The places where pseudo-Anosovs with small dilatation live

Eiko Kin

Tokyo Institute of Technology

(joint work with M. Takasawa and S. Kojima)

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This talk is based on the following papers:

[KT0] E. Kin and M. Takasawa, *The boundary of a fibered face of the magic 3-manifold and the asymptotic behavior of the minimal pseudo-Anosovs dilatations.* preprint (2012) arXiv:1205.2956

[KKT] E. Kin, S. Kojima and M. Takasawa, *Minimal dilatations of pseudo-Anosovs generated by the magic 3-manifold and their asymptotic behavior.* preprint (2011) arXiv:1104.3939

[KT1] E. Kin and M. Takasawa, *Pseudo-Anosovs on closed surfaces having small entropy and the Whitehead sister link exterior*, to appear in "Jounal of the Mathematical Society of Japan"

[KT2] E. Kin and M. Takasawa, *Pseudo-Anosov braids with small entropy and the magic 3-manifold*, Communications in Analysis and Geometry 19, volume 4 (2011), 705-758.

Mapping class groups

$\Sigma = \Sigma_{g,n}$; closed orientable surface of genus g by removing n punctures

$\mathrm{Homeo}_+(\Sigma) = \{f : \Sigma \to \Sigma : \text{ori. pres. homeo. pres. punctures setwise}\}$

$\mathrm{Mod}(\Sigma) = \mathrm{Homeo}_+(\Sigma)/\mathrm{Homeo}_0(\Sigma)$

We focus on elements $\phi \in \mathrm{Mod}(\Sigma)$, called pseudo-Anosov (pA).

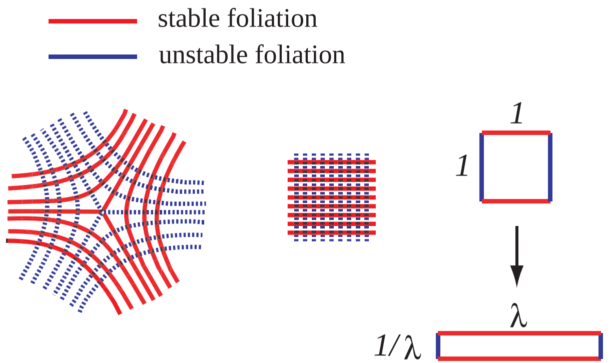
Theorem 1 (Thurston). $\phi \in \mathrm{Mod}(\Sigma)$ *is pseudo-Anosov* $\iff$ $^{\exists} f \in \phi$ *such that* f *is a pseudo-Anosov homeo.*

A **homeomorphism** $f : \Sigma \to \Sigma$ is pseudo-Anosov if $^{\exists}\lambda > 1$, and $^{\exists}\mathcal{F}^s, \mathcal{F}^u$; a pair of transverse measured foliations such that

$$f(\mathcal{F}^s) = \frac{1}{\lambda}\mathcal{F}^s \text{ and } f(\mathcal{F}^u) = \lambda\mathcal{F}^u.$$

The constant λ is called the dilatation of f.

$\mathcal{F}^s$ and $\mathcal{F}^u$ are called the stable and unstable foliation of f.



Invariants of pA mapping classes

Let $f \in \phi$ be a pseudo-Anosov homeomorphism. Then $\lambda(f)$ does not depend on the choice of a representative.

- $\lambda(\phi) := \lambda(f) > 1$; dilatation of ϕ
- $\mathrm{ent}(\phi) := \log \lambda(f)$; entropy of ϕ
- $\mathrm{Ent}(\phi) := |\chi(\Sigma)| \log \lambda(f)$; normalized entropy of ϕ
  $= |\chi(\Sigma)|\ \mathrm{ent}(\phi)$

