Tangle sums and factorization of $A$-polynomials

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Tohoku University

RIMS Seminar in Hakone, 1 June 2012
Plan of this talk

§1. Factorization of $A$-polynomials
§2. Alexander polynomials and epimorphisms
§3. Cyclic surgeries
§1. Factorization of $A$-polynomials

$K$: a knot in $S^3$

$M_K$: the complement of $K$

$i^* : X(M_K) \to X(\partial M_K)$: induced by the $i_\# : \pi_1(\partial M_K) \to \pi_1(M_K)$

$\Lambda \subset R(\partial M_K)$: the set of diagonal representations of $\pi_1(\partial M_K)$

$t|_\Lambda : \Lambda \to X(\partial M_K)$

$p : \Lambda \to \mathbb{C}^* \times \mathbb{C}^*$: taking the left-top entries of $\rho(\mu)$ and $\rho(\lambda)$

$X_1, \cdots, X_k$: irreducible components of $X(M_K)$

$$
X_1 \xrightarrow{i^*} i^*(X_1) \xrightarrow{\text{alg. closure in } X(\partial M_K)} Y_1 \xrightarrow{p \cdot t|_\Lambda^{-1}} D_1
$$

$A_i(L, M)$: the defining equation of $D_i$

**Definition**

The $A$-polynomial of a knot $K$ is defined as

$$
A_K(L, M) = \prod_{i=1}^{k} A_i(L, M).
$$
### Table of Knot Invariants

*Please Cite KnotInfo*

<table>
<thead>
<tr>
<th>Knot Theory Calculators</th>
<th>Knot Atlas</th>
<th>Knot Theory Links</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unknown Values (last updated: 06 Feb 2009)</td>
<td>Acknowledgments</td>
<td></td>
</tr>
</tbody>
</table>

**Build a Knot Table.** Welcome to KnotInfo. Check the desired boxes in the sections below and then click SUBMIT on the page to produce your desired table of knots. If you do not know the name of a particular knot you are interested in, KnotFinder can help you.

**Preferences (Select invariants to hide.)**

New advanced search feature is now available! [Advanced Search]

#### Select knots you want tabulated. [Advanced Search]

Specify crossing numbers. The letters a and n designate alternating and nonalternating knots. *12 crossing knots are grouped.*

<table>
<thead>
<tr>
<th>3-6</th>
<th>7</th>
<th>8a (1-200)</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>11a</td>
<td>11n</td>
<td>12a (801-1000)</td>
<td>12a (1200-1288)</td>
<td>12a (401-600)</td>
</tr>
<tr>
<td>12a (601-800)</td>
<td>12n (401-600)</td>
<td>12a (1001-1200)</td>
<td>12a (201-400)</td>
<td>12n (801-888)</td>
</tr>
<tr>
<td>12n (201-400)</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

#### Names and descriptions. Please select the naming and notational descriptions desired. *Names are linked to diagrams.*

<table>
<thead>
<tr>
<th>Name</th>
<th>DT Notation</th>
<th>Gauss Notation</th>
<th>Fibered</th>
<th>Alternating</th>
<th>Classical Conway Name</th>
<th>Braid Notation</th>
<th>DT Name</th>
<th>Conway Notation</th>
<th>Two-Bridge Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Name Rank</td>
<td>DT Rank</td>
<td>PD Notation</td>
<td>Tetrahedral Census Name</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

#### Three-Dimensional Invariants.

<table>
<thead>
<tr>
<th>Alexander</th>
<th>Crossing Number</th>
<th>Seifert Matrix</th>
<th>Crosscap Number</th>
<th>Tunnel Genus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Braid Index</td>
<td>Determinant</td>
<td>Super Bridge Index</td>
<td>Thurston-Bennequin Number</td>
<td>Unknotting Number</td>
</tr>
</tbody>
</table>

#### Concordance and Four-Dimensional Invariants.

<table>
<thead>
<tr>
<th>Arf Invariant</th>
<th>Topological Concordance Order</th>
<th>Smooth 4D Crosscap Number</th>
<th>Signature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smooth Concordance Genus</td>
<td>Algebraic Concordance Order</td>
<td>Topological 4D Crosscap Number</td>
<td>Signature Function</td>
</tr>
<tr>
<td>Topological Concordance Genus</td>
<td>Smooth Four Genus</td>
<td>Rasmussen Invariant</td>
<td>Smooth Concordance Crosscap Number</td>
</tr>
</tbody>
</table>

