Parametrization of PSL(n,C)-representations of surface group II

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Review of part I

S : a compact orientable surface (genus g, $|\partial S| = b$, $\chi(S) < 0$)

$X_{PSL}(S)$: the PSL$(2, \mathbb{C})$-character variety of S

In part I, we have constructed a map

$$\mathbb{C}^{6g-6+2b} \to X_{PSL}(S)$$

essentially considering the action of PSL$(2, \mathbb{C})$ on $\mathbb{C}P^1$.

In part II, we will construct PGL$(n, \mathbb{C})$-representations using the action on the *flag manifold* $\mathcal{F}_n$ based on a work of Fock and Goncharov. This is a joint work with Xin Nie.

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$\mathrm{PGL}(n,\mathbb{C}) := \mathrm{GL}(n,\mathbb{C})/\mathbb{C}^*$,

$\mathrm{PSL}(n,\mathbb{C}) := \mathrm{SL}(n,\mathbb{C})/\{\xi \mid \xi^n = 1\}$.

These are isomorphic but $\mathrm{PGL}(n,\mathbb{C})$ is convenient for our arguments.

Flag

A (full) *flag* in $\mathbb{C}^n$ is a sequence of subspaces

$$\{0\} = V^0 \subsetneq V^1 \subsetneq V^2 \subsetneq \cdots \subsetneq V^n = \mathbb{C}^n$$

We denote the set of all flags by $\mathcal{F}_n$. $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{PGL}(n,\mathbb{C})$ act on $\mathcal{F}_n$ from the left.

Fact $\mathcal{F}_n \cong \mathrm{GL}(n,\mathbb{C})/B$ *where* $B = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ O & & * \end{pmatrix} \right\}$

We represent $X \in \mathrm{GL}(n, \mathbb{C})$ by n column vectors:

$$X = \begin{pmatrix} x^1 & x^2 & \cdots & x^n \end{pmatrix}. \quad (x^i \in \mathbb{C}^n)$$

An upper triangular matrix acts as

$$X \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ & \ddots & \vdots \\ O & & b_{nn} \end{pmatrix} = \begin{pmatrix} b_{11}x^1 & b_{12}x^1 + b_{22}x^2 & \ldots & b_{1n}x^1 + \cdots + b_{nn}x^n \end{pmatrix}$$

By setting $X^i = \mathrm{span}_{\mathbb{C}}\{x^1, \ldots, x^i\}$, we obtain a map

$$\mathrm{GL}(n, \mathbb{C})/B \to \mathcal{F}_n.$$

This is bijective.

We call an element of $\mathcal{AF}_n := \mathrm{GL}(n, \mathbb{C})/U$ an *affine flag* where

$$U = \left\{ \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ O & & 1 \end{pmatrix} \right\}.$$ (∃ a projection $\mathcal{AF}_n \to \mathcal{F}_n$.)

Generic k-tuples of flags

$X_1, \ldots, X_k$: flags

Take a representative $X_i = (x_i^1 \cdots x_i^n) \in \mathrm{GL}(n, \mathbb{C})$

$(X_1, \ldots, X_k)$ is *generic* if

$$\det(x_1^1 \ldots x_1^{i_1} x_2^1 \ldots x_2^{i_2} \ldots x_k^1 \ldots x_k^{i_k}) \neq 0$$

for any $0 \leq i_1, \ldots, i_k \leq n$ satisfying $i_1 + i_2 + \cdots + i_k = n$.

The genericity does not depend on the choices of the matrices X_i.

Moreover for $X_1, \ldots, X_k \in \mathcal{AF}_n$, the determinant is a well-defined complex number. Denote it by $\det(X_1^{i_1} X_2^{i_2} \ldots X_k^{i_k})$.

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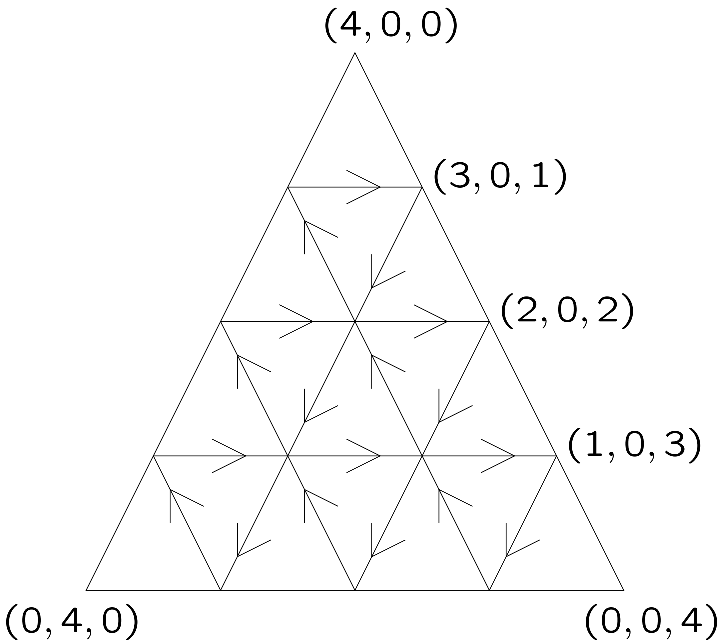
n-triangulation

A triple (i, j, k) of integers satisfying $0 \leq i, j, k \leq n$ and $i + j + k = n$ corresponds to an integral point of a triangle.



We give a ‘counter-clockwise’ orientation to each interior edges of the n-triangulation.

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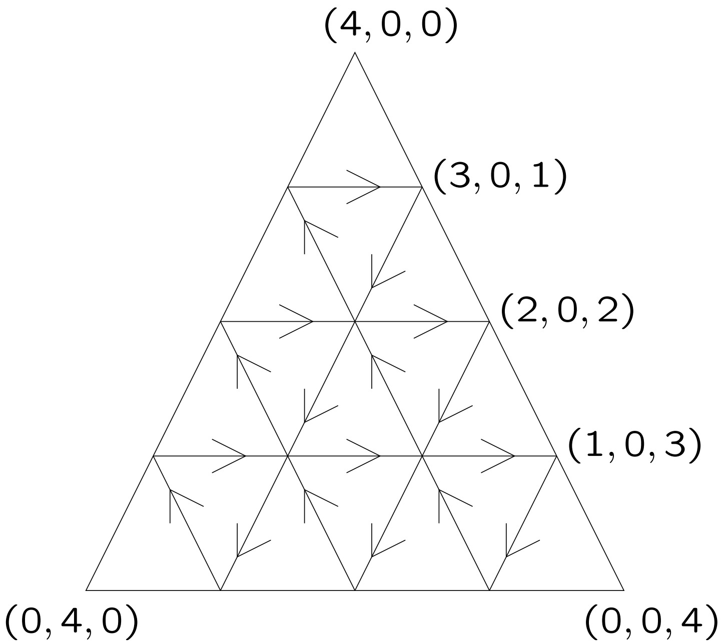

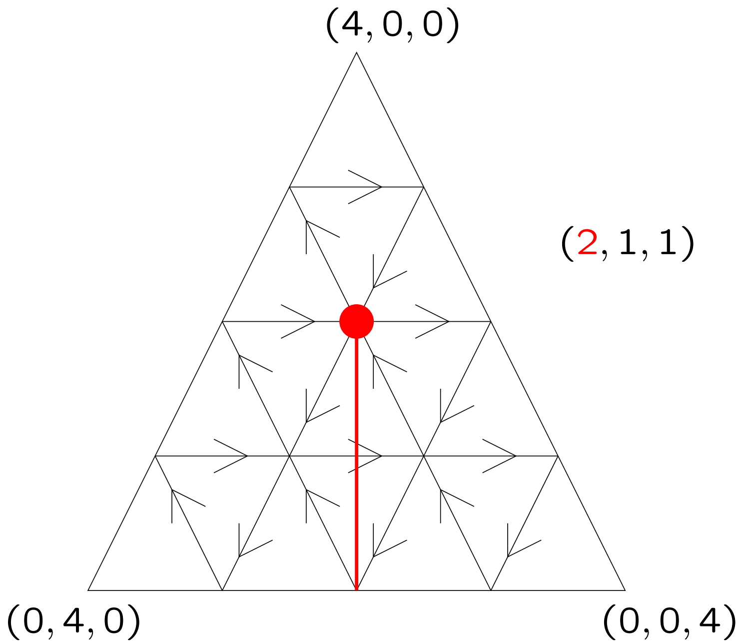

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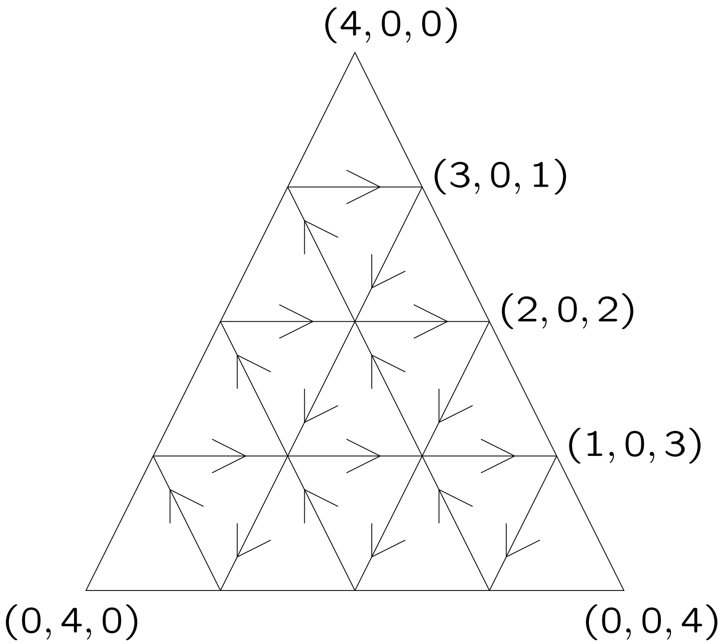

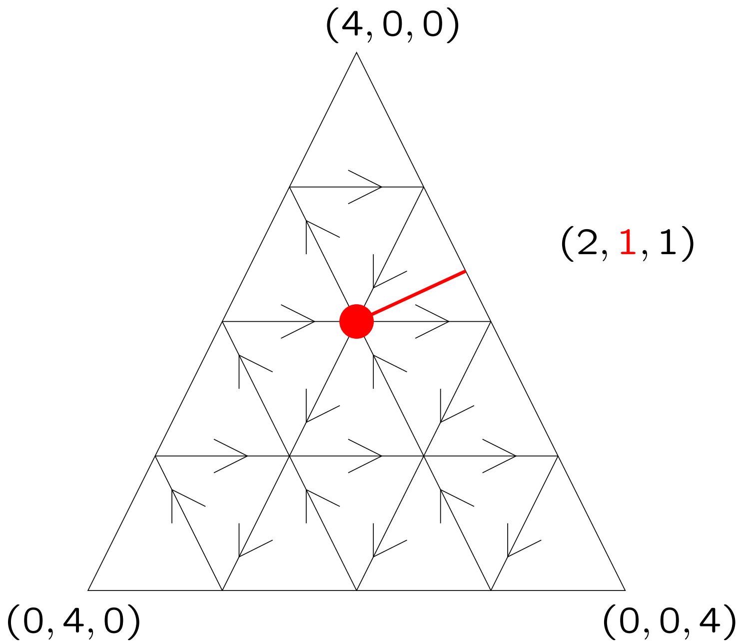

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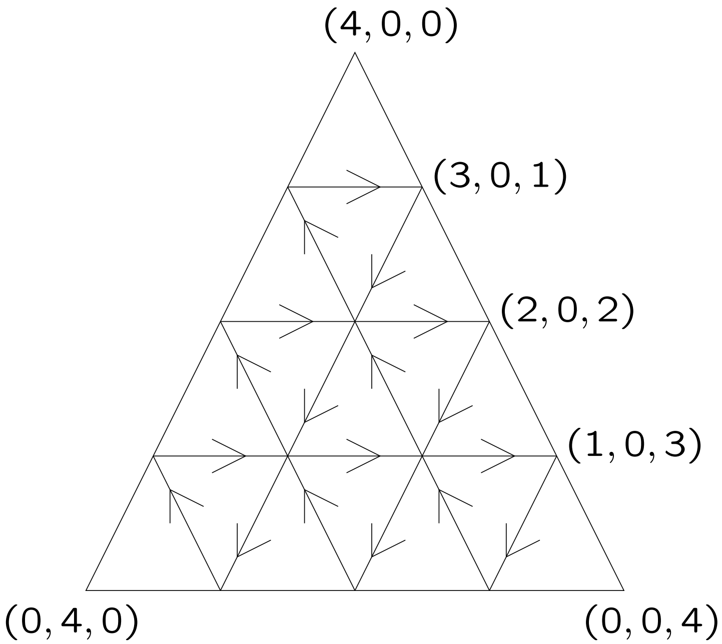

