguchi]**)

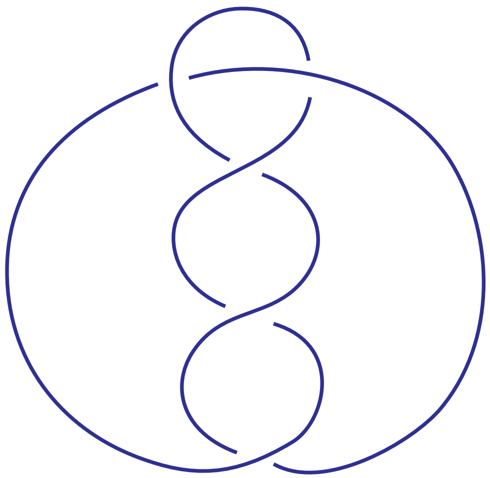
A landscape around $S_0(K)$

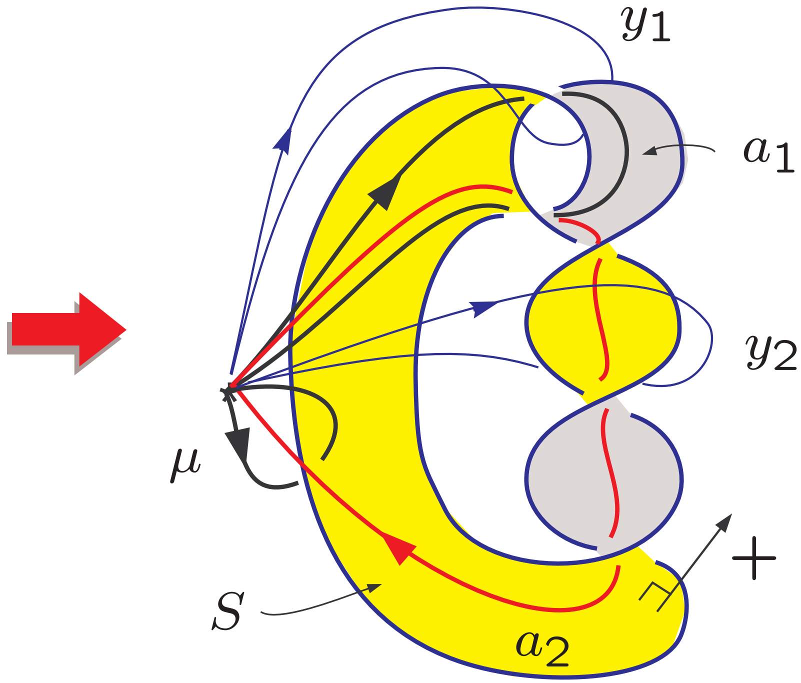
- $\Sigma_2 K$: the 2-fold branched cover of $\mathbf{S}^3$ along K



(**[N-Yamaguchi]** *On the geometry of the slice of trace-free* $\mathrm{SL}_2(\mathbb{C})$*-characters of a knot group*, Mathematische Annalen (DOI: 10.1007/s00208-011-0754-0))

EX. ▶ Take a Seifert surface S via the Seifert algorithm.