Mapping classes and Fibered 3-manifolds

From $\phi \in \mathrm{Mod}(\Sigma)$, we obtain the mapping torus

$$\mathbb{T}(\phi) = \Sigma \times [0,1]/_{(x,0)\sim(f(x),1)},$$

where $f \in \phi$ is a representative

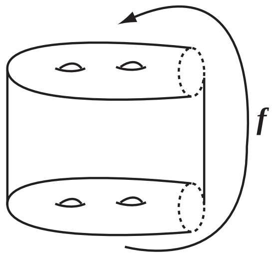


図 1 a fiber Σ of $\mathbb{T}(\phi)$, and a monodromy f of a fibration

Theorem 2 (Thurston). *$\phi \in \mathrm{Mod}(\Sigma)$ is pA $\Longleftrightarrow$ $\mathbb{T}(\phi)$ is a hyperbolic 3-manifold with finite volume*

Minimal dilatations problem

Fix a surface $\Sigma = \Sigma_{g,n}$.

$$Spec(\Sigma) := \{\lambda(\phi) \mid \text{pseudo-Anosov } \phi \in \mathrm{Mod}(\Sigma)\}.$$

★ There exists a minimum of $Spec(\Sigma)$ (Ivanov)

$\delta_{g,n} := \min\{\lambda \mid \lambda \in Spec(\Sigma_{g,n})\}$

Problem 1. *Determine the explicit value of $\delta_{g,n}$. Describe pseudo-Anosov elements which achieve $\delta_{g,n}$.*

The purpose of this talk...

★ find sequences of pseudo-Anosovs with small dialtation

★ Our conjecture: they could have the minimal dilatation

★ These pseudo-Anosovs are coming from a single 3-manifold.

The purpose of this talk...

★ Magic manifold $N = S^3 \setminus$ (3 chain link)

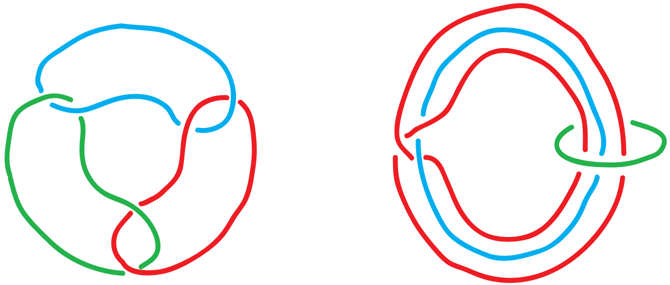


図 2　3 chain link (left), braided link of a 3-braid (right)

- N is a hyperbolic, fibered 3-manifold.

Minimal dilatation $\delta_{0,n}$

D_n: n-punctured disk

$\mathrm{Mod}(D_n)(= \mathrm{Homeo}_+(\mathrm{D_n})/\text{isotopy rel } \partial D \text{ point wise}) < \mathrm{Mod}(\Sigma_{0,n+1})$

$B_n \simeq \mathrm{Mod}(D_n)$

(minimal dilatation of n-braids)

$$\delta(D_n) := \min\{\lambda(\phi) \mid \phi \in \mathrm{Mod}(D_n),\ \text{pseudo-Anosov}\}$$

Clearly, $\delta(D_n) \geq \delta_{0,n+1}$

Question 1. *What is the value of $\delta(D_n)$?*

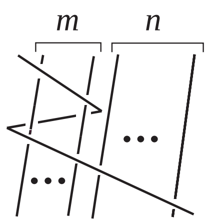


図 3 $\sigma_{m,n} \in B_{m+n+1}$

Theorem 3 (Hironaka-K (2006)). • $\sigma_{m,n}$ *is pA* $\iff |m-n| \geq 2$
• *When* $(m,n) = (g-1, g+1)$,

$$g \log \lambda(\sigma_{g-1,g+1}) < \log(2+\sqrt{3})$$
$$g \log \lambda(\sigma_{g-1,g+1}) \to \log(2+\sqrt{3}) \textit{ as } g \to \infty$$

Corollary 1 (HK (2006)). $\log \delta_{0,n} \asymp 1/n$

★ $\sigma_{g-1,g+1} \in B_{2g+1}$ has the smallest known dilatation (true for $g = 2, 3$)

For $m \geq 3$, $1 \leq p \leq m-1$,

$$T_{m,p} := (\sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_{m-1})^p \sigma_{m-1}^{-2} \in B_m$$