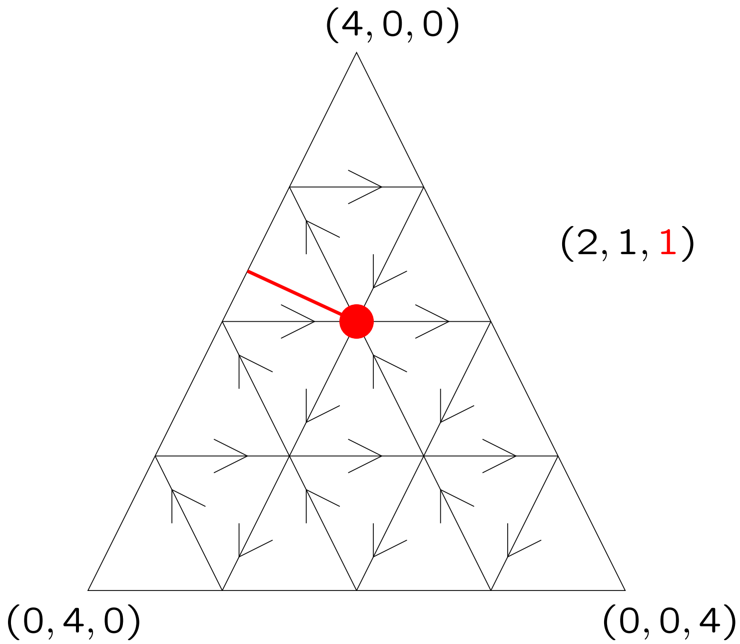

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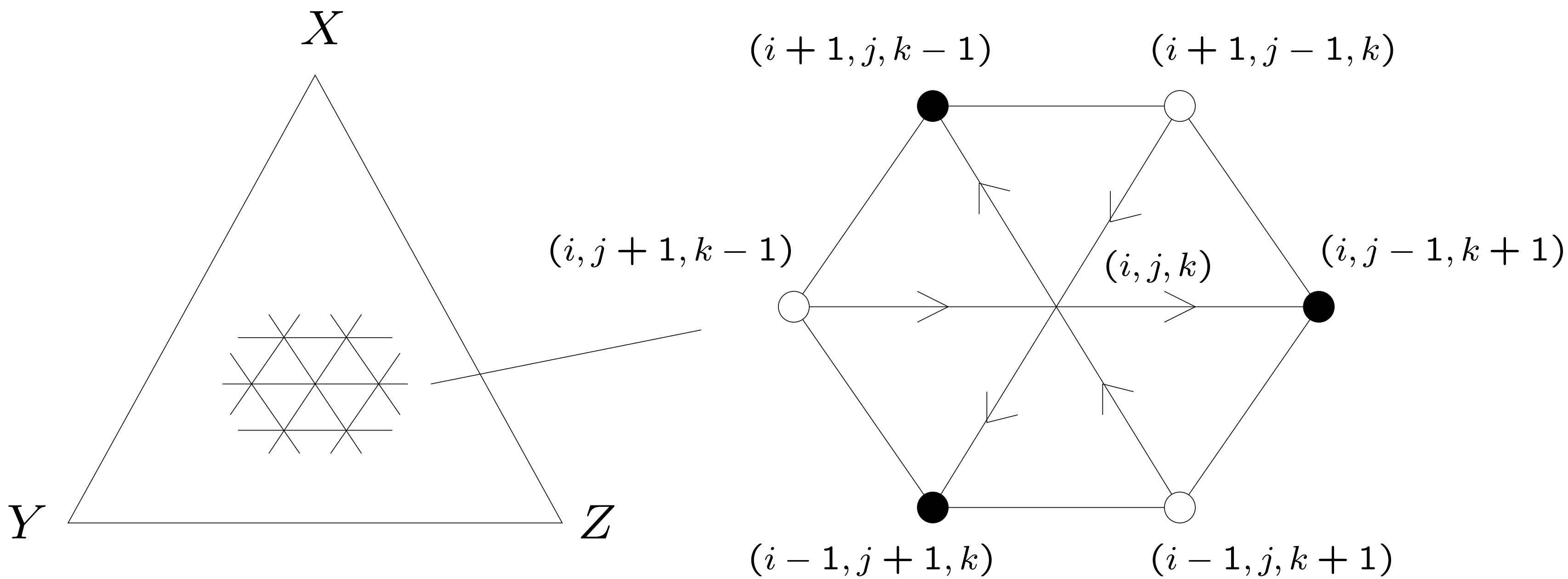
Definition of the triple ratio

$X, Y, Z \in \mathcal{F}_n$: a generic triple of flags

We fix lifts of X, Y, Z to $\mathcal{AF}_n$ and denote $\Delta^{i,j,k} := \det(X^i Y^j Z^k)$.



The *triple ratio* is defined (for $1 \leq i, j, k \leq n-1$) by

$$T^{i,j,k}(X,Y,Z) := \frac{\Delta^{i+1,j,k-1}\Delta^{i-1,j+1,k}\Delta^{i,j-1,k+1}}{\Delta^{i+1,j-1,k}\Delta^{i,j+1,k-1}\Delta^{i-1,j,k+1}}.$$

This does not depend on the choice of the representatives.

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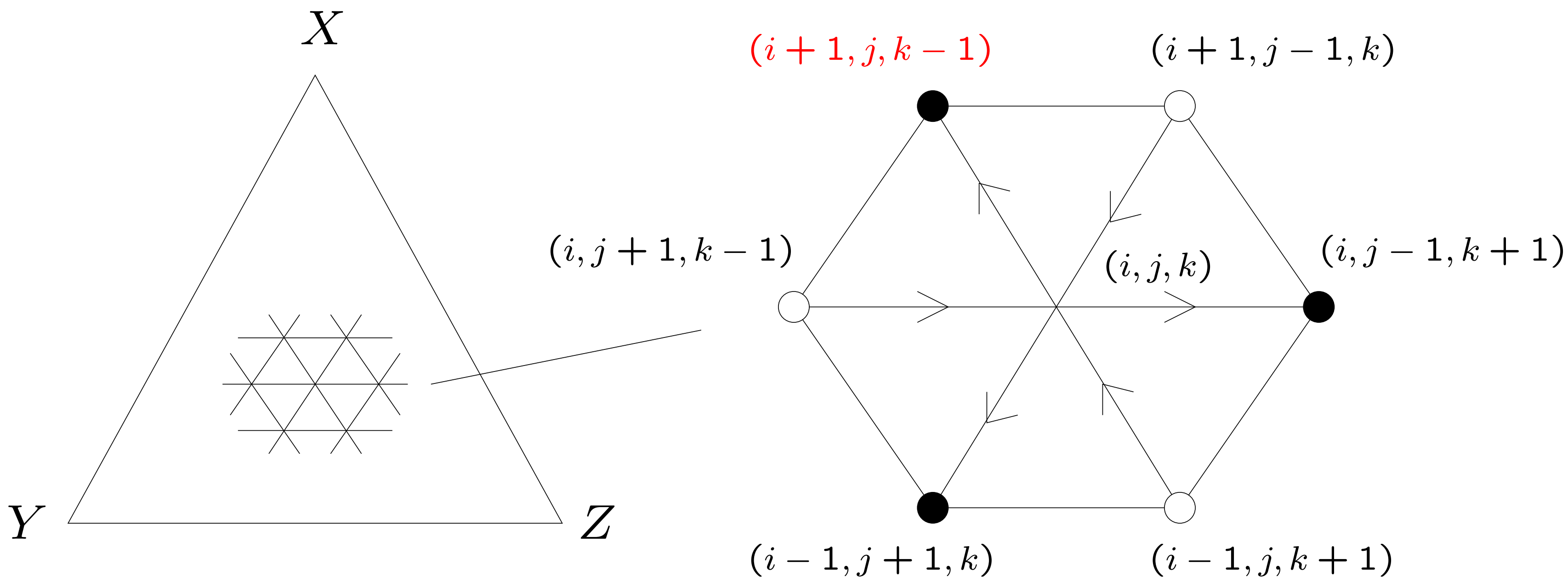

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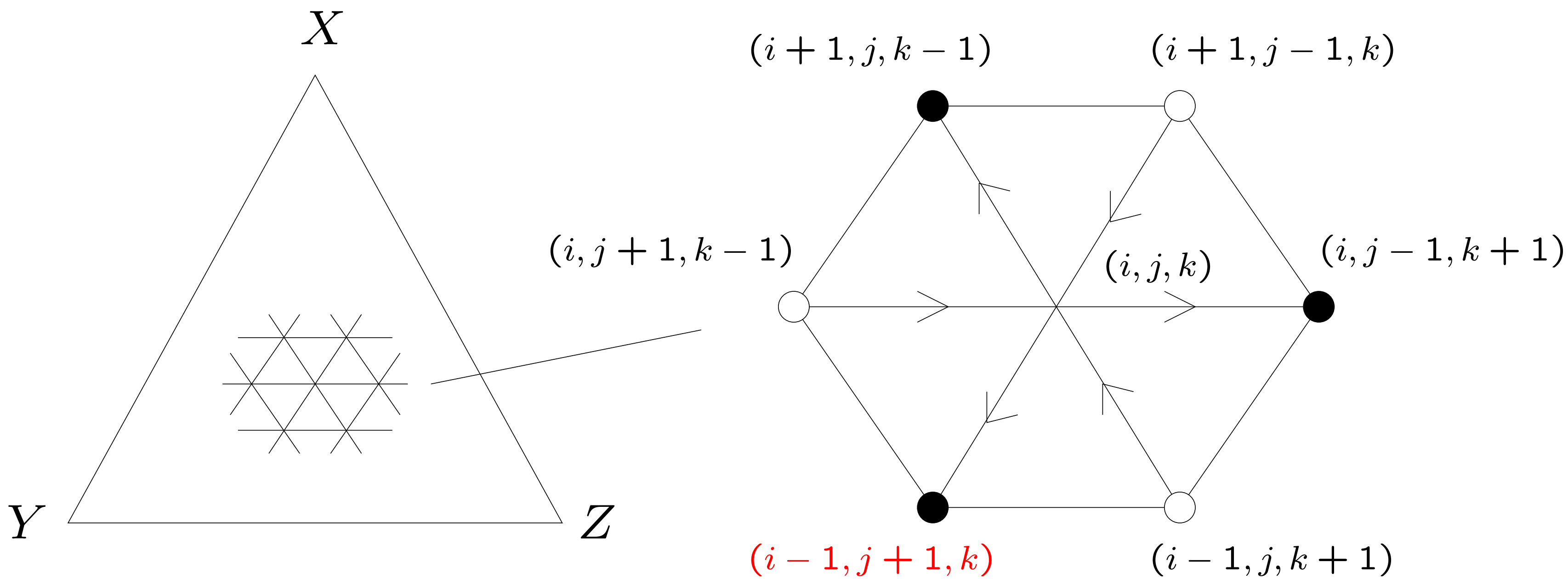