#### Polynomial Invariants.

<table>
<thead>
<tr>
<th>A-Polynomial</th>
<th>Jones Polynomial</th>
<th>Khovanov Polynomial</th>
<th>Conway Polynomial</th>
<th>HOMFLY Polynomial</th>
<th>Khovanov Torsion Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alexander Polynomial</td>
<td>Kauffman Polynomial</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The A-Polynomial

For more information about the A-polynomial we have posted a pdf version of a seminar presentation given by Marc Culler. The original source for the A-polynomial is the paper by Cooper, Culler, Gillet, Long, and P. Shalen, referenced below. We thank Abhijit Champanerkar for helping with the exposition on this page.

There is a map of the $\text{SL}_2(\mathbb{C})$ representation space of a knot complement to $\mathbb{C}^* \times \mathbb{C}^*$, given by evaluating the trace of the representation on the meridian and longitude. The closure of the image is a variety defined by a single polynomial, called the A-Polynomial. Jim Hoste gave us information on 2-bridge knots and Marc Culler provided us with further tables, based on gluing equations. These have not been proved to equal the A-polynomial; the issue is described next.

The set of isometry classes of ideal hyperbolic tetrahedra is paramaterized by the upper half complex plane. Thus, if the complement of a knot is decomposed into tetrahedra, the set of glueings that yield hyperbolic structures on the knot complement is determined by the solutions to gluing equations. The set of gluing equations defines an algebraic variety that maps to the $\text{PSL}_2(\mathbb{C})$ character variety of the knot. To the image variety there is associated an "A-polynomial", which is the $\text{PSL}_2(\mathbb{C})$ version of the classical A-polynomial. In many cases the $\text{PSL}_2(\mathbb{C})$ A-polynomial can be computed directly from the gluing and completeness equations by eliminating the tetrahedral parameters to get a 2-variable polynomial. However, the resulting polynomial depends on the choice of the triangulation and in general only divides the $\text{PSL}_2(\mathbb{C})$ A-polynomial. For an exposition of this alternative viewpoint of A-polynomials, see the appendix by N. Dunfield to Mahler's Measure and the Dilogarithm by Boyd, Rodrigues-Villegas, and Dunfield or "A-polynomial and Bloch invariants of hyperbolic 3-manifolds" by A. Champanerkar.

We have provided three tables of A-polynomials, all linked in Table of A-Polynomials: two-bridge knots. Jim Hoste has provided us with this table of values for 2-bridge knots of 9 crossings or less.

Table of A-Polynomials (Glueing equations approach). This data, based on glueing equations, was provided by Marc Culler.

Table of A-Polynomial: tetrahedral census (Glueing equations approach). This table, also provided by Marc Culler, lists the A-polynomials of knots in the tetrahedral enumeration. There is an overlap in the two tables. Warning: in the overlap, orientations changed for some knots, so one polynomial is related to the other by a change of variable (something like $L \rightarrow L^{-1}$).

Warning: a change of orientation, from a knot to its mirror image, changes the A-polynomial. The data in our tables has not be checked for its match to the choice of orientation in our diagrams. Also, the A-polynomial can be defined so that repeated factors are significant. In our table repeated factors have been removed.
A-Polynomials-gluing Equations

This list of A-polynomials was compiled by Marc Culler. For a table using the labelings of the tetrahedral enumeration, go to A-Polynomials: tetrahedral enumeration, and for one produced by Jim Hoste for two-bridge knots, visit A-polynomials: two bridge enumeration.

Warning: See the description section for the A-polynomial for details regarding the various definitions of the A-polynomial, and possible distinctions between them. A change of orientation, from a knot to its mirror image, changes the A-polynomial. The data in our tables has not been checked for its match to the choice of orientation in our diagrams. Also, the A-polynomial can be defined so that repeated factors are significant. In our tables all repeated factors have been removed.

