On the number of hyperbolic 3-manifolds of a given volume

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Representation spaces, twisted topological invariants and geometric structures of 3-manifolds

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Mostow-Prasad rigidity

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Theorem (Mostow-Prasad rigidity theorem)

Let M_1, M_2 be hyperbolic 3-manifolds. Then,
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- By Mostow-Prasad rigidity the volume of a given hyperbolic 3-manifold is a topological invariant.
- For example, for a given hyperbolic link, the volume is a link invariant.

Volume

Theorem (Jϕrgensen-Thurston)

Let $\mathcal{H}$ be isometry classes of hyperbolic 3-manifolds. Then the volume function $\mathrm{vol} : \mathcal{H} \to \mathbb{R}_{>0}$ is a finite-to-one function. Further, the image $\mathrm{vol}(\mathcal{H})$ is a well-ordered subset of $\mathbb{R}_{>0}$ of order type ω^ω.

By this theorem, for a given $v \in \mathbb{R}_{>0}$, there exists a natural number $N(v) := \sharp \mathrm{vol}^{-1}(v)$.

$N(v)$

Question

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- Gabai-Meyerhoff-Milley proved that the Weeks manifold W is the *unique* smallest volume manifold among all hyperbolic 3-manifolds.
  i.e $N(\mathrm{vol}(W)) = 1$ and for $v < \mathrm{vol}(W)$, $N(v) = 0$
- (As far as I know) prior to our work, this is the only result which gives the exact value of $N(v)$.

Classes of hyperbolic 3-manifolds

There are many interesting classes of hyperbolic 3-manifolds. For example,

- $\mathcal{C}$: cusped manifolds.
- $\mathcal{A}$: arithmetic manifolds.
- $\mathcal{G}$: manifolds with geodesic boundaries.
- $\mathcal{L}$: link complements.

It is also interesting to ask

$$N_{\mathcal{X}}(v) = \sharp\{\mathrm{vol}^{-1}(v) \cap \mathcal{X}\}$$

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- Kojima-Miyamoto detected the smallest compact manifolds with geodesic boundaries and Fujii proved that there are 8 of them. i.e. $N_{\mathcal{CG}}(6.452...) = 8$.

Computer experiments

SnapPy has many good censuses of hyperbolic manifolds.

- Orientable Cusped Census. (at most 8 ideal tetrahedra)
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- Link Exteriors (using Rolfsen's notation).
- (Non) Alternating Knot Exteriors (up to 16 crossings).
- MorwenLinks(up to 14 crossings, about 180k links).

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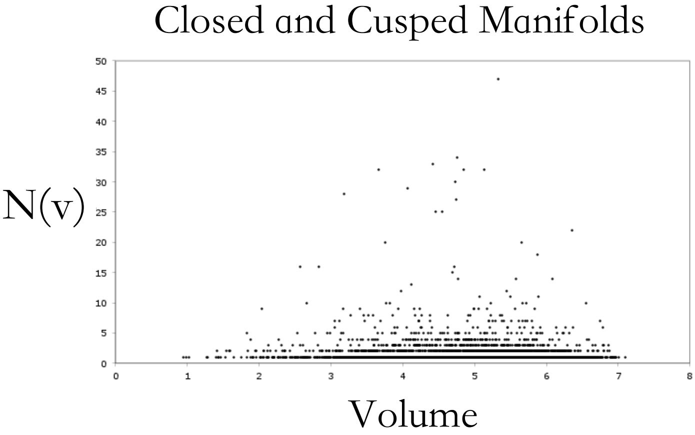
We used the first two censuses and compute $N_{\text{census}}(v)$'s.

Computer experiments



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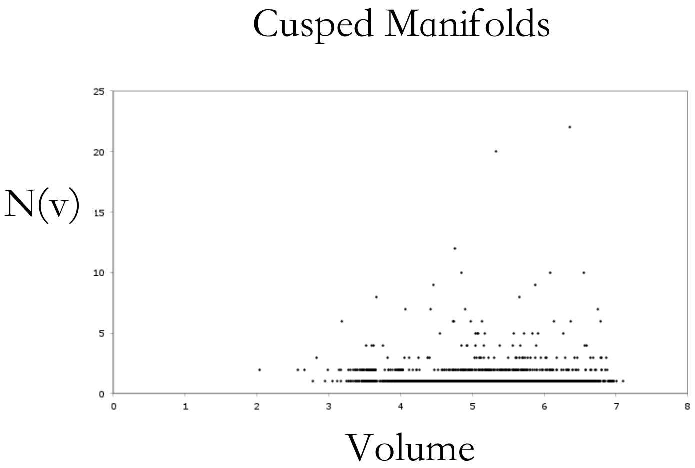


Main theorem 1

Theorem (Unique volume Manifolds)

There exists an infinite sequence of hyperbolic manifolds $\{M_i\}$ such that $N(\mathrm{vol}(M_i)) = 1$.