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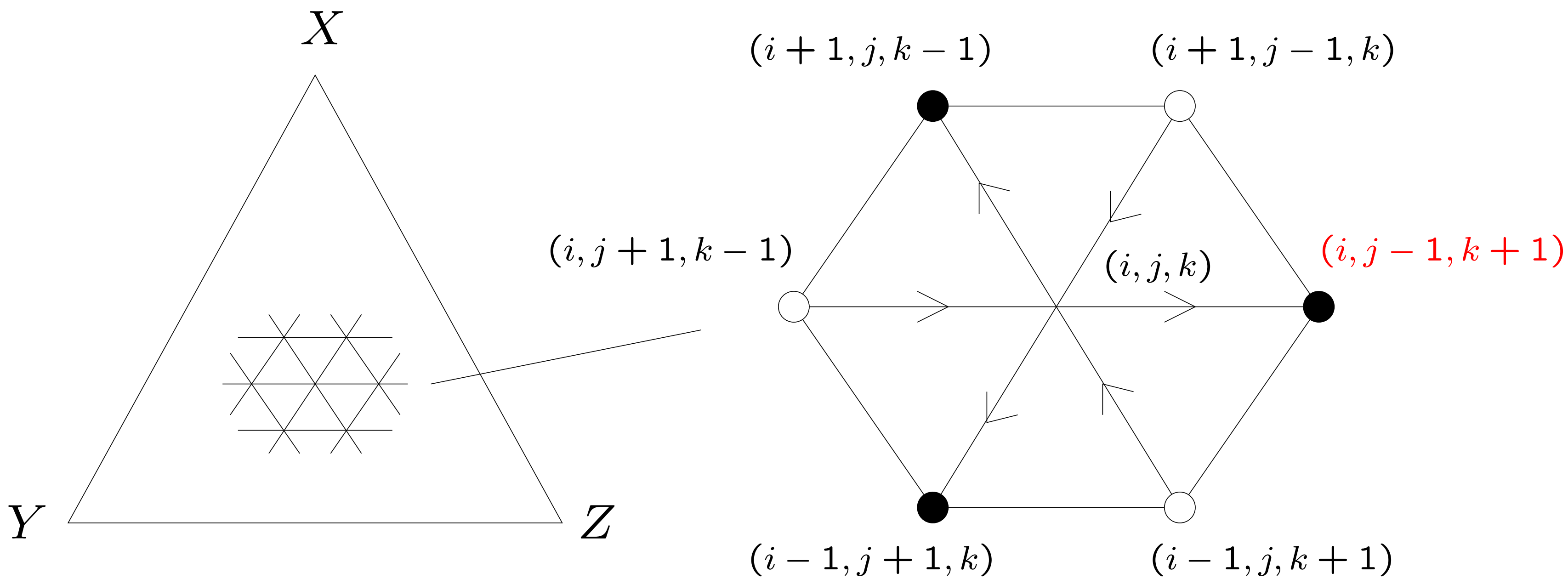

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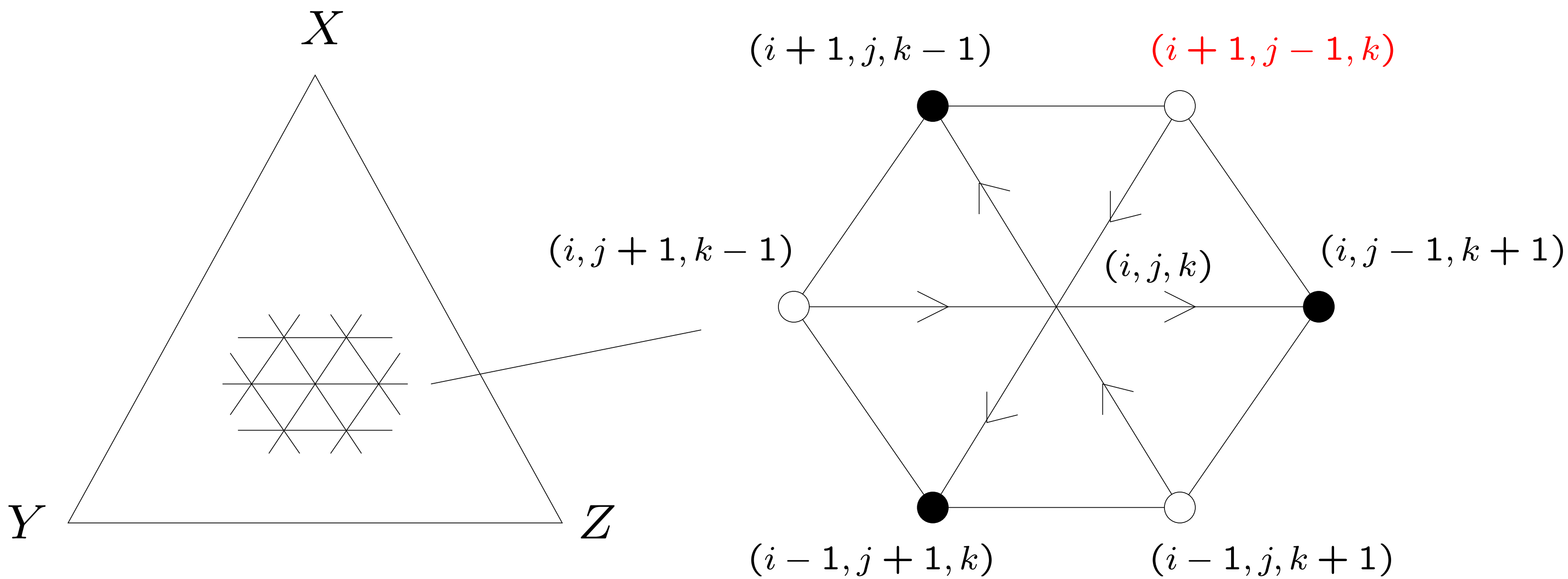

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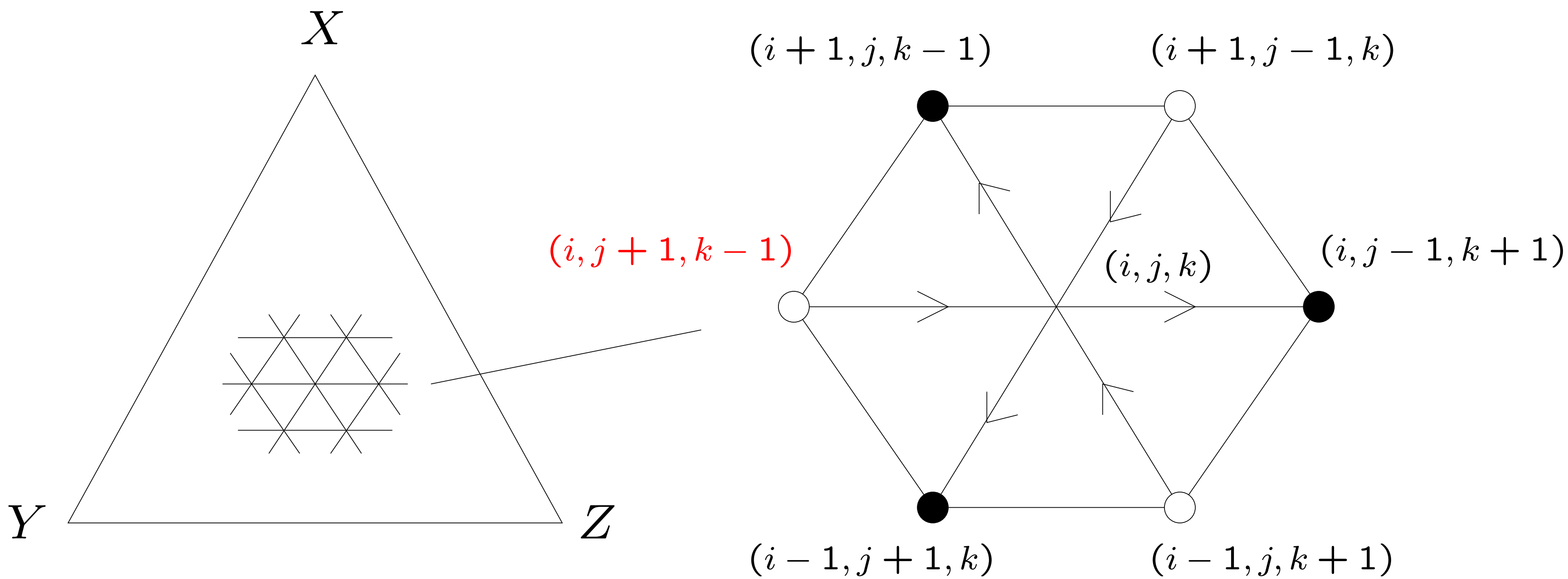

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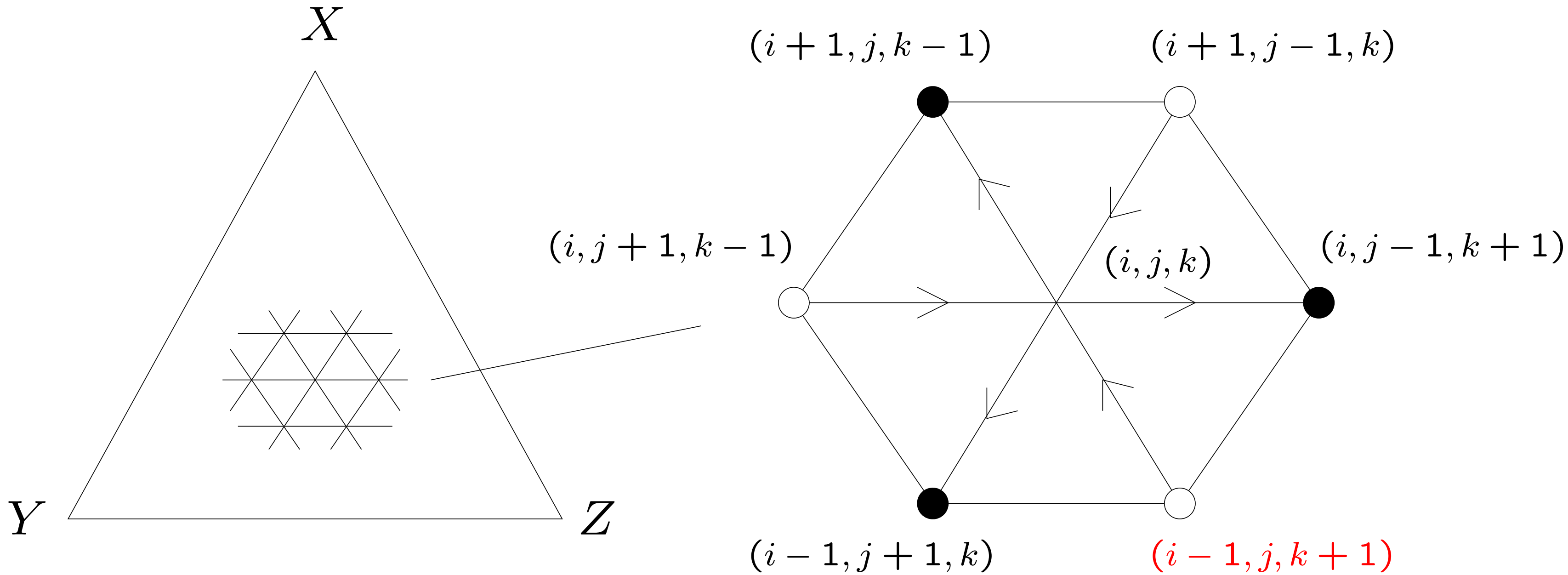
We fix lifts of X, Y, Z to $\mathcal{A}\mathcal{F}_n$ and denote $\Delta^{i,j,k} := \det(X^i Y^j Z^k)$.



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Facts

For a generic triple $X, Y, Z \in \mathcal{F}_n$ and $A \in \mathrm{PGL}(n, \mathbb{C})$, we have

$$T^{i,j,k}(X, Y, Z) = T^{j,k,i}(Y, Z, X) = T^{k,i,j}(Z, X, Y),$$

$$T^{i,j,k}(X, Y, Z) = T^{i,j,k}(AX, AY, AZ).$$

If we let

$$\mathrm{Conf}_k(\mathcal{F}_n) = \mathrm{GL}(n, \mathbb{C}) \backslash \{(X_1, \ldots, X_k) \mid X_1, \ldots, X_k : \text{generic}\},$$

$T^{i,j,k}$ are invariants of $\mathrm{Conf}_3(\mathcal{F}_n)$. Moreover,

Theorem (Fock-Goncharov)

A point of $\mathrm{Conf}_3(\mathcal{F}_n)$ is completely determined by the $\frac{(n-1)(n-2)}{2}$ triple ratios.

We will give a sketch of a proof.

Prop 1

Let (X, Y, Z) be a generic triple of $\mathcal{F}_n$. Then there exists a unique $A \in \mathrm{GL}(n, \mathbb{C})$ and upper triangular matrices B_1, B_2, B_3 up to scalar multiplication s.t.

$$AXB_1 = \begin{pmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{pmatrix}, \quad AYB_2 = \begin{pmatrix} O & & 1 \\ & ⋰ & \\ 1 & & O \end{pmatrix},$$

$$AZB_3 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & O \\ \vdots & & \ddots & \\ 1 & * & & 1 \end{pmatrix}.$$

(Thus the lower triangular part of AZB_3 gives a set of complete invariants of the configuration of generic triples of flags.)

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Prop 2

The lower triangular part of AXB_3 is uniquely determined by the triple ratios $T^{i,j,k}(X,Y,Z)$

From **Prop 1** and **2**, we obtain the Fock-Goncharov's thm.

