Geometric Identities
RIMS Seminar 2012

Greg McShane

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Part I

Introduction

Surfaces

- Σ is a surface
- totally geodesic boundary
- finite volume hyperbolic structure

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- totally geodesic boundary
- finite volume hyperbolic structure

- $\Gamma \simeq \pi_1(\Sigma)$
- $\Lambda =$ limit set of Γ
- $CC(\Lambda)/\Gamma = \Sigma$

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Ortho geodesic is a pair $\alpha, \beta \in \Gamma$, $[\alpha], [\beta] \subset \partial\Sigma$

$$\frac{(\alpha^- - \beta^-)(\alpha^+ - \beta^+)}{(\alpha^- - \beta^+)(\alpha^+ - \beta^-)} = \tanh^2(\frac{1}{2}\ell(\gamma^*))$$

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- $\delta =$ Hausdorff dimension of the limit set.
- $\mathrm{Vol}(\Sigma)$
- $\mathrm{Vol}(\partial\Sigma)$

Length spectrum

- (Weyl) Spectrum of Laplacian determines the area

$$N_\Gamma(t) := |\{\text{eigenvalues of} \Delta_{\mathbb{H}/\Gamma} < t\}| \sim \frac{\mathrm{Vol}(\mathbb{H}/\Gamma)}{4\pi} t$$

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- (Wolpert) Length spectrum determines the isometry type of the surface up to finitely many choices

Trace formula

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$$\sum_n h(\lambda_n) = \frac{\mathrm{Vol}(\mathbb{H}/\Gamma)}{4\pi} \int_{\mathbb{R}} rh(r) \tanh(\pi r) dr + \sum_{[\gamma]} \frac{2\ell(\gamma)}{\sinh(\frac{1}{2}\ell(\gamma))} \hat{h}(\ell(\gamma))$$

where

- λ_n are the eigenvalues of the Laplacian.
- $\ell(\gamma)$ is the length of the geodesic in the homotopy class $[\gamma]$

Simple Length Spectra

- (Wolpert) Simple length spectrum determines the surface up to finitely many choices.
- (Mirkzahani)

$$N(t) := |\{\text{simple geodesics } \ell(\alpha) < t\}| \sim C(\mathbb{H}/\Gamma)t^{6g-6}.$$

g = genus of Σ.

Part II

Identities

Basmajian Identity

Theorem (1992)

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$$\sum_{\alpha^*} \mathrm{Vol}_{n-1}\left(\text{Ball radius} = \sinh^{-1}\left(\frac{1}{\sinh(\ell(\alpha^*))}\right)\right) = \mathrm{Vol}_{n-1}(\partial M)$$

Bridgeman-Kahn Identity

Theorem (2008)

$$2\pi Vol(M) = 8\sum_{\alpha^*} \mathcal{L}\left(\frac{1}{\cosh^2(\ell(\alpha^*)/2)}\right)$$

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$$\mathrm{Li}_2(z) = \sum \frac{z^k}{k^2} = -\int_0^z \frac{\log(1-x)}{x} dx$$

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- Roger's dilogarithm

$$\mathcal{L}(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log|x|\log(1-x),\ x < 1.$$

$$\mathcal{L}'(x) = \frac{1}{2}\left(\frac{\log(1-x)}{x} + \frac{\log(x)}{1-x}\right).$$

Bridgeman-Kahn Identity in general

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Exist F_n such that for any hyperbolic n-manifold M with totally geodesic boundary

$$\text{Vol}(M) = \sum_{\beta} F_n(\ell(\alpha^*))$$

the volume of M is equal to the sum of the values of F_n on the orthospectrum of M.

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- integral formula for F_n in terms of elementary functions.