Theorem (Unique volume Cusped manifolds)

There exists an infinite sequence of cusped hyperbolic manifolds $\{M_i^{\mathcal{C}}\}$ such that $N_{\mathcal{C}}(\mathrm{vol}(M_i^{\mathcal{C}})) = 1$.

- These manifolds are obtained by Dehn filling on m004 and m129 respectively.

(m004 = complement of figure eight knot,
m129 = complement of Whitehead link)

Growth rate

In the above theorem, we discussed the case $N(v)$ is small.

Question

How large can N(v) be?

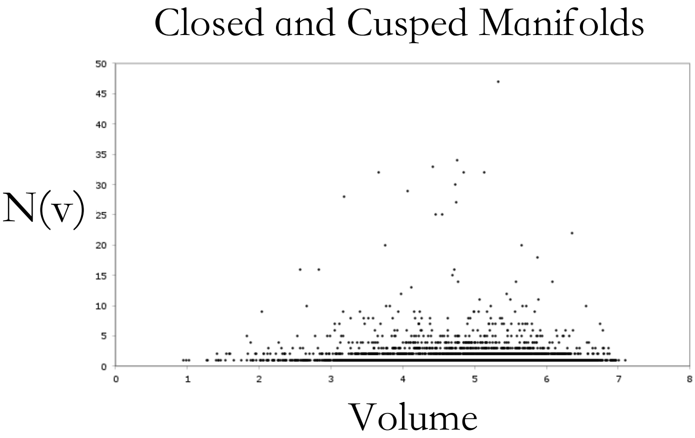
- Wielenberg: For all $n \in \mathbb{N}$, there exists $v \in \mathbb{R}_{>0}$ such that $N(v) > n$.
- Zimmerman: $N_{\mathrm{closed}}(v) > n$.

c.f.

Theorem (Chesebro-DeBlois, 2012)

$C(v)$ can be arbitrary large. Where $C(v)$ is the number of commensurability classes that contain manifolds of volume v.

Computer experiments



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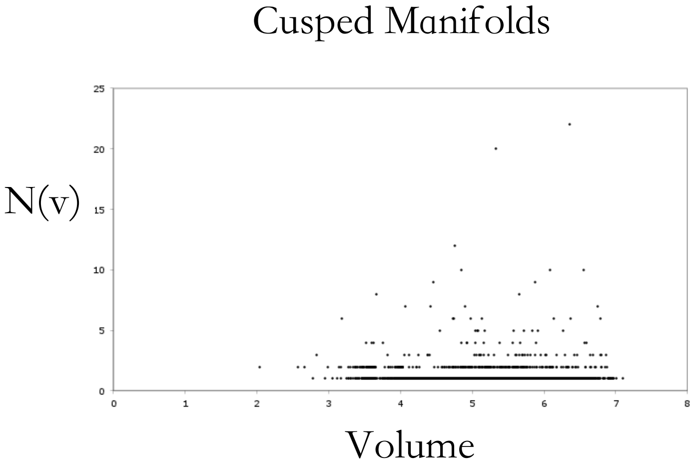

Growth rate

Question

How fast can $N(v)$ grow?

In other words, what can we say about $G(V)$? Where

$$\max_{v \leq V} N(v) \asymp G(V)$$

Known results

Theorem (Belolipetsky, Gelander, Lubotzky, Shalev, 2010)

There exists constants $a, b > 0$ such that for $x \gg 0$,

$$x^{ax} < \max_{x_i \leq x} N_{\mathcal{A}}(x_i) < x^{bx}$$

Theorem (Frigerio, Martelli and Petronio, 2003)

There exists a constant $c > 0$ such that for $x \gg 0$,

$$N_{\mathcal{G}}(x) > x^{cx}$$

($\mathcal{A}$: arithmetic manifolds
$\mathcal{G}$: manifolds with geodesic boundaries)

Main theorem 2

Theorem (Hodgson-M)

There exists $c > 1$ *such that*

$$N_{\mathcal{L}}(x) > c^x$$

($\mathcal{L}$: Link complements)

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Hyperbolic Dehn surgery theorem

- M : hyperbolic 3-manifold with a cusp T.
- $M(a, b)$: manifold after Dehn filling T along slope (a, b).
- $L(a, b)$: length of the slope (a, b) on T.