E.g. When $n=3$, let $T = T^{1,1,1}(X,Y,Z)$, then

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & T+1 & 1 \end{pmatrix}$$

When $n=4$, let $T^{ijk} = T^{i,j,k}(X,Y,Z)$, then

$$I_4, \quad C_4, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & T^{121}+1 & 1 & 0 \\ 1 & (T^{211}+1)T^{121}+1 & (T^{112}+1)T^{211}+1 & 1 \end{pmatrix}.$$

Actually we can construct $A \in$ PGL in **Prop 1** explicitly.

Lem For a generic triple of flags (X, Y, Z), there exists a unique element $A \in \mathrm{GL}(n, \mathbb{C})$ such that

$$AX = \begin{pmatrix} x'_{11} & \cdots & x'_{1n} \\ & \ddots & \vdots \\ O & & x'_{nn} \end{pmatrix}, \quad AY = \begin{pmatrix} O & & y'_{1n} \\ & \iddots & \vdots \\ y'_{n1} & \cdots & y'_{nn} \end{pmatrix}, \quad Az^1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Proof We need to find a matrix $A = (a_{ij})$ satisfying

$$a_{i1}x_1^j + a_{i2}x_2^j + \cdots + a_{in}x_n^j = 0, \quad (j < i)$$

$$a_{i1}y_1^j + a_{i2}y_2^j + \cdots + a_{in}y_n^j = 0, \quad (j < n - i + 1)$$

$$a_{i1}z_1^1 + a_{i2}z_2^1 + \cdots + a_{in}z_n^1 = 1.$$

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\begin{aligned}
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\end{aligned}
$$

This system of linear equations is equivalent to:

$$
\begin{pmatrix}
x_1^1 & \ldots & x_n^1\\
\vdots & & \vdots\\
x_1^{i-1} & \ldots & x_n^{i-1}\\
y_1^1 & \ldots & y_n^1\\
\vdots & & \vdots\\
y_1^{n-i} & \ldots & y_n^{n-i}\\
z_1^1 & \ldots & z_n^1
\end{pmatrix}
\begin{pmatrix} a_{i1}\\ \vdots\\ a_{in} \end{pmatrix}
=
\begin{pmatrix} 0\\ \vdots\\ 0\\ 1 \end{pmatrix}. \quad (i = 1, \ldots, n)
$$

By genericity, the above $n \times n$-matrix is invertible, thus there exists a unique $A \in M(n, \mathbb{C})$. $\square$

Cor A Let $X, Y \in \mathcal{F}_n$ and $z \in \mathbb{C}P^{n-1}$ be a generic triple, and $X', Y' \in \mathcal{F}_n$ and $z' \in \mathbb{C}P^{n-1}$ another generic triple. Then there exists a unique matrix $A \in \mathrm{PGL}(n, \mathbb{C})$ s.t.

$$AX = X', \quad AY = Y', \quad Az = z'.$$

Proof Since there exist unique A_1 and A_2 in $\mathrm{PGL}(n, \mathbb{C})$ s.t.

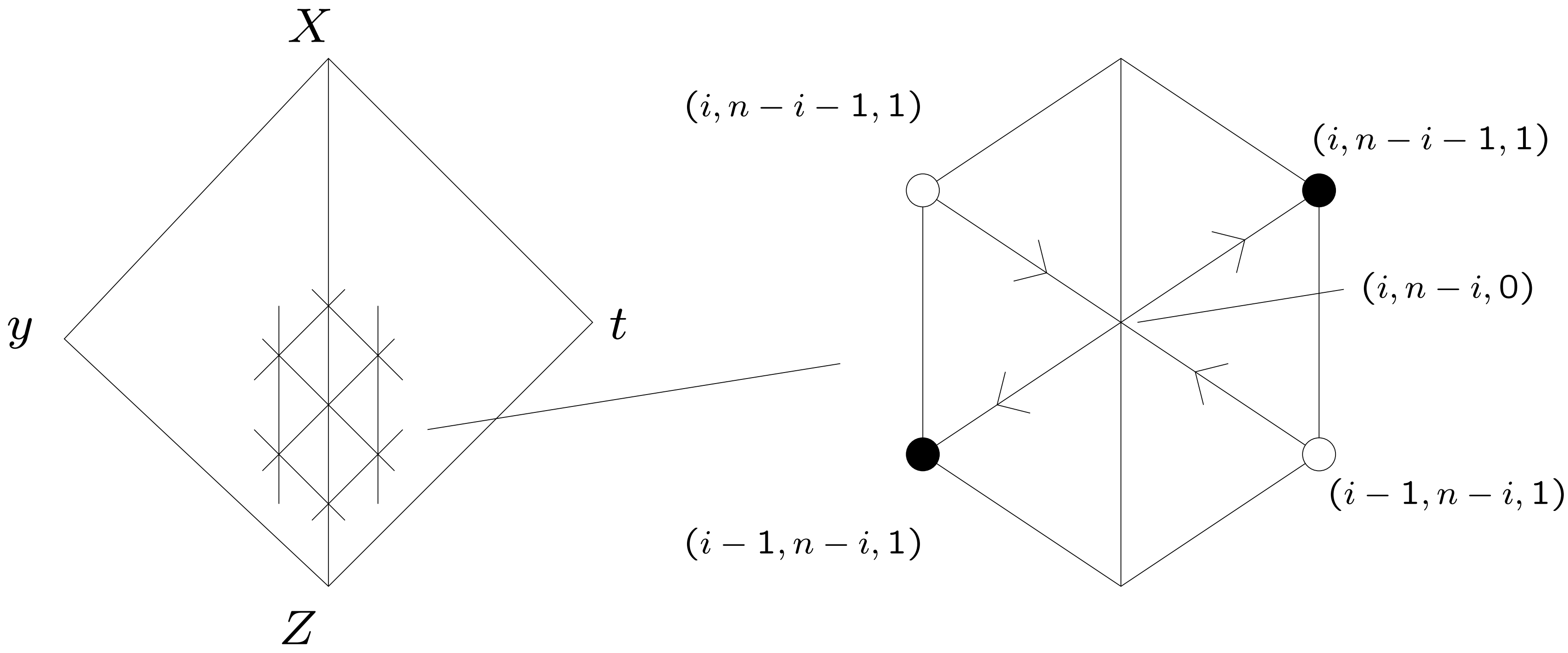
$$X \xrightarrow[A_1]{} I_n \xleftarrow[A_2]{} X', \quad Y \xrightarrow[A_1]{} C_n \xleftarrow[A_2]{} Y', \quad z \xrightarrow[A_1]{} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \xleftarrow[A_2]{} z'.$$

Put $A = A_2^{-1} A_1$. $\square$

Cor B Let $X, Y \in \mathcal{F}_n$ and $z \in \mathbb{C}P^{n-1}$ be a generic triple. For any $\frac{(n-1)(n-2)}{2}$ non-zero complex numbers $\{T^{i,j,k}\}$, there exists a unique $Z \in \mathcal{F}_n$ s.t. $Z^1 = z$ and $T^{i,j,k}(X, Y, Z) = T^{i,j,k}$.

Definition of the edge function

$X, Z \in \mathcal{AF}_n$: affine flags, $\quad y, t \in \mathbb{C}^n$: vectors



We define the *edge function* for $i = 1, \dots, n-1$

$$\delta^i(X, y, Z, t) = -\frac{\Delta^{i,n-i-1,1}(X, Z, t)\Delta^{i-1,n-i,1}(X, Z, y)}{\Delta^{i-1,n-1,1}(X, Z, t)\Delta^{i,n-i-1,1}(X, Z, y)}.$$

This is well-defined for $X, Z \in \mathcal{F}_n$ and $y, t \in \mathbb{C}P^{n-1}$.

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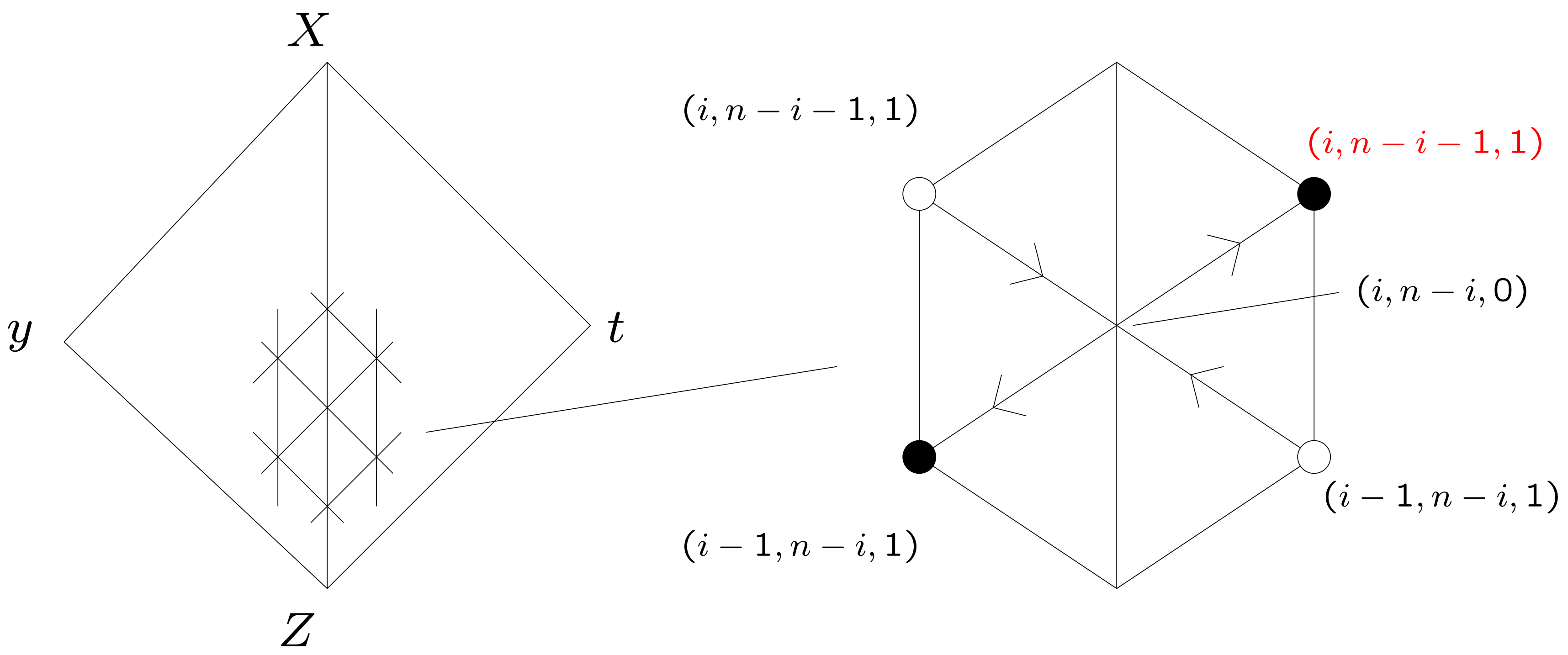

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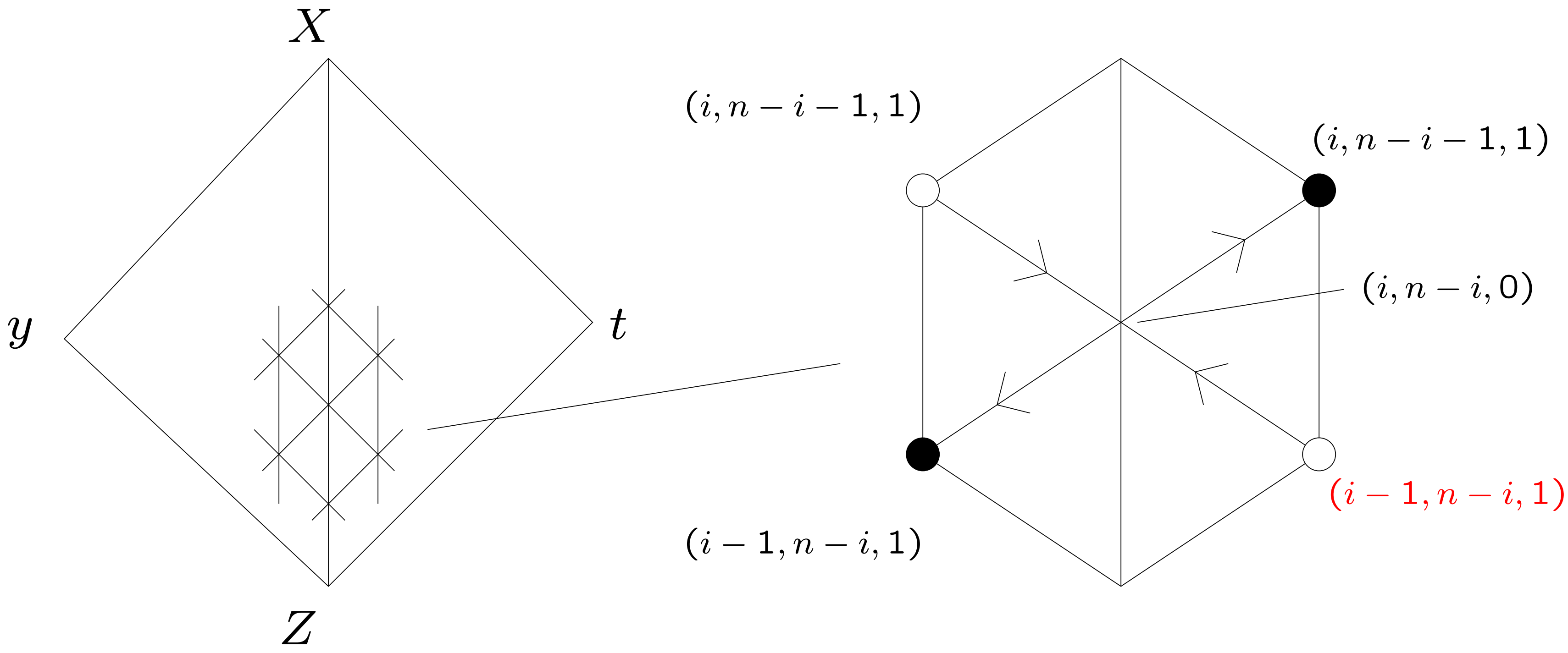