A_L103001 := (1*M^6) + (L^1)*1;
A_L104001 := (1*M^4) + (L^1)*(-1 + 1*M^2 + 2*M^4 + 1*M^6 - 1*M^8) + (L^2)*(1*M^4);
A_L105001 := (1*M^10) + (L^1)*1;
A_L105002 := (1) + (L^1)*(-1 + 2*M^2 + 2*M^4 - 1*M^8 + 1*M^10) + (L^2)*(1*M^4 - 1*M^6 + 2*M^10 + 2*M^12 - 1*M^14) + (L^3)*(1*M^14);
A_L107001 := (1*M^14) + (L^1)*1;
Factorization of $A$-polynomials

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$A_{L104001} := (1*M^4) + (L^1)*(-1 + 1*M^2 + 2*M^4 + 1*M^6 - 1*M^8) + (L^2)*(1*M^4)$;

$A_{L104001} := M^4 + L(-1 + M^2 + 2*M^4 + M^6 - M^8) + L^2*M^4$
A_L104001 := (1\times M^4) + (L^1)*( -1 + 1\times M^2 + 2\times M^4 + 1\times M^6 - 1\times M^8) + (L^2)*(1\times M^4);  
A_L104001 := M^4 + L(-1 + M^2 + 2 M^4 + M^6 - M^8) + L^2 M^4.
Masaharu ISHIKAWA (Tohoku University)

Factorization of $A$-polynomials

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\begin{verbatim}
> factor(A_L109024);

$-L^2 M^6 - 155 M^6 - 186 M^6 - 28 M^6 + 2 M^6 \) + $L^6 \) \( (15158 M^6 - 1487 M^6 - 1259 M^6 + 6935 M^6 - 7947 M^6

- 667 M^6 - 18376 M^6 - 2217 M^6 + 2794 M^6 - 18156 M^6 - 10 M^6 + 116 M^6 - 567 M^6 + 1410 M^6 - 1175 M^6

+ 1995 M^6 - 5847 M^6 - 5519 M^7 + 6387 M^7 - 2536 M^7 + 1234 M^7 - 327 M^7 - 1030 M^7 - 423 M^7 - 135 M^7

- 11 M^8 + 1 M^8 + L (10 M^4 + 79 M^4 + 243 M^4 + 27 M^4 - 252 M^4 + 1030 M^4 - 543 M^4 + 135 M^4

- 674 M^4 - 842 M^4 + 287 M^4 - 1017 M^4 + 245 M^4 - 199 M^4 + 245 M^4 + 1 M^4 + 497 M^4 + 203 M^4 + 59 M^6

+ L^18 (-5 M^5 + 20 M^5 - 25 M^5 - 15 M^5 + 78 M^5 - 24 M^5 + 33 M^5 - 66 M^5 - 13 M^5 + 3 M^5 + L^13 M^5)

\end{verbatim}

Tim: 2.1s  Dyre: 4.00M  Available: 993M
Factorization of $A$-polynomials

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\[-45 \, M^{30} + 125 \, M^{32} - 104 \, M^{34} - 235 \, M^{36} + 683 \, M^{38} - 491 \, M^{40} - 649 \, M^{42} + 1321 \, M^{44} - 159 \, M^{45} - 737 \, M^{48} + 523 \, M^{50} \]

\[-143 \, M^{52} - 57 \, M^{54} + 193 \, M^{56} - 181 \, M^{58} + 36 \, M^{60} - 24 \, M^{52} + 3 \, M^{54} \]

\[+ L^{13} (-4 \, M^{38} + 15 \, M^{38} - 15 \, M^{40} - 23 \, M^{42} + 73 \, M^{44} - 10 \, M^{46} - 41 \, M^{48} + 33 \, M^{50} - 14 \, M^{52} + 3 \, M^{54} ) + L^{14} \, M^{44} \]

\[\text{factor}(A_{L109037}); \]