Identity for embedded pants

Σ has a single boundary component of length $\ell(\delta) \geq 0$

- Punctured torus $\ell(\delta) = 0$

$$\sum_{\alpha} \frac{1}{1 + e^{\ell(\alpha)}} = \frac{1}{2}$$

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- One-holed torus

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- One-holed genus g

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P on a holed torus is pants with waist δ and legs α, α

Luo-Tan

Theorem (2010)

$$\sum_P f(P) + \sum_T g(T) = 2\pi Vol(M)$$

where

$$f(P) := 4\sum_{i\neq j}[2\mathcal{L}(\frac{1-x_i}{1-x_iy_j}) - 2\mathcal{L}(\frac{1-y_j}{1-x_iy_j}) - \mathcal{L}(y_j) - \mathcal{L}(\frac{(1-x_i)^2y_j}{(1-y_j)^2x_i})]$$

$$g(T) := 4\pi^2 + 8\sum_A[2\mathcal{L}(\frac{1-x_A}{1-x_Ay_A}) - 2\mathcal{L}(\frac{1-y_A}{1-x_Ay_A}) - 2\mathcal{L}(y_A) - \mathcal{L}(\frac{(1-x_A)^2y_A}{(1-y_A)^2x_A})]$$

Part III

Proofs

Decompositions

Given an identity :
what is the associated decomposition of the surface ?

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what is the associated decomposition of the surface ?
Decomposition:

some space $X = (\sqcup\{\text{geometric pieces}\}) \sqcup \{\text{negligible}\}$

- $X = \partial\Sigma$
- $X = \partial\mathbb{H}$,
  negligible $= \Lambda$
- $X =$ unit tangent bundle Σ,
  negligible $=$ geodesics that stay in convex core.

Limit set

$\Lambda :=$ limit set.

Theorem (Ahlfors)

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Proposition

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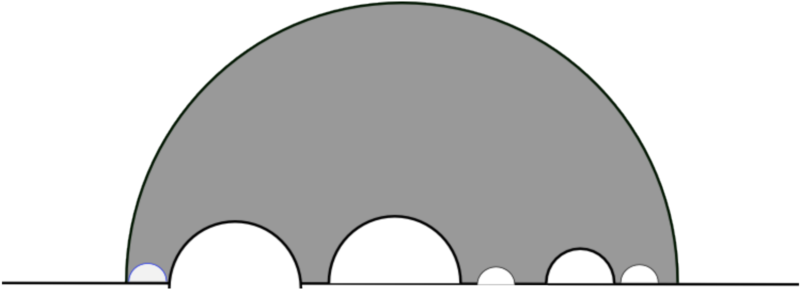
Theorem (Birman-Series)

Let K_x be the set of endpoints x such that $[x_0, x]$ projects to a simple geodesic. Then K_x is Hausdorff dimension 0.

Convex core



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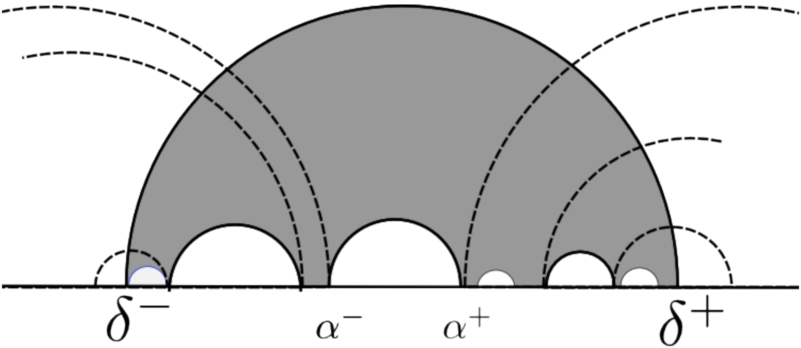

Caligari's Chimneys

Proposition

Let M be a compact hyperbolic n-manifold with totally geodesic boundary S. Let M_S be the covering space of M associated to S. Then M_S has a canonical decomposition into a piece of zero measure, together with two chimneys of height l_i for each number l_i in the orthospectrum.