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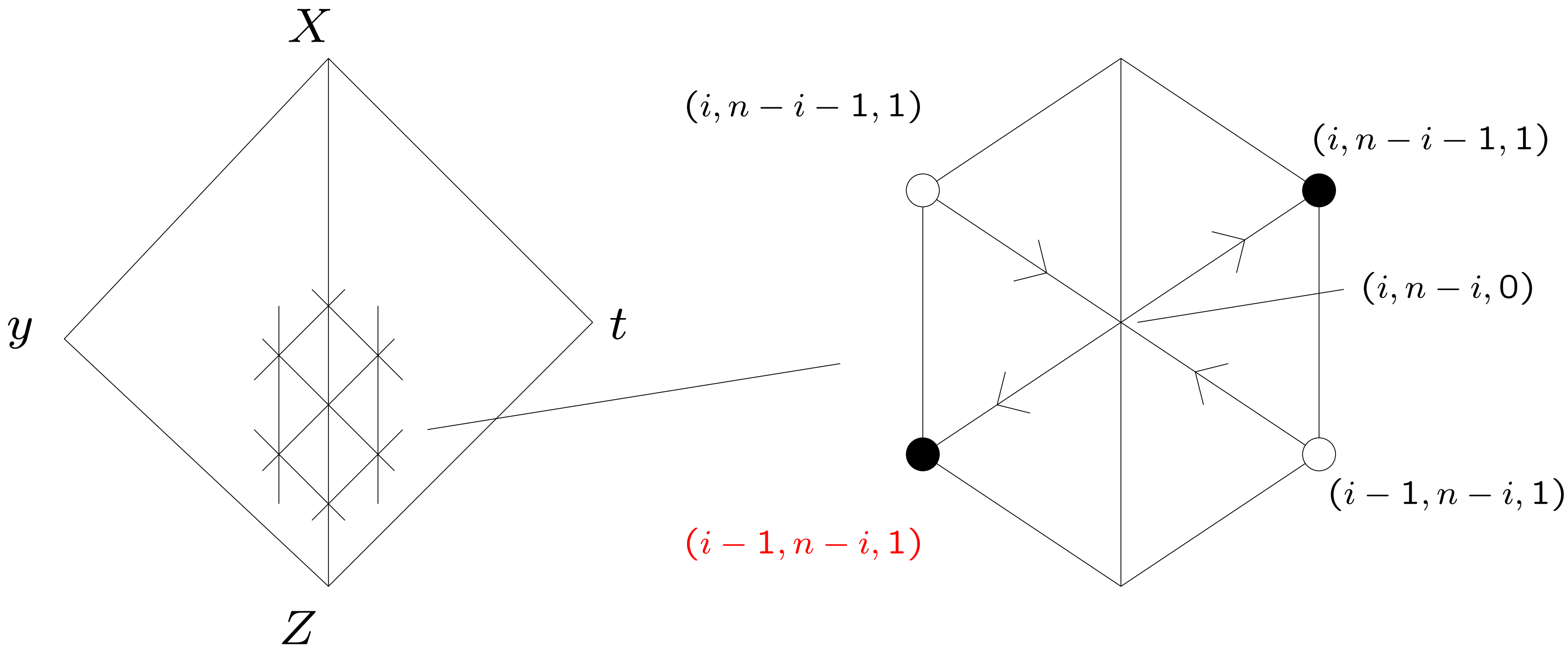

We define the *edge function* for $i = 1, \ldots, n-1$

$$\delta^i(X, y, Z, t) = -\frac{\Delta^{i,n-i-1,1}(X, Z, t)\Delta^{i-1,n-i,1}(X, Z, y)}{\Delta^{i-1,n-1,1}(X, Z, t)\Delta^{i,n-i-1,1}(X, Z, y)}.$$

This is well-defined for $X, Z \in \mathcal{F}_n$ and $y, t \in \mathbb{C}P^{n-1}$.

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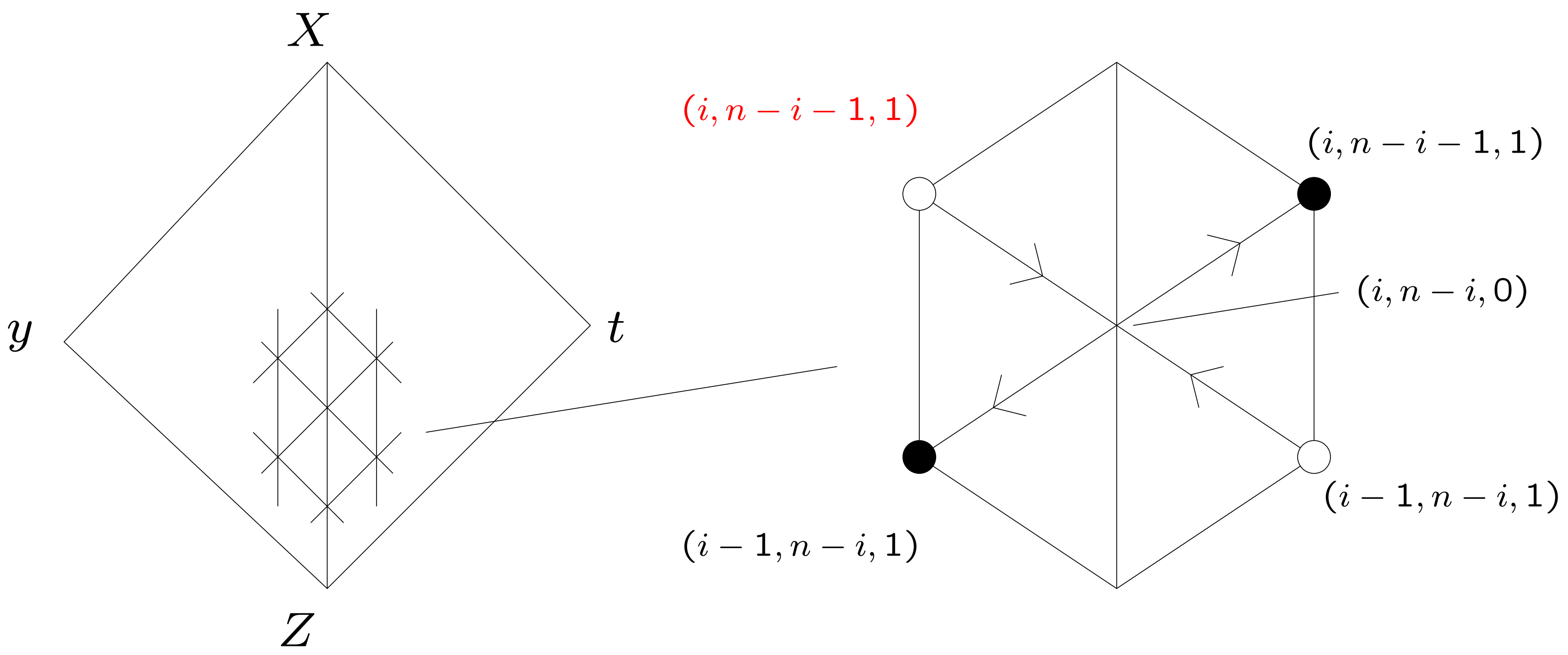

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This is well-defined for $X, Z \in \mathcal{F}_n$ and $y, t \in \mathbb{C}P^{n-1}$.

For a quadruple $X, Y, Z, T \in \mathcal{F}_n$, we simply denote

$$\delta^i(X, Y, Z, T) := \delta^i(X, Y^1, Z, T^1).$$

This satisfies

$$\delta^i(AX, AY, AZ, AT) = \delta^i(X, Y, Z, T).$$

Thus they are functions on $\mathrm{Conf}_4(\mathcal{F}_n)$.

For (X, Y, Z, T), we have $2 \times \frac{(n-1)(n-2)}{2}$ triple ratios from (X, Y, Z) and (X, Z, T) and $(n-1)$ edge functions.

Theorem (Fock-Goncharov)

These $(n-1)(n-2) + (n-1) = (n-1)^2$ invariants completely determine a point of $\mathrm{Conf}_4(\mathcal{F}_n)$.

Lem C Let $X, Z \in \mathcal{F}_n$ and $y \in \mathbb{C}P^{n-1}$. For any $d_1, \dots, d_{n-1} \in \mathbb{C}^*$, there exits a unique $t \in \mathbb{C}P^{n-1}$ s.t.

$$\delta^i(X, y, Z, t) = d_i. \quad (i = 1, \dots, n-1)$$

Proof By **Cor A**, we can assume that

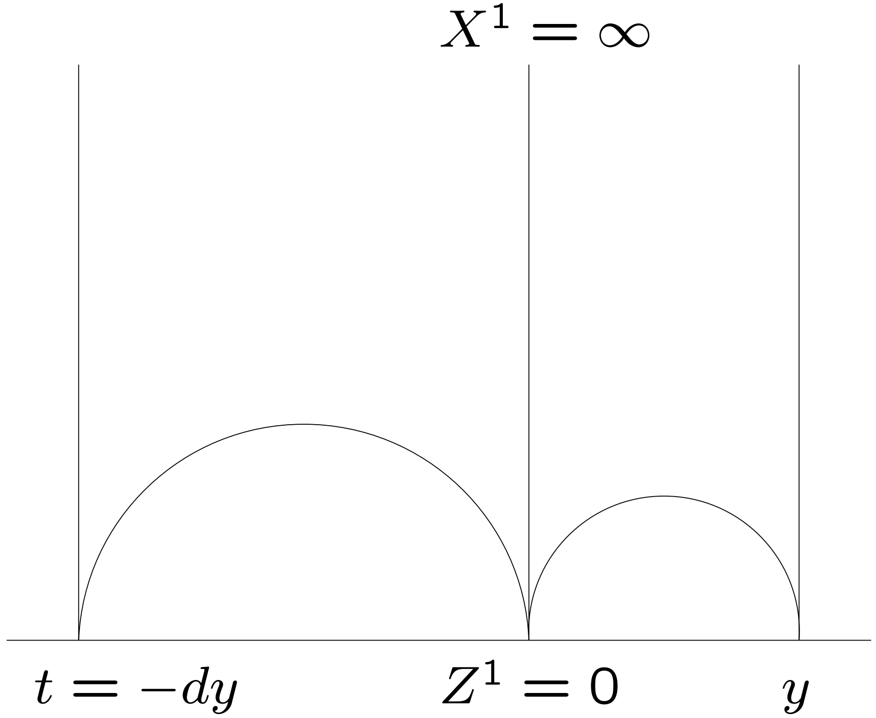
$$X = \begin{pmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} O & & 1 \\ & ⋰ & \\ 1 & & O \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Then

$$\delta^i(X, y, Z, t) = -\frac{\begin{vmatrix} I_i & O & \vdots \\ O & O & t_{i+1} \\ O & C_{n-i-1} & \end{vmatrix} \cdot \begin{vmatrix} I_{i-1} & O & \vdots \\ O & O & y_i \\ O & C_{n-i} & \vdots \end{vmatrix}}{\begin{vmatrix} I_{i-1} & O & \vdots \\ O & O & t_i \\ O & C_{n-i} & \vdots \end{vmatrix} \cdot \begin{vmatrix} I_i & O & \vdots \\ O & O & y_{i+1} \\ O & C_{n-i-1} & \vdots \end{vmatrix}} = -\frac{t_{i+1}}{t_i} \cdot \frac{y_i}{y_{i+1}}$$

Thus $t \in \mathbb{C}P^{n-1}$ is uniquely determined by $\delta^1, \dots, \delta^{n-1}$. $\square$

When $n = 2$, if we regard $[y_1 : y_2] \in \mathbb{C}P^1$ as $y = y_1/y_2 \in \mathbb{C} \cup \{\infty\}$ we have the following picture:



$$\delta^1(X, y, Z, t) = -\frac{y_2}{y_1} \cdot \frac{t_1}{t_2}$$
$$\therefore t = -dy$$

where $d = \delta^1(X, y, Z, t)$.