\[(-L + \frac{M}{4}) (-L^{2} + L^{2} + L^{3} + 2 \, M^{4} - M^{5} - L^{6} - M^{8} - L) (3 \, M^{4} + L^{2} - L^{13} + 2 \, M^{8} - 2 \, M^{16} - 2 \, L^{2} + 2 \, M^{14} + L + 3 \, L^{2} + M^{14} + M^{8} + 2 \, M^{6} + L^{10} + L^{12} + 2 \, M^{12} + 2 \, M^{6} + 2 \, M^{2} - 2 \, M^{2} - L^{7} - L^{8} - L^{2} + 2 \, L^{2} + M^{14} + L + M^{10} - 7 \, L^{2} + M^{10} ) (31 \, L^{2} + M^{34} + 34 \, L^{2} + M^{58} + 1091 \, L^{3} + M^{22} + 4 \, L^{5} + M^{8} - 30 \, M^{18} - L - 25 \, M^{8} - 89 \, L^{4} + M^{14} + 105 \, L^{3} + M^{14} - 254 \, M^{16} + 2 \, L^{2} - 2 \, M^{14} + L^{8} + 329 \, M^{12} + 2 \, M^{20} + 41 \, L^{2} + M^{20} + 363 \, L^{2} + M^{32} - 1195 \, L^{4} + M^{16} - 69 \, L^{6} + M^{24} + 841 \, L^{4} + M^{12} + 164 \, M^{8} + 37 \, L^{6} + 34 - 6 \, L^{6} + 42 \, M^{48} + 1604 \, L^{3} + 16 + 12 \, M^{2} + 16 - 113 \, L^{2} + 36 - 45 \, L^{5} + 10 - 442 \, M^{14} + 9 \, L^{3} + 980 \, M^{12} + 35 \, M^{26} + 45 \, L^{2} + 38 - 376 \, L^{3} + M^{18} - 37 \, M^{12} + 12 \, L^{2} + 1085 \, L^{2} + 96 \, L^{2} + 23 - 4 \, L^{3} + 32 - 742 \, L^{2} + 22 - 637 \, L^{2} + 24 + 449 \, L^{4} + M^{14} + 625 \, M^{10} + 3 + 242 \, L^{4} + M^{5} - 277 \, L^{3} + M^{6} + 8 \, L^{2} + 6 + 15 \, L^{4} + 8 \, L^{2} + 6 \, M^{50} - 6 \, L^{52} - 6 \, L^{52} - 6 \, M^{54} - 186 \, M^{5} + 42 \, L^{3} - 12083 \, L^{3} - 20 - 258 \, L^{2} + 20 \, L^{2} - 2 \, M^{4} - M^{24} + 3 \, L^{2} + + 190 \, L^{3} + M^{24} - 45 \, L^{3} + M^{46} + 4 \, L^{3} + M^{48} - 1236 \, L^{3} + M^{24} - 751 \, L^{3} + 26 + 1073 \, L^{3} + M^{38} - 140 \, L^{3} + M^{40} - 303 \, L^{3} + M^{24} + 583 \, L^{3} + M^{24} - 1091 \, L^{3} + M^{26} - 1005 \, L^{3} + M^{23} + 947 \, L^{3} + M^{20} + 1433 \, L^{3} + M^{32} - 14 - 15 \, L^{3} + M^{54} + 841 \, L^{3} + M^{44} - 471 \, L^{1} + M^{45} - 136 \, L^{1} + M^{45} + 242 \, L^{1} + M^{50} - 89 \, L^{1} + M^{52} + L^{3} + M^{56} + 1723 \, L^{4} + M^{34} + 1290 \, L^{4} + M^{36} - 919 \, L^{4} + M^{38} - 1196 \, L^{4} + M^{40} + 449 \, L^{4} + M^{42} - 1614 \, L^{4} + M^{42} - 1044 \, L^{4} + M^{42} + 1840 \, L^{4} + M^{28} - 1044 \, L^{4} + M^{40} - 1614 \, L^{4} + M^{42} + 1728 \, L^{4} + M^{42} - 27 \, L^{5} + M^{54} - 930 \, L^{5} + M^{44} + 526 \, L^{5} + M^{46} + 164 \, L^{5} + M^{48} - 277 \, L^{5} + M^{50} + 105 \, L^{5} + M^{52} + 5 \, M^{5} + 1091 \, L^{5} + M^{34} - 12083 \, L^{5} + M^{36} - 376 \, L^{5} + M^{38} + 1604 \, L^{5} + M^{40} - 442 \, L^{5} + M^{42} + 1433 \, L^{5} + M^{24} + 947 \, L^{5} + M^{26} - 1005 \, L^{5} + M^{28} - 1091 \, L^{5} + M^{30} + 583 \, L^{5} + M^{32} + 1073 \, L^{5} + M^{34} - 751 \, L^{5} + M^{20} - 1236 \, L^{5} + M^{22} + 190 \, L^{5} + M^{12} - 308 \, L^{5} + M^{14} - 140 \, L^{5} + M^{16} - 742 \, L^{6} + M^{34} + 829 \, L^{6} + M^{36} + 34 \, L^{6} + M^{38} - 254 \, L^{6} + M^{40} + 85 \, L^{6} + M^{42} + 363 \, L^{6} + M^{42} - 588 \, L^{6} + M^{26} - 96 \, L^{6} + M^{28} + 1085 \, L^{6} + M^{30} - 687 \, L^{6} + M^{32} + 45 \, L^{6} + M^{18} - 113 \, L^{6} + M^{20} + 31 \, L^{6} + M^{22} - 6 \, L^{6} + M^{16} + 4 \, L^{7} + M^{24} + 41 \, L^{7} + M^{26} - 30 \, L^{7} + M^{38} + 12 \, L^{7} + M^{40} - 2 \, L^{7} + M^{42} + 4 \, L^{7} + M^{24} - 15 \, L^{7} + M^{26} + 12 \, L^{7} + M^{28} + 35 \, L^{7} + M^{30} - 69 \, L^{7} + M^{32} - 8 \, L^{7} + M^{32} + 4 \, L^{7} + M^{22} - 471 \, L^{10} + 4 \, M^{4} + 42 \, L^{7} + M^{10} ) \]
• \(9_{24}\) is given by the sum of tangles \(1/3 + (-1/3)\) and \(5/2\).