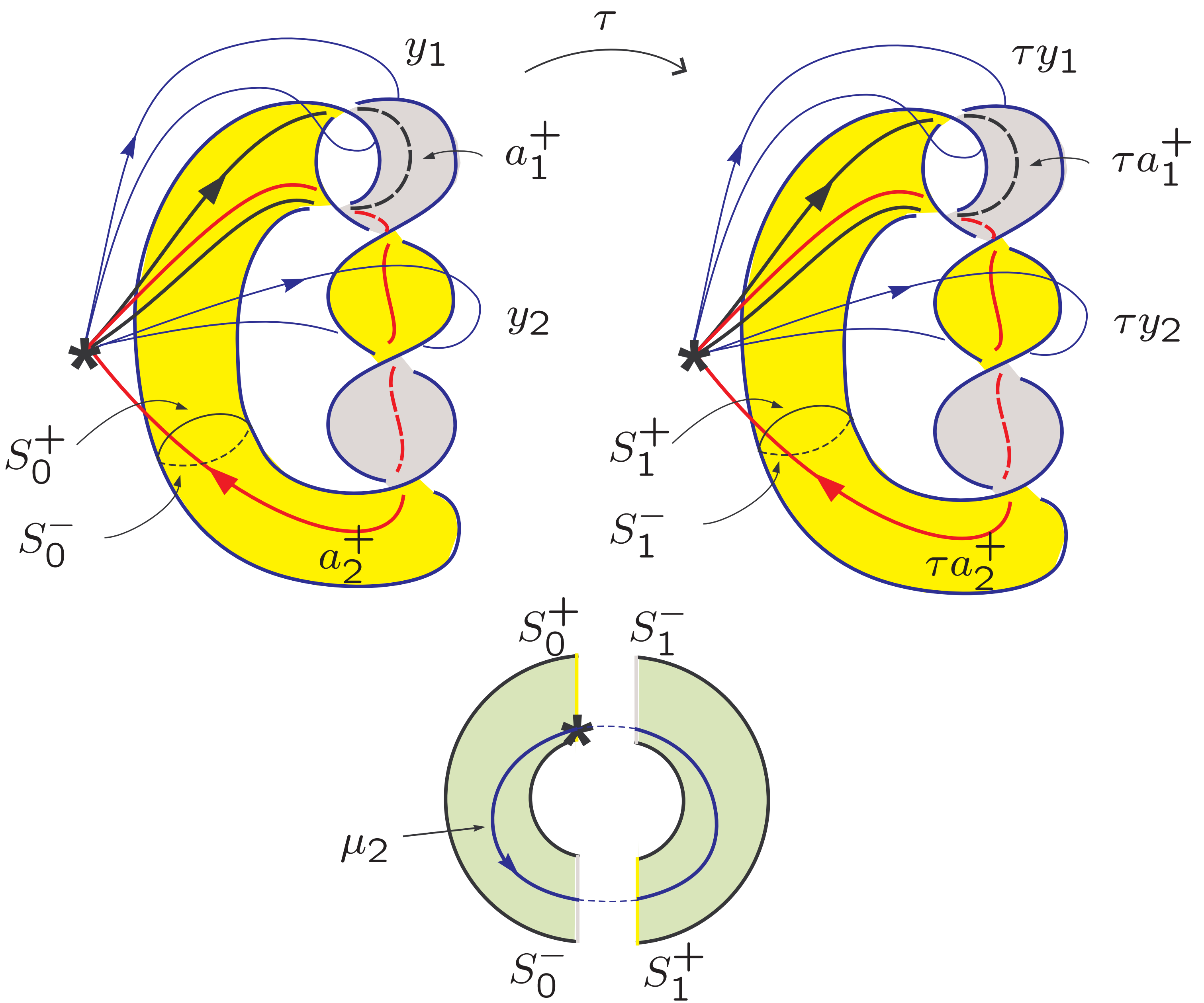


▶ $G(5_2) = \left\langle y_1, y_2, \mu \;\middle|\; \mu a_1^+ \mu^{-1} = a_1^-,\ \mu a_2^+ \mu^{-1} = a_2^- \right\rangle$

a Lin's presentation of a knot group $G(K)$

▶ $\mathbf{C}_2 K$: the 2-fold cyclic cover of $\mathbf{E}_K$

$\pi_1(\mathbf{C}_2 5_2) = \left\langle \widetilde{y_1}, \widetilde{y_2}, \tau\widetilde{y_1}, \tau\widetilde{y_2}, \mu_2 \;\middle|\; \mu a_1^+ \mu^{-1} = a_1^-,\ \mu a_2^+ \mu^{-1} = a_2^- \right\rangle$



τ
y_1
a_1^+
y_2
S_0^+
S_0^-
a_2^+
τy_1
τa_1^+
τy_2
S_1^+
S_1^-
τa_2^+
S_0^+
S_1^-
μ_2
S_0^-
S_1^+

▶ $\pi_1(\Sigma_2 5_2) \cong \pi_1(\mathbf{C}_2 K)/\langle\langle \mu_2 \rangle\rangle$

$$= \left\langle \widetilde{y_1}, \widetilde{y_2}, \tau\widetilde{y_1}, \tau\widetilde{y_2} \;\middle|\; \tau a_i^- = a_i^+, \tau a_i^+ = a_i^- \;\; (i = 1, 2) \right\rangle$$