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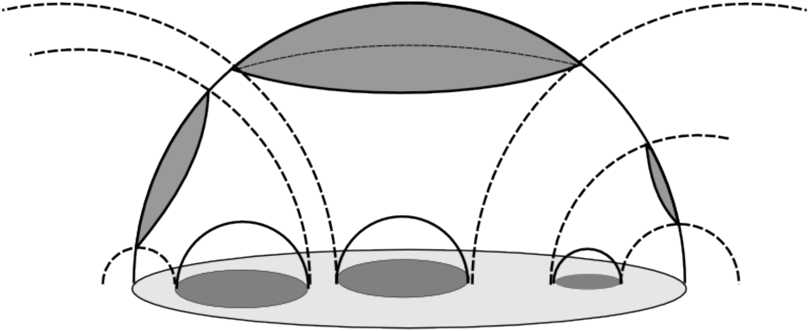
Picture in $\mathbb{H}^3$



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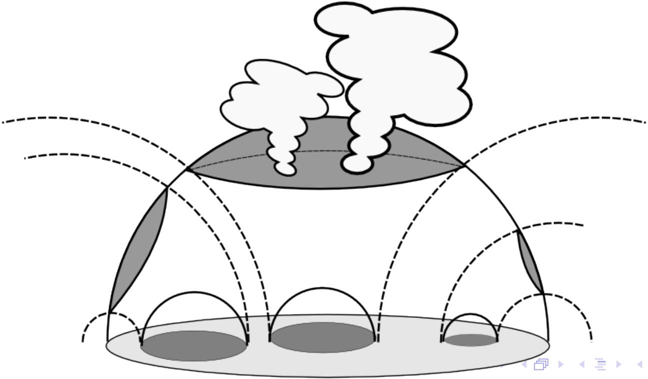

Caligari's Chimneys

The boundary of M_S consists of a copy of S, together with a union of totally geodesic planes.
Plane is the top of a chimney, with base a round disk in S, and these chimneys are pairwise disjoint and embedded.
Since M is geometrically finite, the limit set has measure zero, and therefore these chimneys exhaust all of M_S except for a subset of measure zero. Every oriented ortho geodesic in $\alpha \subset M$ lifts to a unique geodesic arc with initial point in M_S . This arc is the core of a unique chimney in the decomposition, and all chimneys arise this way.

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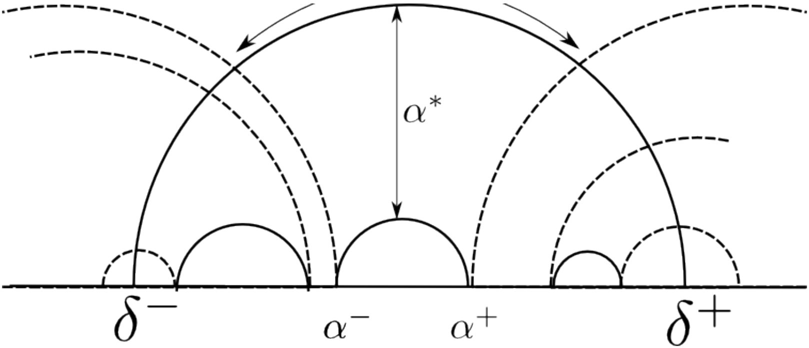
Basmajian

- $\partial M = (\sqcup \text{leopard spots}) \sqcup \text{projection of} \Lambda$
- $\text{Vol}(\partial M) = \sum \text{Vol}(\text{leopard spots})$



Bridgeman-Kahn

- Group unit tangent vectors v, u of $CC(\Lambda)$
  such that the geodesics γ_v, γ_u
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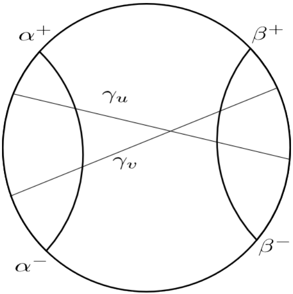