• \(9_{37}\) is given by the sum of tangles \(1/3 + (-1/3)\) and \(5/3\).
Theorem (Mattman-Shimokawa-I., 2011)

Suppose that $N(T)$ and $N(S + T)$ are knots and $N(S)$ is a split link in $S^3$. Then

$$A_{N(T)}^\circ (L, M) \mid A_{N(S+T)}(L, M)$$

Here $A_{K}^\circ (L, M)$ is the product of factors of $A_{K}(L, M)$ containing the variable $L$. 
## §2. Alexander polynomials and epimorphisms

<table>
<thead>
<tr>
<th>RTY</th>
<th>$A_K$ fac.</th>
<th>type</th>
<th>Alex. poly.</th>
<th>epi.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8_{10}$</td>
<td>$1/3, 3/2, -1/3$</td>
<td>$3_1$</td>
<td>A</td>
<td>$(3_1)^3$</td>
</tr>
<tr>
<td>$8_{11}$</td>
<td>$[2, -2, 3, 2, -2]$</td>
<td>$3_1$</td>
<td>B</td>
<td>$(3_1)(6_1)$</td>
</tr>
<tr>
<td>$9_{24}$</td>
<td>$1/3, 5/2, -1/3$</td>
<td>$4_1$</td>
<td>A</td>
<td>$(3_1)^2(4_1)$</td>
</tr>
<tr>
<td>$9_{37}$</td>
<td>$1/3, 5/3, -1/3$</td>
<td>$4_1$</td>
<td>B</td>
<td>$(4_1)(6_1)$</td>
</tr>
<tr>
<td>$10_{21}$</td>
<td>$[2, -2, 5, 2, -2]$</td>
<td>$5_1$</td>
<td>B</td>
<td>$(5_1)(6_1)$</td>
</tr>
<tr>
<td>$10_{40}$</td>
<td>$[2, 2, 3, -2, -2]$</td>
<td>$3_1$</td>
<td>B</td>
<td>$(3_1)(8_8)$</td>
</tr>
</tbody>
</table>

(continued)
### Table: Factorizations of RTя knots (2nd page)