(e.g, $T_{6,1} = \sigma_1^2 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_5^{-2} = \sigma_1^2 \sigma_2 \sigma_3 \sigma_4 \sigma_5^{-1}$)

By forgetting the 1st strand of $T_{m,p}$, we can define $T'_{m,p} \in B_{m-1}$

Theorem 4 (KT2). *Let $g \geq 2$.*

(1) $\sigma_{g-1,g+1}$ *is conjugate to* $T'_{2g+2,2}$

(2) $S^3 \setminus \widehat{T_{2g+2,2}} \simeq$ *magic manifold* N, *where* $\widehat{b}$ *denotes the braided link of a braid* b

We can prove more (see [KT2])

- $T_{m,p}$ is pseudo-Anosov $\Longleftrightarrow$ $\gcd(m-1,p)=1$
- If $T_{m,p}$ is pseudo-Anosov, then $S^3 \setminus \widehat{T_{m,p}} \simeq N$

Remark 1 (potential candidates with the smallest dilatation (KT2))**.** *Pseudo-Anosov m-braids with the smallest known dilatation are of the form $T_{m,p}$ or $T'_{m+1,p}$. (True for $m \leq 8$.)*

★ The places where the braids $T_{m,p}$ live?

Thurston norm of hyperbolic 3-manifolds M

Thurston norm $\|\cdot\| : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$;

For an integral class $a \in H_2(M, \partial M; \mathbb{Z})$, define

$$\|a\| = \min_F \{|\chi(F)|\},$$

where the minimum is taken over all oriented surface F embedded in M, such that $a = [F]$ and F has no components of non-negative Euler characteristic.

★ The surface F which realizes this minimum is denoted by F_a.

★ The norm $\|\cdot\|$ defined on integral classes admits a unique continuous extension $\|\cdot\| : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$ which is linear on the ray through the origin.

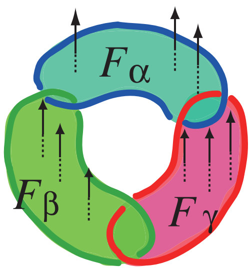
★ The unit ball U_M w.r.t to $\|\cdot\|$ is a compact, convex polyhedron.

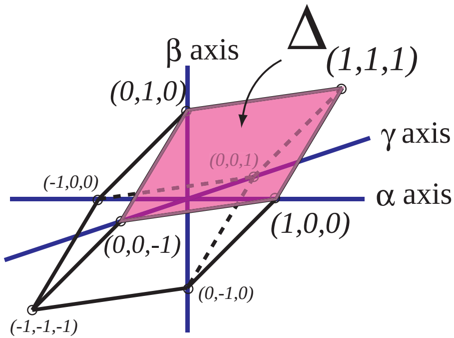
The places where the braids $T_{m,p}$ live

Consider the Thurston norm $\|\cdot\| : H_2(N, \partial N; \mathbb{R}) \to \mathbb{R}$

$\alpha := [F_\alpha]$, $\beta := [F_\beta]$, $\gamma := [F_\gamma] \in H_2(N, \partial N; \mathbb{Z})$

$\|\alpha\| = \|\beta\| = \|\gamma\| = 1$





Every top dimensional face Δ of ∂U_N is a fibered face

$C_\Delta :=$ a cone over Δ through 0

for any $^\forall a \in int(C_\Delta)$: integral class, the minimal representative F_a (i.e, $a = [F_a]$) becomes a fiber of a fibration of N

Take a particular fibered face

$$\Delta = \{(X, Y, Z) \mid X + Y - Z = 1,\ X \geq 0,\ Y \geq 0,\ X \geq Z,\ Y \geq Z\}.$$

- When $\gcd(m-1, p) = 1$, we can talk about the integral class, say $a_{m,p} \in H_2(N, \partial N; \mathbb{Z})$, associated to the monodromy $T_{m,p}$