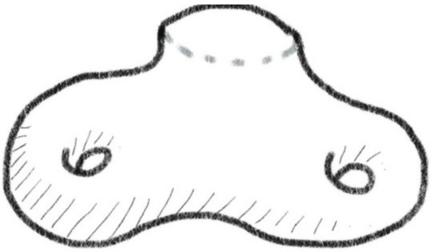
$\text{Vol}(\text{unit tangent bundle}M) = \sum \text{Vol}(\text{tetrahedra})$
Would be an rectangle cross $\mathbb{R}$ but we truncate when the geodesic leaves the convex core $CC(\Lambda)$.

Pants

$$\sum_{\alpha} 2\log\left(\frac{1+e^{\frac{1}{2}(\ell(\alpha)+\ell(\beta)-\ell(\delta))}}{1+e^{\frac{1}{2}(\ell(\alpha)+\ell(\beta)+\ell(\delta))}}\right) = \ell(\delta)$$

What is the associated decomposition of the surface ?

Pre proof



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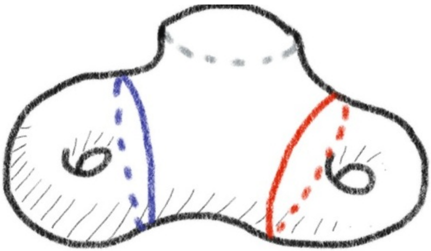

Gap decomposition of δ

Define $X \subset \delta$ to be the set of x starting points for $\gamma_x :=$ geodesic leaving δ at right angles which

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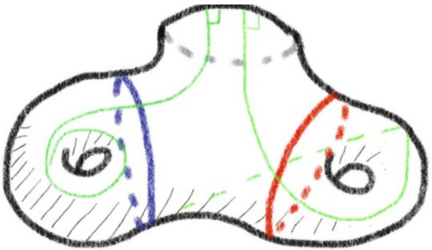

Gap decomposition of δ

The geodesic ray γ_x

- either exits a pair of pants by one of the boundaries α, β.
- or spirals to one of the boundaries α, β.

Lemma

There are a pair of intervals $\subset \delta$ which contain no point of X

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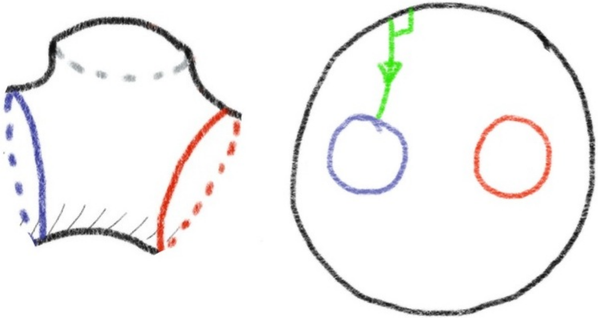
Decomposition

$$\partial M = (\sqcup \text{gaps}) \sqcup \text{projection of } K \subset \Lambda$$

$K =$ endpoints of certain simple ortho geodesics

Proof



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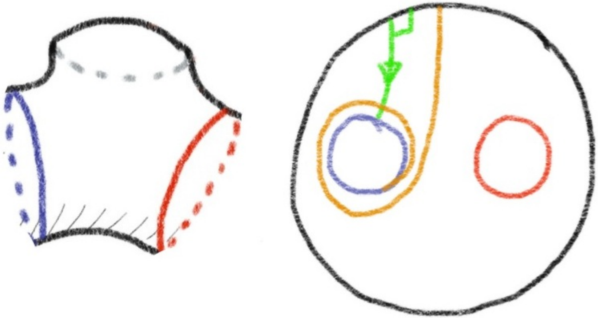