<table>
<thead>
<tr>
<th>RTя</th>
<th>$A_K$ fac.</th>
<th>type</th>
<th>Alex. poly.</th>
<th>epi.</th>
</tr>
</thead>
<tbody>
<tr>
<td>10\text{59}</td>
<td>$2/5, 3/2, -2/5$</td>
<td>$3_1$</td>
<td>$(3_1)(4_1)^2$</td>
<td>$\rightarrow 4_1$</td>
</tr>
<tr>
<td>10\text{62}</td>
<td>$1/3, 5/4, -1/3$</td>
<td>$5_1$</td>
<td>$(3_1)^2(5_1)$</td>
<td>$\rightarrow 3_1$</td>
</tr>
<tr>
<td>10\text{65}</td>
<td>$1/3, 7/4, -1/3$</td>
<td>$5_2$</td>
<td>$(3_1)^2(5_2)$</td>
<td>$\rightarrow 3_1$</td>
</tr>
<tr>
<td>10\text{67}</td>
<td>$1/3, 7/5, -1/3$</td>
<td>$5_2$</td>
<td>$(5_2)(6_1)$</td>
<td>No</td>
</tr>
<tr>
<td>10\text{74}</td>
<td>$1/3, 7/3, -1/3$</td>
<td>$5_2$</td>
<td>$(5_2)(6_1)$</td>
<td>$\rightarrow 5_2$</td>
</tr>
<tr>
<td>10\text{77}</td>
<td>$1/3, 7/2, -1/3$</td>
<td>$5_2$</td>
<td>$(3_1)^2(5_2)$</td>
<td>$\rightarrow 3_1$</td>
</tr>
<tr>
<td>10\text{98}</td>
<td>$1/3, T_0, -1/3$</td>
<td>$3_1 # 3_1$</td>
<td>$(3_1)^2(6_1)$</td>
<td>$\rightarrow 3_1$</td>
</tr>
<tr>
<td>10\text{99}</td>
<td>$1/3, T_1, -1/3$</td>
<td>$3_1 # 3_1^\text{mir}$</td>
<td>$(3_1)^4$</td>
<td>$\rightarrow 3_1$</td>
</tr>
<tr>
<td>10\text{143}</td>
<td>$1/3, 3/4, -1/3$</td>
<td>$3_1$</td>
<td>$(3_1)^3$</td>
<td>$\rightarrow 3_1$</td>
</tr>
<tr>
<td>10\text{147}</td>
<td>$1/3, 3/5, -1/3$</td>
<td>$3_1$</td>
<td>$(3_1)(6_1)$</td>
<td>No</td>
</tr>
</tbody>
</table>
Lemma.

Let $K = N(R + T + \mathcal{Y})$ be an $RT\mathcal{Y}$ knot with $R = R(p/q)$ and $q > 0$. Then

(i) $q > 1$.

(ii) If $K$ is of type A then $\Delta_K(t) = \Delta_{N(T)}(t) \Delta_{D(R)}(t)^2$.

(iii) If $K$ is of type B then $\Delta_K(t) = \Delta_{N(T)}(t) \Delta_{N(R+R(1/1)+\mathcal{Y})}(t)$.

(iv) The knot determinant of $K$ is divisible by $q^2$.

Proposition

Let $K$ be a prime knot of 10 or fewer crossings. Suppose that $K$ is not $8_{18}$, $9_{40}$, $10_{82}$, $10_{87}$, or $10_{103}$. Then $K$ is $RT\mathcal{Y}$ with $N(T)$ a non-trivial knot of 10 or fewer crossings if and only if it is in the above table.
Definition

An epimorphism \( \phi : \pi_1(M_{K_1}) \to \pi_1(M_{K_2}) \) is said to be preserving peripheral structures if \( \phi(\pi_1(\partial M_{K_1})) \subset \pi_1(\partial M_{K_2}) \).