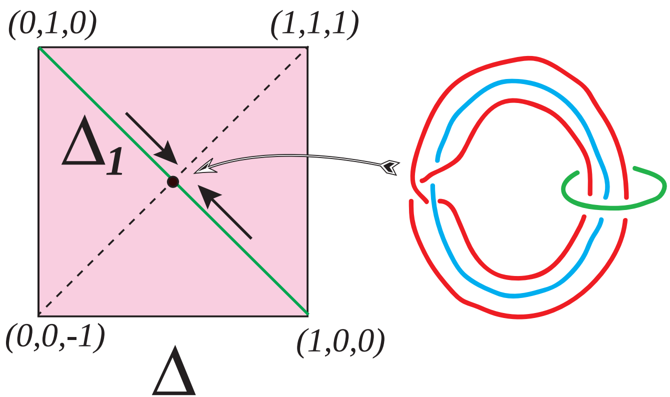
Where do the braids $T_{m,p}$ live? Answer (see [KT2])

- (the projective class) $\overline{a_{m,p}} \in \Delta_1 \subset \Delta$, where $\Delta_1 = \{(X, Y, 0) \in \Delta\}$

(Recall : the braid by forgetting the 1st strand of $T_{2g+2,2}$ is conjugate to $\sigma_{g-1,g+1}$)

$$\lim_{g \to \infty} \overline{a_{2g+2,2}} = (1/2, 1/2, 0) \sim (1, 1, 0)$$

★ The monodromy associated to $(1, 1, 0)$ is a 3-braid with the dilatation $2 + \sqrt{3}$. (Geometric proof of $g \log \lambda(\sigma_{g-1,g+1}) \to \log(2 + \sqrt{3})$ as $g \to \infty$)



Minimal dilatation $\delta_{g,n}$, $g > 1$

Theorem 5 (Tsai 2009)**.** *For any fixed* $g > 1$, $\log \delta_{g,n} \asymp \dfrac{\log n}{n}$.

★ This is in contrast with the cases $g = 0, 1$.

$^{\exists} c_g > 0$ such that

$$\frac{\log n}{c_g n} < \log \delta_{g,n} < \frac{c_g \log n}{n} \quad \left(\Longleftrightarrow \frac{1}{c_g} < \frac{n \log \delta_{g,n}}{\log n} < c_g\right)$$

★ What is the value of c_g?

(Examples by Tsai)

Given $g \geq 2$, ${}^{\exists}\{f_{g,n} : \Sigma_{g,n} \to \Sigma_{g,n}\}_{n \in \mathbb{N}}$ such that $\log \lambda(f_{g,n}) \asymp \frac{\log n}{n}$

$$\lim_{n\to\infty} \frac{n \log \lambda(f_{g,n})}{\log n} = 2(2g+1). \text{ (So } \limsup_{n\to\infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2(2g+1).)$$

See [KT0]

Thm A. [KT0] $^{\exists}\infty$ ly many g's such that if we fix such a g, then

$$\limsup_{n\to\infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2.$$

Thm B. [KT0] $^{\forall}g \geq 2$, $^{\exists}\{n_i\}_{i=0}^{\infty}$ with $n_i \to \infty$ such that

$$\limsup_{i\to\infty} \frac{n_i \log \delta_{g,n_i}}{\log n_i} \leq 2.$$

Sketch of proof of Theorem B

(useful formula) Let $a = (x, y, z) \in int(C_\Delta)$ be a primitive fibered class.

(1) $\|a\| = x + y - z$.

(2) the number of the boundary components of the mini. representative $F_a = F_{(x,y,z)}$ is equal to

$$\gcd(x, y + z) + \gcd(y, z + x) + \gcd(z, x + y).$$

(3) the dilatation $\lambda_{(x,y,z)}$ is the largest real root of

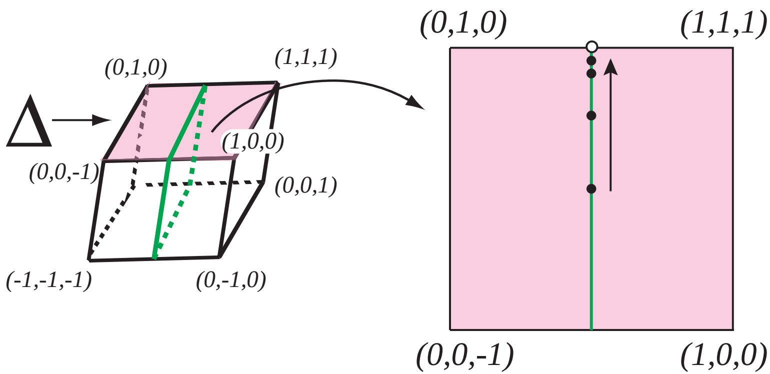
$$f_{(x,y,z)}(t) = t^{x+y-z} - t^x - t^y - t^{x-z} - t^{y-z} + 1.$$