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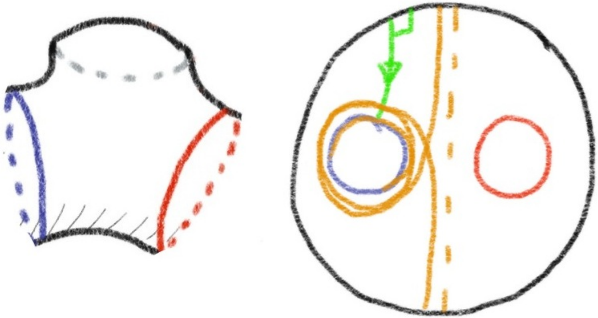

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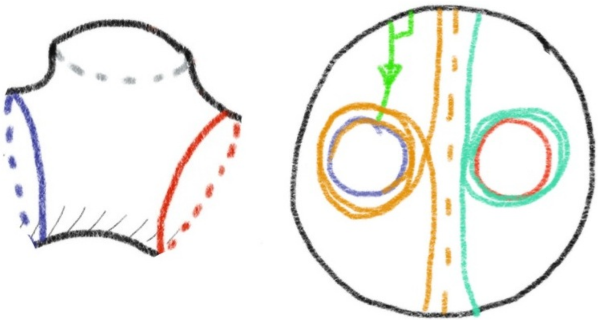

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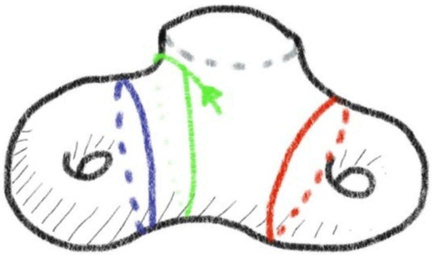
- Get a decomposition of the unit tangent bundle of Σ into measurable pieces.
- What is the contribution of an embedded pair of pants to the volume of the unit tangent bundle of Σ?
- What is the probability that a geodesic segment has it's first intersection in a pair of pants

Tan's lassoo decomposition



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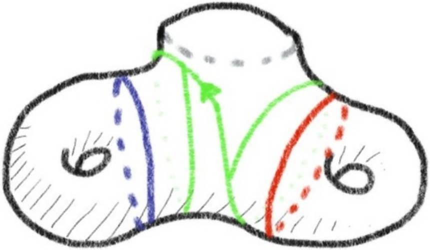
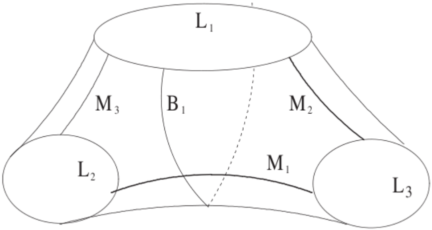


$$f(P) := 4\sum_{i\neq j}[2\mathcal{L}(\frac{1-x_i}{1-x_iy_j}) - 2\mathcal{L}(\frac{1-y_j}{1-x_iy_j}) - \mathcal{L}(y_j) - \mathcal{L}(\frac{(1-x_i)^2y_j}{(1-y_j)^2x_i})]$$

Tan's lassoo functions

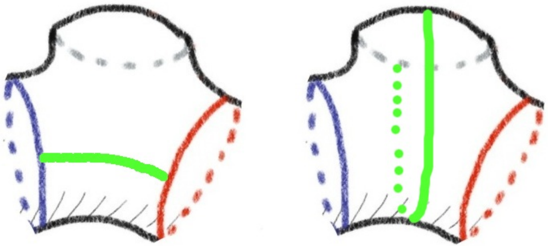


$$
\begin{aligned}
f(P) &= 4\pi^2 \\
&- 8\left\{\sum \mathcal{L}(\cosh^{-2}(M_i/2)) + \mathcal{L}(\cosh^{-2}(B_i/2))\right\} \\
&+ \sum\nolimits_{i \neq j} La(L_i, M_j) \\
&= \mathrm{Vol}(P) - \mathrm{Vol}(\text{just an arc}) - \mathrm{Vol}(\text{makes a lasso})
\end{aligned}
$$

$$
La(x, y) = \mathcal{L}(x) - \mathcal{L}\left(\frac{1-x}{1-xy}\right) + \mathcal{L}\left(\frac{1-y}{1-xy}\right).
$$