Theorem (Hoste-Shanahan, 2010)

Suppose that there exists an epimorphism \( \phi : \pi_1(M_{K_1}) \to \pi_1(M_{K_2}) \) preserving peripheral structures. Then

- \( \phi(\mu_1) = \mu_2 \) and \( \phi(\lambda_1) = \lambda_2^d \) for some \( d \in \mathbb{Z} \).
- \( A_{K_2}(L, M) \mid (L^d - 1)A_{K_1}(L^d, M) \).
<table>
<thead>
<tr>
<th>RTЯ</th>
<th>$A_K$ fac.</th>
<th>type</th>
<th>Alex. poly.</th>
<th>epi.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$9_{24}$</td>
<td>$1/3, 5/2, -1/3$</td>
<td>4_1</td>
<td>$\text{A}$</td>
<td>$(3_1)^2(4_1) \rightarrow 3_1$</td>
</tr>
<tr>
<td>RTRY</td>
<td>$A_K$ fac.</td>
<td>type</td>
<td>Alex. poly.</td>
<td>epi.</td>
</tr>
<tr>
<td>------</td>
<td>-----------</td>
<td>------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>8_{11}</td>
<td>$[2, -2, 3, 2, -2]$</td>
<td>$3_1$</td>
<td>B</td>
<td>$(3_1)(6_1)$</td>
</tr>
<tr>
<td>9_{37}</td>
<td>$1/3, 5/3, -1/3$</td>
<td>$4_1$</td>
<td>B</td>
<td>$(4_1)(6_1)$</td>
</tr>
</tbody>
</table>

Masaharu ISHIKAWA (Tohoku University)
Fact (Kitano-Suzuki, 2008)

There exists an epimorphism \( \phi : \pi_1(M_{937}) \to \pi_1(M_{41}) \) such that

<table>
<thead>
<tr>
<th>( \pi_1(M_{937}) )</th>
<th>8182, 7283, 9493, 3435, 1515, 5657, 2728, 4948</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\mu_1, \lambda_1) )</td>
<td>(1, 8793152461)</td>
</tr>
<tr>
<td>( (\mu_2, \lambda_2) )</td>
<td>(2, 1234)</td>
</tr>
<tr>
<td>( \phi )</td>
<td>1 \leftrightarrow 2, 2 \leftrightarrow 3, 3 \leftrightarrow 141, 4 \leftrightarrow 3, 5 \leftrightarrow 1, 6 \leftrightarrow 141, 7 \leftrightarrow 4, 8 \leftrightarrow 1, 9 \leftrightarrow 4</td>
</tr>
<tr>
<td>( \phi(\lambda) )</td>
<td>( \bar{4321} = -\lambda )</td>
</tr>
</tbody>
</table>

By Hoste-Shanahan, \( A_{41}(L, M) \mid A_{937}(L, M) \).
$K_{2,k} : (2, k)$-torus knot.

**Corollary (Mattman-Shimokawa-I., 2011)**

Let $K$ be the 2-bridge knot described below, where $k > 2$ is odd and $n > 1$. Then $\pi_1(M_K)$ admits no epimorphism onto $\pi_1(M_{K_2,k})$ preserving peripheral structure, although $A_{K_2,k}(L, M) | A_K(L, M)$.

- First assertion follows from González-Acuña - Ramírez.
- Second assertion is a corollary of our factorization.
§3. Cyclic surgeries

$M$: a compact, connected, irreducible and $\partial$-irreducible 3-manifold whose boundary $\partial M$ is a torus.

$R(M) = \operatorname{Hom}(\pi_1(M), SL(2, \mathbb{C}))$

$X(M)$: the character variety of $M$

- $R(M) \ni \rho \mapsto \chi_\rho \in X(M)$: the character of $\rho$

$\gamma \in \pi_1(M)$

- $I_\gamma : X(M) \to \mathbb{C}$: the regular function defined by

\[ I_\gamma(\chi_\rho) = \chi_\rho(\gamma) \]

- $f_\gamma : X(M) \to \mathbb{C}$: defined by $f_\gamma = I_\gamma^2 - 4$. 

Definition

A 1-dimensional algebraic subset $X_1$ of $X(M)$ is called a norm curve if $f_\alpha$ is not constant for any $\alpha \in H_1(\partial M, \mathbb{Z}) \setminus \{0\}.$

- $X_1$: a norm curve
- $\tilde{X}_1$: the smooth model of the projective completion of $X_1$
- $\alpha \in \pi_1(\partial M)$
- $\|\alpha\|_{x_1}$: the degree of $f_\alpha$ on $\tilde{X}_1$

Lemma

$\| \cdot \|_{x_1}$ is a norm.
\(X_1^{(i)}\) : irreducible component of \(X_1\)
\(d_1^{(i)}\) : the degree of the map \(\mathfrak{i}^*|_{X_1^{(i)}} : X_1^{(i)} \to X(\partial M)\)