- For $g \geq 2$ and $p \geq 0$, take a fibered class

$$a_{(g,p)} = (p+g+1, 2p+1, p-g) \in int(C_\Delta).$$

If $a_{(g,p)}$ is primitive, then $F_{a_{(g,p)}} \simeq \Sigma_{g,2p+4}$.

- $^\forall g \geq 2$, $^\exists\{a_{(g,p_i)}\}_{i=0}^{\infty}$ such that $a_{(g,p_i)}$ is primitive, $p_i \to \infty$, and $\overline{a_{(g,p_i)}} \to (1/2, 1, 1/2) \in \partial\Delta$ as $i \to \infty$



The next proposition implies Theorem B.

Proposition 1.

$$\lim_{i\to\infty} \frac{\|a_{(g,p_i)}\| \log \lambda(a_{(g,p_i)})}{\log \|a_{(g,p_i)}\|} = 2.$$

Remark. (Fried, S. Matsumoto, McMullen) Ω: a fibered face of a hyperbolic fibered 3-manifold.

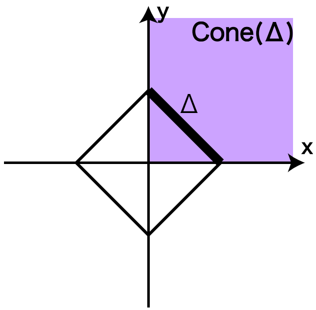
(1) ent : $int(C_\Omega(\mathbb{Z})) \to \mathbb{R}$ admits a continuous extension

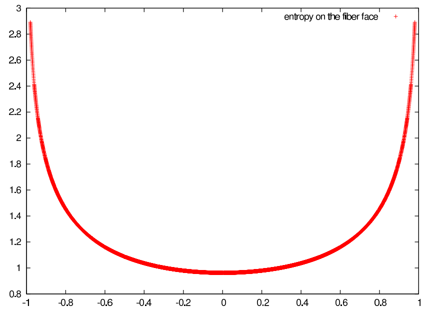
$$\text{ent} : int(C_\Omega) \to \mathbb{R}$$

(2) $\text{Ent}(\cdot) = \|\cdot\| \, \text{ent}(\cdot) : int(C_\Omega) \to \mathbb{R}$ is constant on each ray through 0.

(3) $\text{ent}|_{int(\Omega)} : int(\Omega) \to \mathbb{R}$ is strictly convex, and if $a \in int(\Omega)$ goes to $\partial\Omega$, then $\text{ent}(a) \to \infty$.

So, $\text{ent}|_{int(\Omega)}$ has the minimum at a unique point.





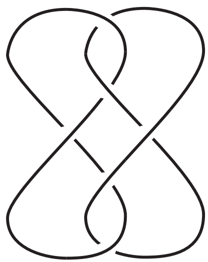
Minimal dilatation $\delta_{1,n}$

★ $\log \delta_{1,n} \asymp 1/n$ (Tsai)

Theorem 6 (KKT).

$$\limsup_{n\to\infty} |\chi(\Sigma_{1,n})| \log \delta_{1,n} \leq 2 \log \delta(D_4) \approx 1.6628$$

★ We study the monodromies of fibrations of the whitehead link exterior $\simeq N(1)$. R.H.S is the minimum of $\mathrm{ent}|_{int(\Omega)}$ for $N(1)$.



How can one get $\|\cdot\|$ and $\mathrm{ent}(\cdot)$ for the Dehn filling $N(r)$?