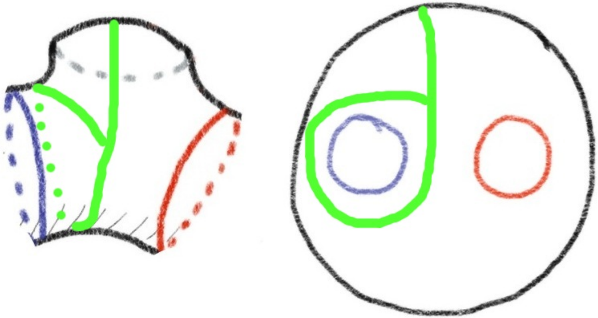
Just an arc

$$f(P) \quad = \quad \mathrm{Vol}(P) \quad - \quad \mathrm{Vol}(\text{just an arc}) \quad - \quad \mathrm{Vol}(\text{makes a lasso})$$



Makes a lassoo

$$f(P) \quad = \quad \mathrm{Vol}(P) \quad - \quad \mathrm{Vol}(\text{just an arc}) \quad - \quad \mathrm{Vol}(\text{makes a lasso})$$



Applications

$l :=$ length shortest orthogeodesic then

$$\mathrm{Vol}_n(M) \geq F_n(l)$$

where $F_n(t) =$

Theorem

There exists

- *A function* $H_n : \mathbb{R}_+ \to \mathbb{R}_+$
- *Constants* $C_n > 0$

∂M *totally geodesic then*

$$Vol_n(M) \geq H_n(Vol_{n-1}(\partial M)) \geq C_n Vol_{n-1}(\partial M)^{\frac{n-2}{n-1}}$$

Applications

For $S \subset \mathbb{H}$ an ideal n-gon,

- hyp area $(n-2)\pi$
- n cusps

the Length Spectrum Identity is a finite summation relation. associated relations give an infinite list of finite relations including the classical identities of Euler, Abel etc

Theorem

$$\sum_{i,j} \mathcal{L}([x_i, x_{i+1}, x_j, x_{j+1}]) = \sum_{\alpha} \mathcal{L}\left(\frac{1}{\cosh^2(l_\alpha/2)}\right) = \frac{(n-3)\pi^2}{6}$$

We now consider the Poincaré disk model

- x_i, $i = 1, \ldots, n$ vertices
- l_{ij} = length of the orthogeodesic $x_i x_{i+1}$ $x_j x_{j+1}$

$$[x_i, x_{i+1}, x_j, x_{j+1}] = \cosh^{-2}(\frac{1}{2} l_{ij})$$

Euler reflection

$$\mathcal{L}(x) + \mathcal{L}(1-x) = \mathcal{L}(1) = \frac{\pi^2}{6}$$
$$\mathcal{L}(x) + \mathcal{L}(1/x) = 2\mathcal{L}(-1) = -\frac{\pi^2}{6}$$

- The ideal quadrilateral has 4 cusps two ortholengths l_1, l_2.
- Cut into quadrilaterals lengths $\infty, \infty, \frac{1}{2}l_1, \frac{1}{2}l_2$.

$$\cosh^{-2}(\frac{1}{2}l_1) + \cosh^{-2}(\frac{1}{2}l_2) = 1$$
$$\Rightarrow \mathcal{L}\left(\cosh^{-2}(\frac{1}{2}l_1)\right) + \mathcal{L}\left(\cosh^{-2}(\frac{1}{2}l_2)\right) = \frac{(4-3)\pi^2}{6}$$