Parametrization of reps of $\pi_1(S)$

S : a bordered surface, T : an ideal triangulation of S

For each triangle of T, assign $\frac{(n-1)(n-2)}{2}$ complex numbers corresponding to the triple ratios.

For each edge of T, assign $(n-1)$ complex numbers corresponding to the edge functions.

Using **Cor B** and **Lem C**, we can construct a developing map $\partial_\infty \widetilde{T} \to \mathcal{F}_n$ from these parameters as follows.

($\widetilde{T}$ is the triangulation lifted from T to the universal cover $\tilde{S}$ and $\partial_\infty \widetilde{T}$ is its ideal boundary.)

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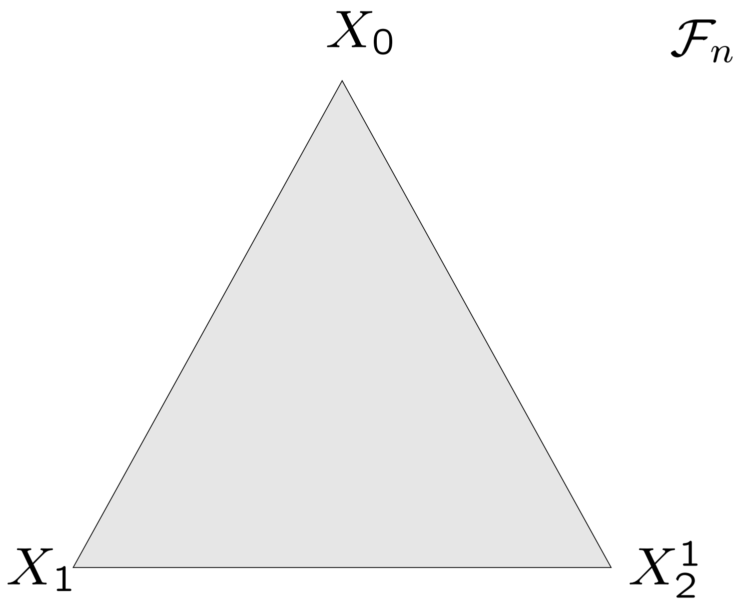
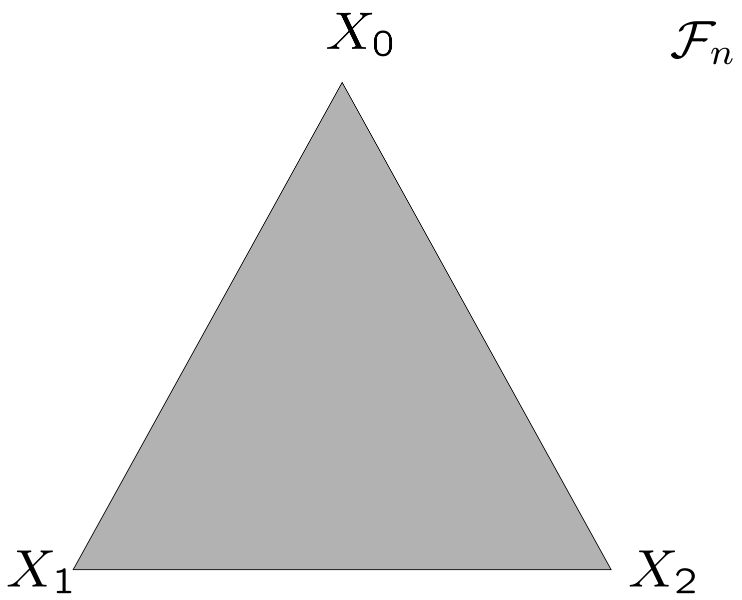
Construction of the developing map



Take $X_0, X_1 \in \mathcal{F}_n$ and $X_2^1 \in \mathbb{C}P^{n-1}$ arbitrarily.

Construction of the developing map



Lift $X_2^1 \in \mathbb{C}P^{n-1}$ to $X_2 \in \mathcal{F}_n$ by **Cor B** according to the triple ratio parameters.

(**Cor B** Let $X, Y \in \mathcal{F}_n$ and $z \in \mathbb{C}P^{n-1}$ be a generic triple. For any $\frac{(n-1)(n-2)}{2}$ non-zero complex numbers $\{T^{i,j,k}\}$, there exists a unique $Z \in \mathcal{F}_n$ s.t. $Z^1 = z$ and $T^{i,j,k}(X,Y,Z) = T^{i,j,k}$.)

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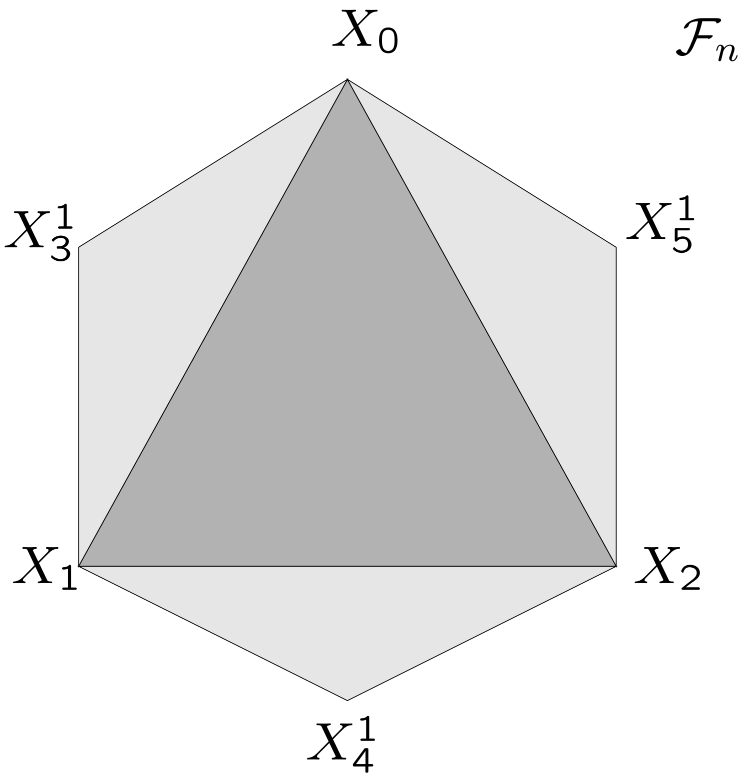

Define $X_3^1, X_4^1, X_5^1 \in \mathbb{C}P^{n-1}$ by **Lem C** according to the edge functions.

(**Lem C** Let $X, Z \in \mathcal{F}_n$ and $y \in \mathbb{C}P^{n-1}$. For any $d_1, \ldots, d_{n-1} \in \mathbb{C}^*$, there exits a unique $t \in \mathbb{C}P^{n-1}$ s.t.

$$\delta^i(X, y, Z, t) = d_i. \quad (i = 1, \ldots, n-1) \quad)$$

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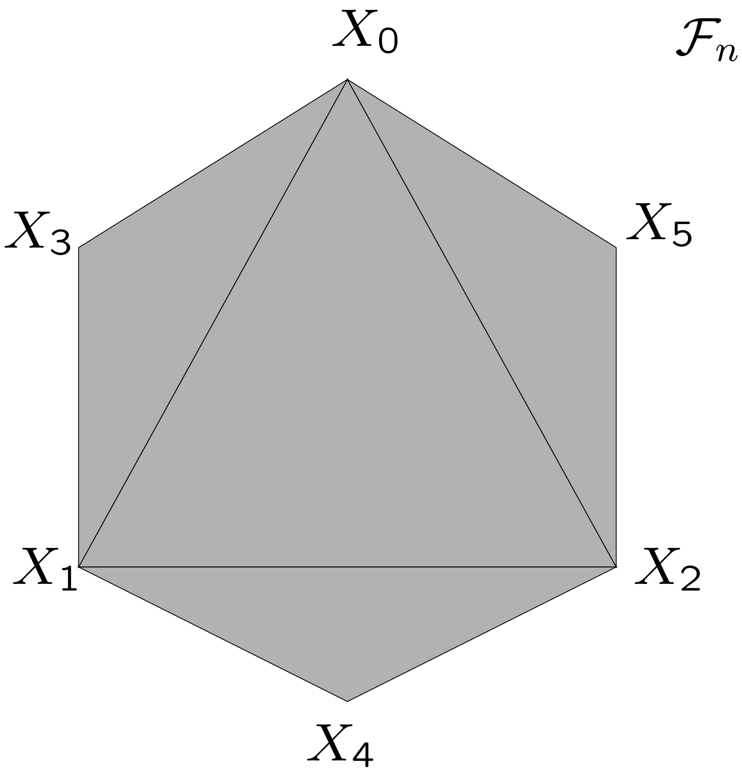

Lift $X_3^1, X_4^1, X_5^1 \in \mathbb{C}P^{n-1}$ to $X_3, X_4, X_5 \in \mathcal{F}_n$ by **Cor B** according to the triple ratios.

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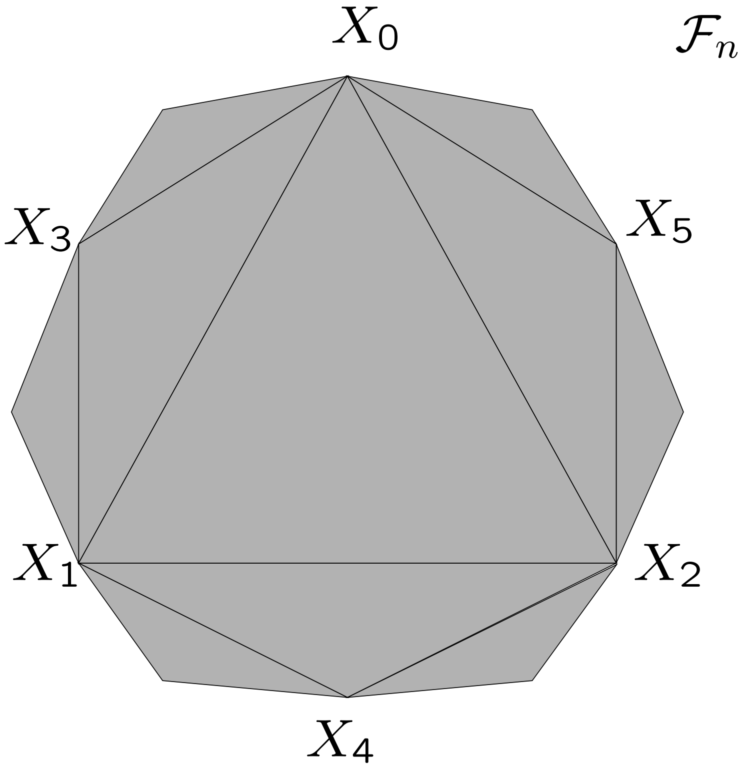
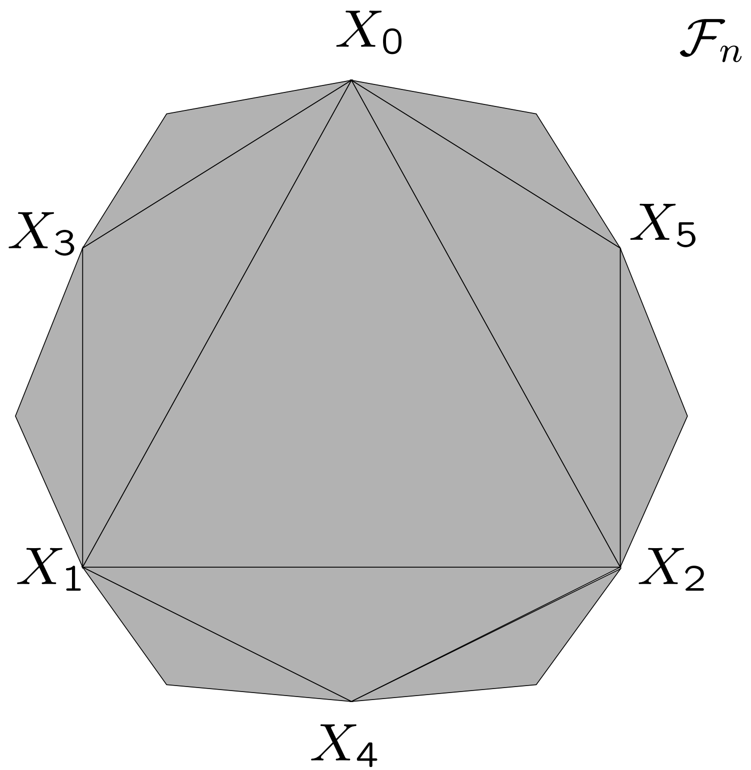


Iterate these procedures, we obtain a developing map $\partial_\infty \widetilde{T} \to \mathcal{F}_n$.