Theorem (Thurston)

Then there exist a constant $C_1 = C_1(M)$ such that the Dehn filling $M(a, b)$ is hyperbolic whenever $L(a, b) > C_1$, and $\mathrm{vol}(M(a, b)) \to \mathrm{vol}(M)$ as $L(a, b) \to \infty$

Neuman-Zagier asymptotic formula

- A : area of the horotori
- $Q(a,b) = L(a,b)^2/A$
- $\Delta(a,b) = \mathrm{vol}(M) - \mathrm{vol}(M(a,b)) > 0$

Theorem (Neumann-Zagier)

There exist a constant $C_2 = C_2(M) > 0$ such that,

$$\left| \frac{\pi^2}{\Delta(a,b)} - Q(a,b) \right| < C_2$$

Key idea

Key idea

(a_0, b_0) : pair of relatively prime integers such that

(i) $Q(a,b) = Q(a_0,b_0) = Q_0$ has few integer solutions,

(ii) there is large enough 2-sided gap around Q_0 in the set of possible value of $Q(x,y)$ for (x,y) relatively prime integers.

$\Rightarrow$ There are few Dehn fillings $M(a,b)$ with the same volume as $M(a_0,b_0)$

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$\Rightarrow$ some number theory proves the existence of a sequence $\{(a_i, b_i)\}$ such that

(i) $(Q_0 :=)Q(a, b) = Q(a_i, b_i) \Rightarrow (a, b) = (a_i, b_i)$

(ii) there is large enough 2-sided gap around Q_0 in the set of possible value of $Q(x, y)$ for (x, y) relatively prime integers.

$\Rightarrow M(a_i, b_i)$ is a unique volume manifold among all Dehn fillings of m004 and m003.

Smallest cusped manifolds

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$\exists \varepsilon > 0$ such that for all manifolds N with

$$2.029... - \varepsilon < \mathrm{vol}(N) < 2.029...$$

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can be obtained from m004 or m003 by a Dehn filling.
$\Rightarrow$ If $M(a, b)$ is a unique volume manifold among all Dehn fillings of m004 or m003, then $M(a, b)$ is unique among all hyperbolic 3-manifolds.
(m004: the figure eight knot complement
m003: the sister of m004)

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Hyperbolic graph

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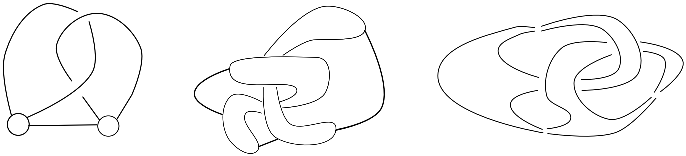
where $\mathcal{N}(v)$ is an open regular neighborhood of v.
Then N_G is a manifold with 3-punctured sphere boundaries, one corresponds to each vertex of G.

Definition

A spacial graph G is hyperbolic if N_G admits complete hyperbolic structure (with parabolic meridians) of finite volume with totally geodesic boundaries.

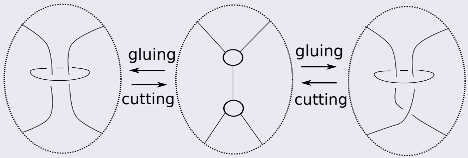
Example (Intuitive picture) of a hyperbolic graph.



Volume preserving moves

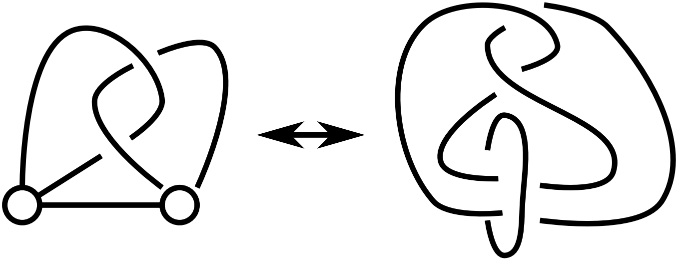
Lemma

The following moves on hyperbolic graphs in S^3 are volume preserving.



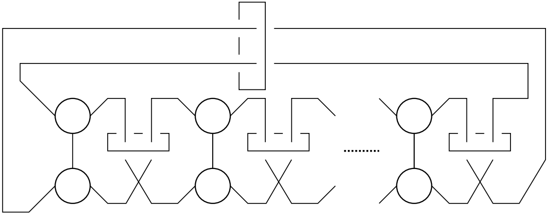
This lemma relates hyperbolic graphs with hyperbolic links.

Example.



Two complements have the same volume.

We apply one of the moves to the following graph.



Then we get possibly distinct 2^n links of a same volume.

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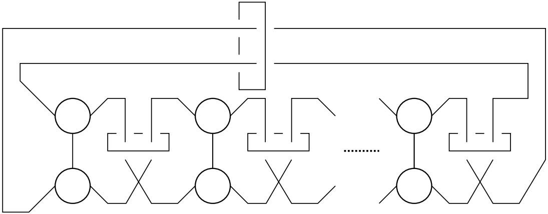
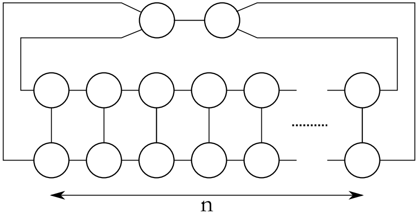


Then we get possibly distinct 2^n links of a same volume.