Symplectic volumes

Weil-Petersson volumes and cone surfaces, (2005)

- Mapping class group $\mathcal{MCG}$.
- Teichmuller space $= \mathcal{T}(\Sigma)$, ω_{WP} – $\mathcal{MCG}$-invar. symplectic form.
- Moduli space $= \mathcal{T}(\Sigma)/\mathcal{MCG}$, – symplectic vol. form

Symplectic volume of the moduli space of a surface

- = a number for surface with marked points.
  Wolpert (1982), Penner, Harer-Zagier
- = a polynomial for surface with boundary.
  Nakanishi-Naatanen (2001), Mirzakhani(2003).

$$\begin{aligned} \text{torus, one hole, } V_1(l_1) &= \frac{1}{24}(4\pi^2 + l_1^2) \\ \text{torus, two hole, } V_1(l_1, l_2) &= \frac{1}{192}(4\pi^2 + l_1^2 + l_2^2)(12\pi^2 + l_1^2 + l_2^2) \end{aligned}$$

Symplectic volume of a once punctured torus

Fenchel Nielsen coordinates $\ell(\alpha), \tau(\alpha)$

$$
\begin{aligned}
\int_{\mathcal{T}/\mathcal{MCG}} 1. d\ell(\alpha) d\tau(\alpha) &= \int_{\mathcal{T}/\mathcal{MCG}} \sum_{\alpha} \left(\frac{2}{1+e^{\ell(\alpha)}} \right) d\ell(\alpha) d\tau(\alpha) \\
&= \int_{\mathcal{T}/\text{Dehn twist}} \left(\frac{2}{1+e^{\ell(\alpha)}} \right) d\ell(\alpha) d\tau(\alpha) \\
&= \int_0^\infty \int_0^{\ell(\alpha)} \frac{2}{1+e^{\ell(\alpha)}} d\tau(\alpha) d\ell(\alpha) \\
&= \int_0^\infty \frac{2\ell(\alpha)}{1+e^{\ell(\alpha)}} d\ell(\alpha) \\
&= \int_0^\infty 2 \sum x(-1)^k e^{-(k+1)x} dx \\
&= \frac{\pi^2}{6}
\end{aligned}
$$

Symplectic volumes

$$\begin{aligned} V_1(l_1) &= \frac{1}{24}(4\pi^2 + l_1^2) \\ V_1(l_1, l_2) &= \frac{1}{192}(4\pi^2 + l_1^2 + l_2^2)(12\pi^2 + l_1^2 + l_2^2) \end{aligned}$$

$$\begin{aligned} \frac{d}{dl_2} V_1(l_1, l_2) &= \frac{1}{96} l_2(16\pi^2 + 2l_1^2 + 2l_2^2) \\ \frac{d}{dl_2}\bigg|_{2\pi i} V_1(l_1, l_2) &= \frac{2\pi i}{96}(8\pi^2 + 2l_1^2) \\ &= \frac{2\pi i}{4.24}(4\pi^2 + l_1^2) = \frac{2\pi i}{4} V_1(l_1) \end{aligned}$$

Do,Norbury

Cone point = geodesic boundary with complex length $i\theta$
Use cone surface with a cone point of angle $0 < \theta < 2\pi$:

- to interpolate the forgetful map $(\Sigma_g, p) \to \Sigma_g$
- study degeneration of associated fibration (Schumacher-Trappani)

$$\Sigma_g \to \mathcal{T}(\Sigma_{g,1})/\mathcal{MCG} \to \mathcal{T}(\Sigma_g)/\mathcal{MCG}$$

- Volume should go to zero (Schumacher-Trappani + some work)

$$V_g(\pm 2\pi) = 0$$

- But what happens to
  - the topology of the moduli space $\mathcal{T}(\Sigma)_\theta$
  - the dynamics of $\mathcal{MCG}$

  as $\theta \to 2\pi$.