Construction of the developing map



By **Cor A**, we can also obtain a representation $\pi_1(S) \to \mathrm{PGL}(n, \mathbb{C})$ explicitly.

(**Cor A** Let $X, Y \in \mathcal{F}_n$ and $z \in \mathbb{C}P^{n-1}$ be a generic triple, and $X', Y' \in \mathcal{F}_n$ and $z' \in \mathbb{C}P^{n-1}$ another generic triple. Then there exists a unique matrix $A \in \mathrm{PGL}(n, \mathbb{C})$ s.t.

$$AX = X', \quad AY = Y', \quad Az = z'.)$$

In particular, representations of the π_1 of a pair of pants are parametrized by $2\times\frac{(n-1)(n-2)}{2}+3\times(n-1) = n^2-1$ parameters.

The remaining problem is how to glue the representations along boundaries.

We can glue two representations along their boundaries iff their monodromies along the boundaries are conjugate, in other words, iff they have same eigenvalues.

We have to compute the eigenvalues of the monodromy along a boundary curve from the triple ratios and edge functions.

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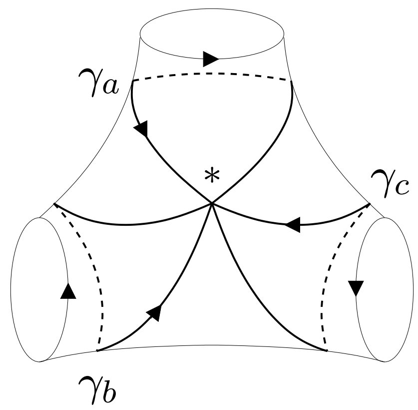
Computation of eigenvalues

P : a pair of pants

Fix $\gamma_a, \gamma_b, \gamma_c \in \pi_1(P)$ as in the figure.

$\rho : \pi_1(P) \to \mathrm{GL}(n, \mathbb{C})$: a rep

$e_{a,1}, \ldots, e_{a,n}$: the eigenvalues of $\rho(\gamma_a)$



Assume that $e_{a,i}$'s are distinct.

v_a^i : the eigenvector corresponding to $e_{a,i}$

Similarly, define $e_{b,i}$, v_b^i, etc.

Let $X_a^i = \mathrm{span}_{\mathbb{C}}\{v_a^1, \ldots, v_a^i\}$.

This defines a flag $X_a = \{X_a^1 \subsetneq X_a^2 \subsetneq \cdots \subsetneq X_a^n\}$.

Define X_b and X_c similarly. By definition,

$$\rho(\gamma_a)X_a = X_a, \quad \rho(\gamma_b)X_b = X_b, \quad \rho(\gamma_c)X_c = X_c.$$

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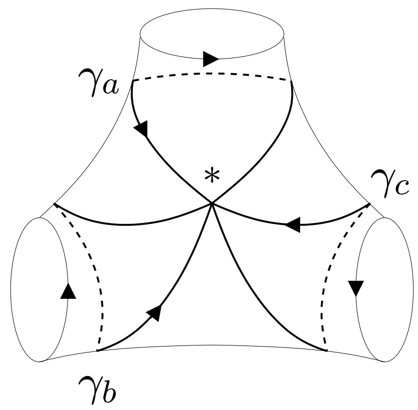

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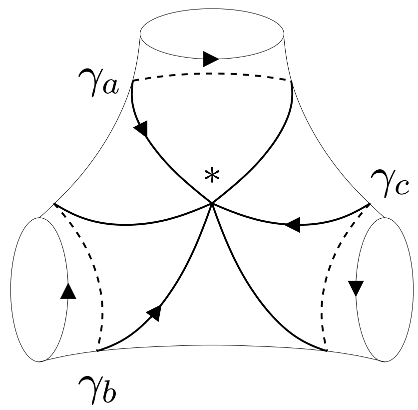

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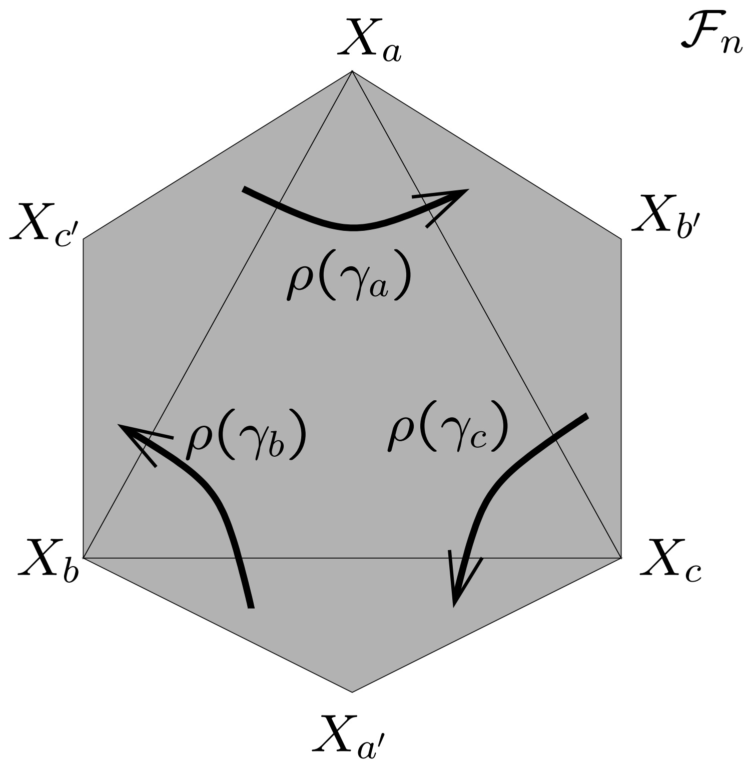
$$\rho(\gamma_a)X_a = X_a, \quad \rho(\gamma_b)X_b = X_b, \quad \rho(\gamma_c)X_c = X_c.$$

Define

$$X_{a'} = \rho(\gamma_c)X_a, \quad X_{b'} = \rho(\gamma_a)X_b, \quad X_{c'} = \rho(\gamma_b)X_c.$$



We have $\rho(\gamma_a)X_{c'} = \rho(\gamma_a)\rho(\gamma_b)X_c = \rho({\gamma_c}^{-1})X_c = X_c$.

Similarly $\rho(\gamma_b)X_{a'} = X_a$ and $\rho(\gamma_c)X_{b'} = X_b$.

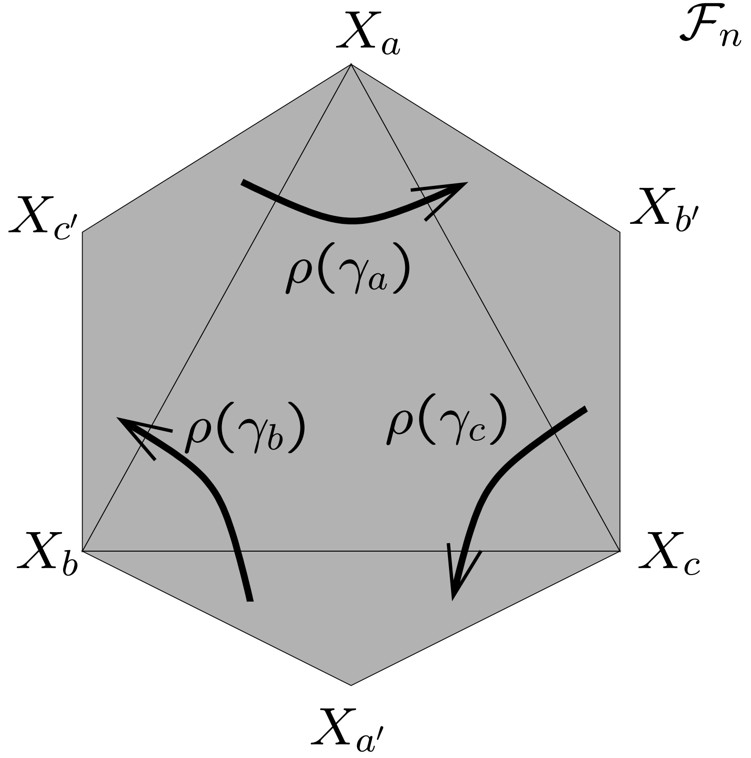
We have

$$\rho(\gamma_a)(X_a, X_{c'}, X_b) = (X_a, X_c, X_{b'}),$$

$$\rho(\gamma_b)(X_{a'}, X_c, X_b) = (X_a, X_{c'}, X_b),$$

$$\rho(\gamma_c)(X_a, X_c, X_{b'}) = (X_{a'}, X_c, X_b).$$

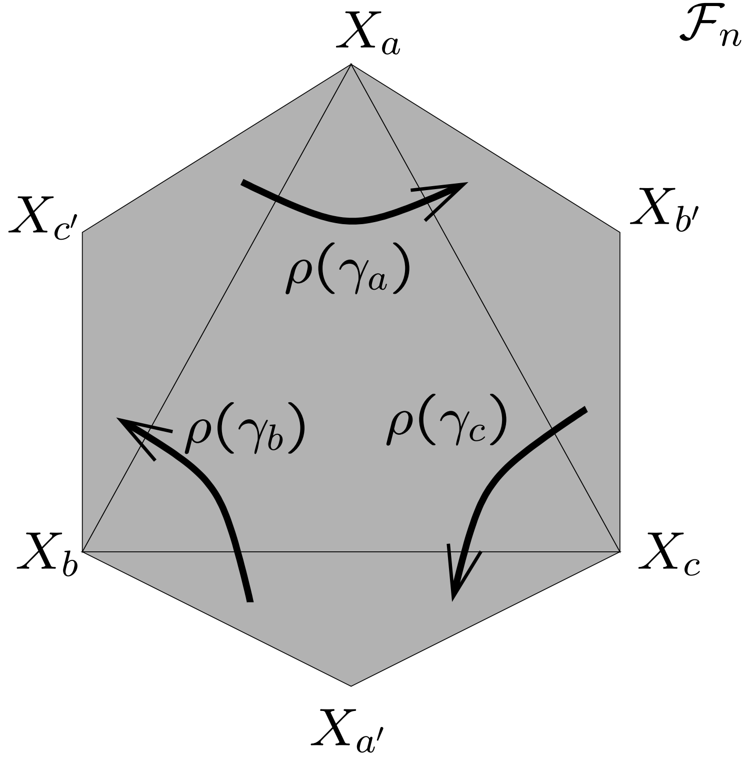
Thus these triples are in the same GL$(n, \mathbb{C})$-orbit. Thus they have same triple ratios.



We assume that (X_a, X_b, X_c) and $(X_a, X_{c'}, X_b)$ are generic triples.