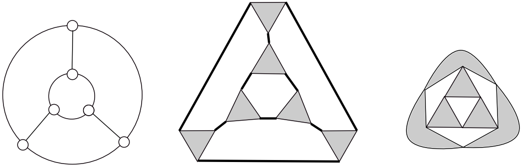
- We distinguish these manifold by computing the moduli of cusps and edges of canonical decomposition.

Our graph

The graph comes from a planar graph

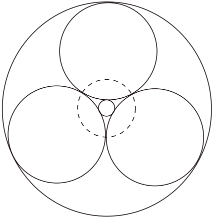


The complements of planar hyperbolic graphs admit useful polyhedral decompositions.



Remark.
This decomposition is same as the decomposition of a *fully augmented link* found by Agol-Thurston.

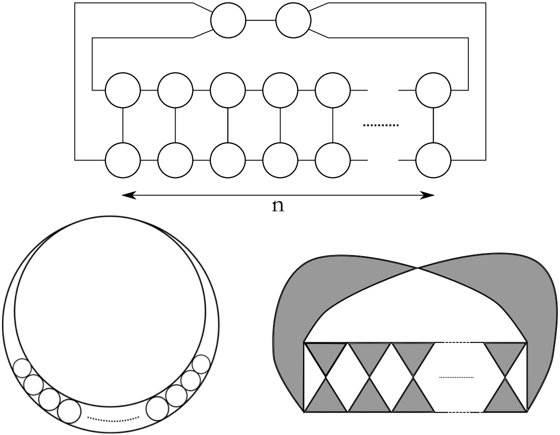
Since each dihedral angle is $\pi/2$, this decomposition gives a circle packing on $\partial\mathbb{H}^3 \cong S^2$.



- This circle packing enables us to compute modulus of each cusp.

Our graph

Our graph, its polyhedral decomposition and corresponding circle packing.

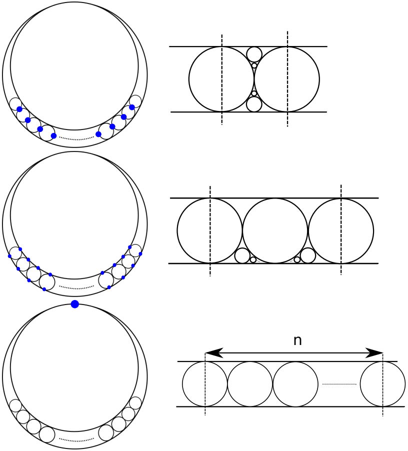


Moduli of cusps

For our graph, there are 3 different types of annuli cusps.

By gluing annuli cusps together we get a torus cusp.

$\Rightarrow$
We can compute cusp moduli.



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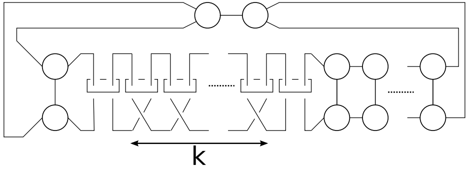
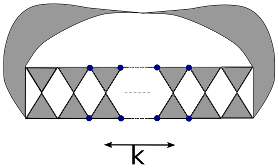
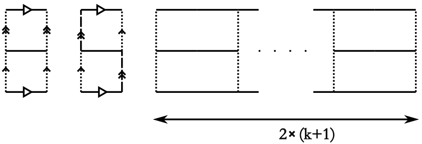
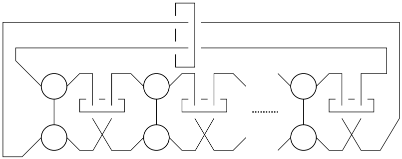




This graphs has 3 types of tori cusps and their shapes are



For each link that we obtain after gluing the 3-punctured sphere of



we can assign (horoball) volume to each cusp in terms of the moduli.
It gives us a way to fix a canonical decomposition.
(SnapPy demo.)

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4. What is the largest volume $v < v_\omega = 2.029883...$ of a closed hyperbolic 3-manifold which does not arise from Dehn filling of m004 or m003? (This would allow us to make the above results explicit.)
   Guess: $v = 2.02885309.. = \mathrm{vol}(\mathrm{m006}(-5, 2))$.

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5. Does there exist $C > 0$ such that $N_{\mathcal{L}}(x) > x^{Cx}$?

Thank you for your attention.

Great Ocean Road (Australia)