★ For the computation of the Thurston norm and the entropy function of $N(r)$, use a natural injection $\iota : H_2(N(r), \partial N(r)) \to H_2(N, \partial N(r))$ whose image is $S(r) := \{(X, Y, Z) \in H_2(N, \partial N) \mid r = \frac{Z+X}{-Y})\}$, see [KKT]

Minimal dilatation $\delta_g := \delta_{g,0}$

★ $\log \delta_g \asymp 1/g$ (Penner 1991)

Theorem 7 (Hironaka, Aaber-Dunfield, KT1)**.**

$$\limsup_{g\to\infty} |\chi(\Sigma_{g,0})| \log \delta_g \leq 2\log(\tfrac{3+\sqrt{5}}{2}) = 2\log\delta(D_3)$$

★ Hironaka $\cdots$ $N(\frac{1}{-2}) \simeq S^3 \setminus \widehat{\sigma_1\sigma_2^{-1}}$

★ AD, KT $\cdots$ $N(\frac{3}{-2}) \simeq S^3 \setminus (-2,3,8)$-pretzel link

R.H.S is the minimum of $\mathrm{ent}|_{int(\Omega)}$ for both $N(\frac{1}{-2})$ and $N(\frac{3}{-2})$

Aside: infinitely many twins

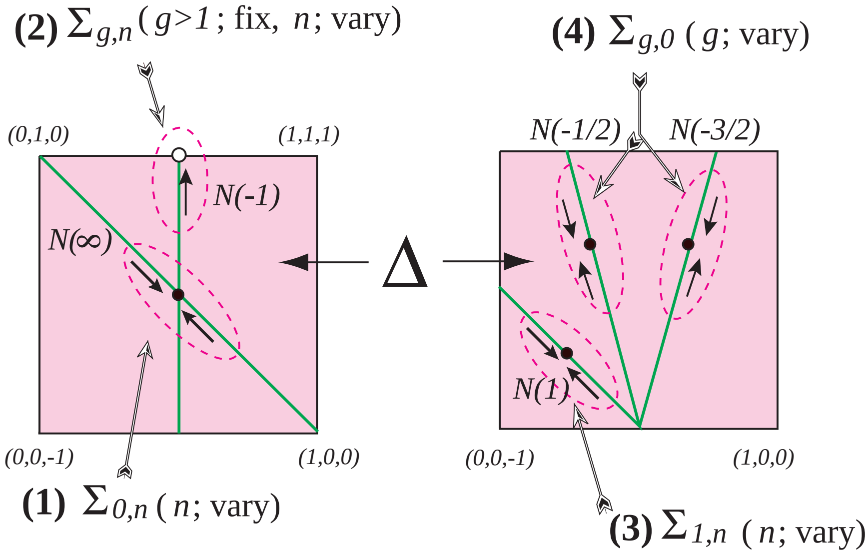
★ $N(\frac{1}{-2})$ and $N(\frac{3}{-2})$ are twins. (They are entropy equivalent)

Hyperbolic fibered 3-manifolds M and M' are entropy equivalent

$\Longrightarrow$ the minimum of $\mathrm{ent}|_{int(\Omega)}$ for M is equal to that for M'.

★ $N(r)$ and $N(-r-2)$ are entropy equivalent for "almost all" $r \in \mathbb{Q}$, see [KKT]

Places where pseudo-Anosovs defined on $\Sigma_{g,n}$ with the smallest known dilatation live



(1) $\log \delta_{0,n} \asymp 1/n$ (2) For any fixed $g \geq 2$, $\log \delta_{g,n} \asymp \dfrac{\log n}{n}$

(3) $\log \delta_{1,n} \asymp 1/n.$ (4) $\log \delta_g \asymp 1/g$

Question 2. *Let* $f_{g,n} : \Sigma_{g,n} \to \Sigma_{g,n}$ *be a pseudo-Anosov homeo. which achieves* $\delta_{g,n}$*. It is true that* $\mathbb{T}(f_{g,n}) \simeq N$*, or* $\mathbb{T}(f_{g,n})$ *is the manifold obtained from* N *by Dehn filling cusps along a fiber of* N*?*