We define the triple ratio parameters by

$$T^{i,j,k}_{a,b,c} := T^{i,j,k}(X_a, X_b, X_c),$$
$$U^{i,j,k}_{a,c,b} := T^{i,j,k}(X_a, X_{c'}, X_b).$$



and the edge functions

$$\delta^i_{a,b} := \delta^i(X_a, X^1_c, X_b, X^1_{c'})$$
$$\delta^i_{b,c} := \delta^i(X_b, X^1_a, X_c, X^1_{a'})$$
$$\delta^i_{c,a} := \delta^i(X_c, X^1_b, X_a, X^1_{b'})$$

We use the following notation:

$$T^{i,j,k}_{a,b,c} = T^{j,k,i}_{b,c,a} = T^{k,i,j}_{c,a,b}, \quad U^{i,j,k}_{a,c,b} = U^{j,k,i}_{c,b,a} = U^{k,i,j}_{b,a,c}, \quad \delta^i_{b,a} = \delta^{n-i}_{a,b}.$$

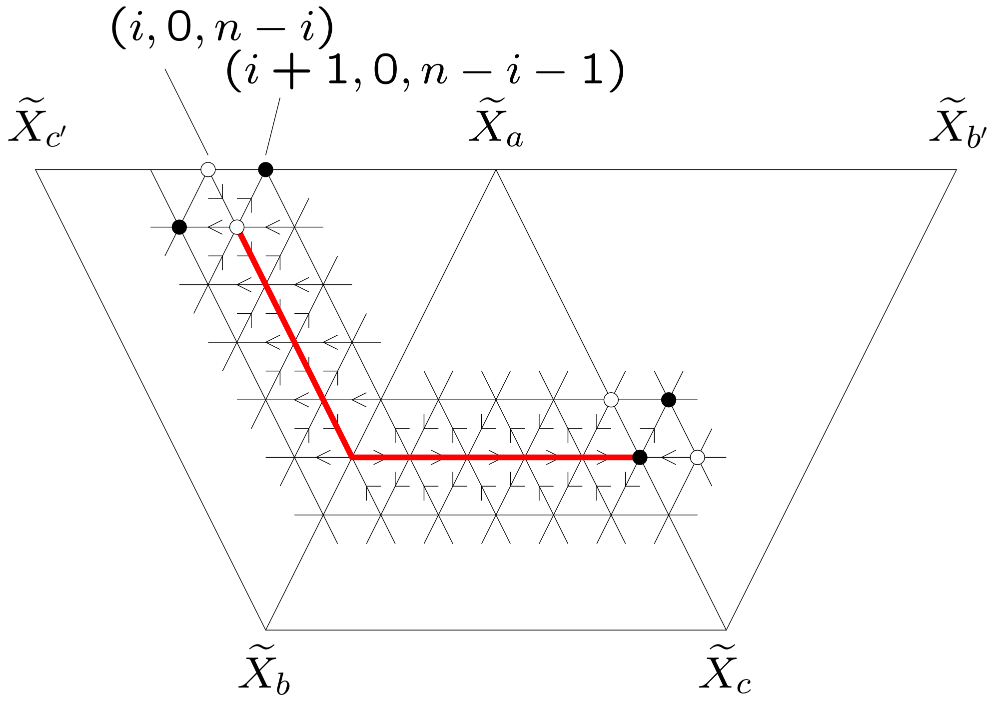
Thm

We have

$$\frac{e_{a,i+1}}{e_{a,i}} = \delta^i_{a,b}\delta^i_{a,c} \prod_{l=1}^{n-1-i} T^{i,l,n-i-l}_{a,b,c} U^{i,l,n-i-l}_{a,c,b},$$

for $i = 1, \ldots, n-1$.

The right hand side is the product of the triple ratios and the edge functions on the red line.



Sketch of proof

We fix a lift $\widetilde{X}_a \in \mathcal{AF}_n$ of X_a. (Fix $\widetilde{X}_b$ and $\widetilde{X}_c$ similarly).

For $0 \leq i, j, k \leq n$ satisfying $i + j + k = n$, we denote

$$\Delta^{i,j,k}_{a,c',b} = \mathrm{det}(\widetilde{X}^i_a \widetilde{X}^j_{c'} \widetilde{X}^k_b), \quad \Delta^{i,j,k}_{a,c,b'} = \mathrm{det}(\widetilde{X}^i_a \widetilde{X}^k_c \widetilde{X}^j_{b'}), \text{ etc.}$$

Consider the product of the triple ratios and the edge functions corresponding to the vertices on the red line. These are written in terms of $\Delta^{i,j,k}_{*,*,*}$, and most of them cancel out:

$$\delta^i_{a,b}\delta^i_{a,c} \prod_{l=1}^{n-1-i} T^{i,l,n-i-l}_{a,b,c} U^{i,l,n-i-l}_{a,c,b} = \frac{\Delta^{i+1,0,n-i-1}_{ac'b} \Delta^{i-1,1,n-i}_{ac'b}}{\Delta^{i,0,n-i}_{ac'b} \Delta^{i,1,n-i-1}_{ac'b}} \frac{\Delta^{i,0,n-i}_{acb'} \Delta^{i,1,n-i-1}_{acb'}}{\Delta^{i+1,0,n-i-1}_{acb'} \Delta^{i-1,1,n-i}_{acb'}}$$

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$$= \frac{\Delta^{i+1,0,n-i-1}_{ac'b} \Delta^{i-1,1,n-i}_{ac'b}}{\Delta^{i,0,n-i}_{ac'b} \Delta^{i,1,n-i-1}_{ac'b}} \frac{\Delta^{i,0,n-i}_{acb'} \Delta^{i,1,n-i-1}_{acb'}}{\Delta^{i+1,0,n-i-1}_{acb'} \Delta^{i-1,1,n-i}_{acb'}}$$

On the other hand, we have

$$\det\rho(\gamma_a)\cdot\det(\widetilde{X}_a^i\widetilde{X}_{c'}^j\widetilde{X}_b^k) = \det((\rho(\gamma_a)\widetilde{X_a})^i(\rho(\gamma_a)\widetilde{X}_{c'})^j(\rho(\gamma_a)\widetilde{X}_b)^k)$$
$$= \frac{e_{a,1}\cdots e_{a,i}}{e_{c,1}\cdots e_{c,j}}\det(\widetilde{X}_a^i\widetilde{X}_c^j\widetilde{X}_{b'}^k).$$

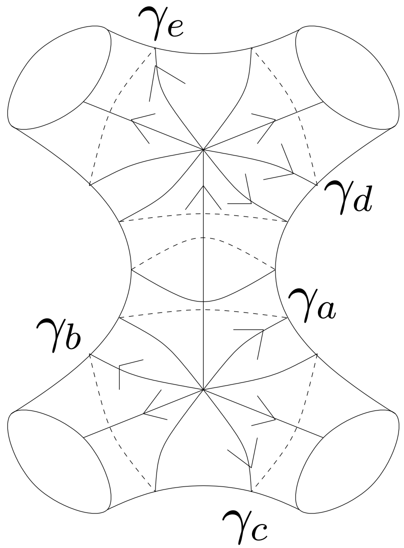
Thus we have

$$\frac{\Delta_{ac'b}^{i+1,0,n-i-1}\Delta_{ac'b}^{i-1,1,n-i}}{\Delta_{ac'b}^{i,0,n-i}\Delta_{ac'b}^{i,1,n-i-1}} = \frac{e_{a,i+1}}{e_{a,i}}\frac{\Delta_{acb'}^{i+1,0,n-i-1}\Delta_{acb'}^{i-1,1,n-i}}{\Delta_{acb'}^{i,0,n-i}\Delta_{acb'}^{i,1,n-i-1}}.$$

Therefore

$$\delta_{a,b}^i\delta_{a,c}^i\prod_{l=1}^{n-1-i}T_{a,b,c}^{i,l,n-i-l}U_{a,c,b}^{i,l,n-i-l}$$
$$= \frac{\Delta_{ac'b}^{i+1,0,n-i-1}\Delta_{ac'b}^{i-1,1,n-i}}{\Delta_{ac'b}^{i,0,n-i}\Delta_{ac'b}^{i,1,n-i-1}}\cdot\frac{\Delta_{acb'}^{i,0,n-i}\Delta_{acb'}^{i,1,n-i-1}}{\Delta_{acb'}^{i+1,0,n-i-1}\Delta_{acb'}^{i-1,1,n-i}}$$
$$= \frac{e_{a,i+1}}{e_{a,i}}. \quad \square$$

Twist parameters



Let $S = P \cup P'$ be a four-holed sphere. We fix a system of generators $\gamma_a, \gamma_b, \gamma_c, \gamma_d, \gamma_e \in \pi_1(S)$ as in the figure. Let $\rho : \pi_1(S) \to \mathrm{PGL}(n, \mathbb{C})$.

We need to assume some genericity conditions but I omit them here. Let $v_a^1, \ldots, v_a^n$ be the eigenvectors of $\rho(\gamma_a)$.

Define flags X_a and Y_a by

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We have $\rho(\gamma_a)X_a = X_a$ and $\rho(\gamma_a)Y_a = Y_a$.

By assumption, (X_a, Y_a, X_b^1) and (X_a, Y_a, X_e^1) are generic. Thus we can define the twist parameters along γ_a by

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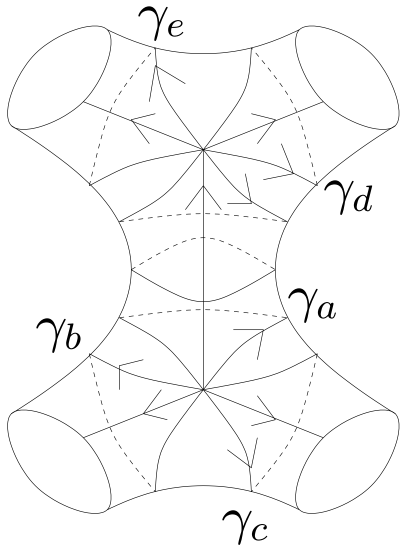
Twist parameters



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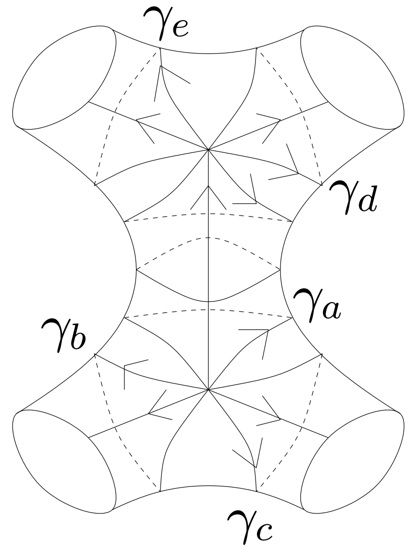

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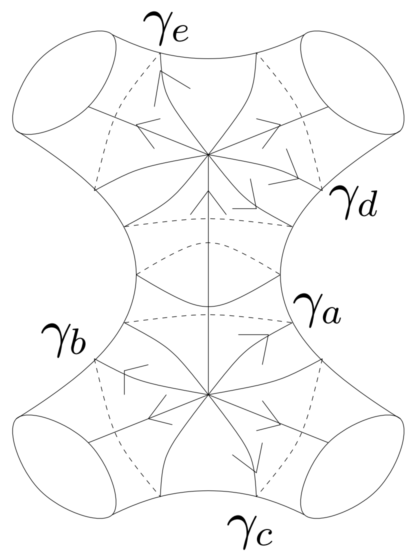

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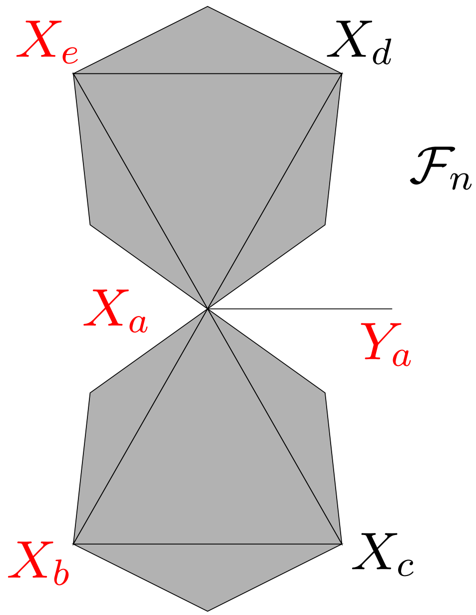
The twist parameter $\delta^i(X_a, X_b^1, Y_a, X_e^1)$ describes the relative position of the two developing maps.



Combining with the triple ratio parameters and the edge functions from two pairs of pants, we can construct a developing map for S, and thus a $\mathrm{PGL}(n, \mathbb{C})$-representation.

F-N coordinates of $\mathrm{PGL}(n, \mathbb{C})$-representations

S : closed, genus $g > 1$, C : a pants decomposition of S

For each pair of pants $S \setminus C$, we assign

- $2 \times \frac{(n-1)(n-2)}{2}$ triple ratio parameters, and
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For each pants curve, we assign

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For each pants curve of C, we have to impose $(n-1)$ relations to have same eigenvalues up to scalar. These relations are explicitly given by **Thm**:

$$\frac{e_{a,i+1}}{e_{a,i}} = \delta^i_{a,b}\delta^i_{a,c} \prod_{l=1}^{n-1-i} T^{i,l,n-i-l}_{a,b,c} U^{i,l,n-i-l}_{a,c,b},$$

Thus we have

- $(2g-2)((n-1)(n-2)+3(n-1))+(3g-3)(n-1)$ parameters
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Thank you for your attention.