▶ The map $\Phi : R_0(K) := \{\rho \in R(K) \mid \mathrm{tr}(\rho(\mu)) = 0\} \to R(\Sigma_2 K)$

$$\Phi(\rho)(\gamma) := \sqrt{-1}^{\,p'_*[\gamma]} \rho(p_* \gamma), \; \gamma \in \pi_1(\mathbf{C}_2 K)$$

where (1) $p_* : \pi_1(\mathbf{C}_2 K) \to G(K)$ is induced by the projection

$$p_*(\mu_2) = \mu^2, \; p_*(\widetilde{y_i}) = y_i, \; p_*(\tau\widetilde{y_i}) = \mu y_i \mu^{-1}.$$

(2) $p'_* : H_1(\mathbf{C}_2 K; \mathbb{Z}) \to \langle 2\mu \rangle \subset \langle \mu \rangle = H_1(\mathbf{E}_K; \mathbb{Z})$

EX. $\widehat{\Phi} : R_0(5_2) \to X(\Sigma_2 5_2)$

$$\begin{aligned} \Phi(\chi_\rho)(\mu_2) &= \Phi(\rho)(\mu_2) = \sqrt{-1}^{\,p'_*[\mu_2]} \rho(p_* \mu_2) \\ &= \sqrt{-1}^{\,2} \rho(\mu^2) = -(-E) = E \;\; \text{(well-defined)} \end{aligned}$$

▶ The map Φ gives $\widehat{\Phi} : S_0(K) \to X(\Sigma_2 K)^\tau \subset X(\Sigma_2 K)$

(2-fold branched, branched at **metabelian characters**)

An application to degree 0 abelian knot contact homology

- $$HC_0^{\mathrm{ab}}(K) \cong \frac{\mathbb{Z}[a_{12},\cdots,a_{nn-1}]}{\left\langle a_{kl}+a_{ij}a_{il}+a_{jl} \;\middle|\; \begin{array}{l} (i,j,k) \text{ range over all } n \text{ crossings} \\ l \in \{1,\cdots,n\} \end{array} \right\rangle}$$

- $$F_2(K) = \left\{ (x_{12},\cdots,x_{nn-1}) \in \mathbb{C}^{\binom{n}{2}} \;\middle|\; \begin{array}{l} x_{ka} - x_{ij}x_{ia} + x_{ja} = 0 \ \textbf{(F2)} \\ (i,j,k)\text{: any Wirtinger triple} \\ a \in \{1,\cdots,n\} \end{array} \right\}$$

- $$\mathbf{C}[V] := \frac{\mathbb{C}[z_1,\cdots,z_N]}{\sqrt{\langle \text{polynomials vanishing on } V \subset \mathbb{C}^N \rangle}}$$ **the coordinate ring of V**

- $g : HC_0^{\mathrm{ab}}(K) \otimes \mathbb{C} \to \mathbf{C}[F_2(K)]$, $g(a_{ij}) := -x_{ij}$, $g(1) = 1$ gives an isomorphism:

Proposition

$$\left(HC_0^{\mathrm{ab}}(K) \otimes \mathbb{C}\right) / \sqrt{0} \cong \mathbf{C}[F_2(K)].$$

Corollary of the main theorem

If there does not exist a **ghost character** in $F_2(K)$

➡ $F_2(K) \cong X(\Sigma_2 K)^\tau$

➡ $\left(HC_0^{\mathrm{ab}}(K) \otimes \mathbb{C}\right)/\sqrt{0} \cong \mathbf{C}[X(\Sigma_2 K)^\tau]$

This is a partial answer to Ng's conjecture:

Conjecture [Ng] (a weak version)

$$\left(HC_0^{\mathrm{ab}}(K) \otimes \mathbb{C}\right)/\sqrt{0} \cong \mathbf{C}[X(\Sigma_2 K)].$$

Remark [N-Yamaguchi]

K: (p, q)-torus knot, 2-bridge knots and pretzel knots

➡ $\widehat{\Phi}$ is surjective

moreover no ghost characters (the reason is omitted in this talk)

➡ $\left(HC_0^{\mathrm{ab}}(K) \otimes \mathbb{C}\right)/\sqrt{0} \cong \mathbf{C}[X(\Sigma_2 K)^\tau] = \mathbf{C}[X(\Sigma_2 K)]$

ありがとうございました (Thank you!)

Section 4:

Computer experiments on $S_0(K)$ **and** $HC_0^{\mathrm{ab}}(K)$

Let's calculate on Maple $S_0(K)$ and $HC_0^{\mathrm{ab}}(K)$ and get a table of them using it!