**Definition**

The \(A\)-polynomial of \(X_1\) with multiplicity is defined as

\[
A_{d_1}^{d}(L, M) = \prod_{i=1}^{k} A_1^{(i)}(L, M)^{d_1^{(i)}}.
\]

**Theorem (Boyer-Zhang, 2001)**

\[\| \cdot \|_{X_1} = \| \cdot \|_{A_1^d}.\]

**Example:**

\[
A_{4_1}(L, M) = M^4 + L(-1 + M^2 + 2M^4 + M^6 - M^8) + L^2M^4
\]
The next results follow immediately from CGLS.

**Theorem**

Suppose that $N(S + T)$ is a knot, $N(T)$ is a hyperbolic knot and $N(S)$ is a split link in $S^3$. Let $X_0$ be the irreducible component of $X(M_{N(T)})$ containing the character of a discrete faithful representation of $\pi_1(M_{N(T)})$. If $\alpha$ is not a strict boundary class of $N(T)$ associated with an ideal point of $X_0$ and satisfies $\|\alpha\|_{X_0} > \|\mu\|_{X_0}$ then $\pi_1(M_{N(S+T)}(\alpha))$ is not cyclic as well as $\pi_1(M_{N(T)}(\alpha))$ is not.

**Corollary**

Suppose further that $N(S + T)$ is a small knot. If every $\alpha \in H_1(\partial M_{N(T)}; \mathbb{Z}) \setminus \{0\}$ except strict boundary classes of $N(T)$ satisfies $\|\alpha\|_{X_0} > \|\mu\|_{X_0}$ then $N(S + T)$ has no non-trivial cyclic slope.
A 1-dimensional algebraic subset \( Y \) of \( X(M) \) is called an \( r \)-curve if \( \| \cdot \|_Y \) is non-zero, not a norm curve and \( \| \alpha \|_Y = 0 \) only when \( \alpha = r \).

The \( A \)-polynomial of the \((p, q)\)-torus knot has the factor \( 1 + LM^{pq} \) or \( L + M^{pq} \).

Hence the \((p, q)\)-torus knot has the \( r \)-curve with slope \( r = pq \).
$K$: a knot in $S^3$
$M_K$: the complement of $K$.

**Proposition**

Suppose that $X(M)$ contains an algebraic curve $X_1$ consisting of two $r$-curves, with different slopes, containing the characters of irreducible representations. If $\alpha$ is not the slopes of these curves and satisfies $\|\alpha\|_{X_1} > \|\mu\|_{X_1}$ then $\pi_1(M(\alpha))$ is not cyclic.

- $Y$ consists of reducible representations
  \[ \Rightarrow A_Y(L, M) = L - 1 \] (mentioned in CCGLS).
- If $K$ is small and $\|\alpha\|_{X_1} > \|\mu\|_{X_1}$ for any $\alpha \neq \mu$ then $K$ has no non-trivial cyclic slope.
The list of $r$-curves (The torus knots are removed from the list)

<table>
<thead>
<tr>
<th>$8_{11}$</th>
<th>$L + M^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8_{21}$</td>
<td>$L + M^2$</td>
</tr>
<tr>
<td>$9_{23}$</td>
<td>$L + M^{18}$</td>
</tr>
<tr>
<td>$9_{37}$</td>
<td>$L - M^4$</td>
</tr>
<tr>
<td>$9_{38}$</td>
<td>$(1 - M)^2(1 + M)^2$</td>
</tr>
<tr>
<td>$9_{41}$</td>
<td>$1 + LM^2$</td>
</tr>
<tr>
<td>$9_{46}$</td>
<td>$1 + LM^2$</td>
</tr>
<tr>
<td>$9_{48}$</td>
<td>$L - M^4$</td>
</tr>
<tr>
<td>$10_{61}$</td>
<td>$1 - LM^{12}$</td>
</tr>
<tr>
<td>$10_{139}$</td>
<td>$1 - LM^{20}$</td>
</tr>
<tr>
<td>$10_{140}$</td>
<td>$1 - L$</td>
</tr>
<tr>
<td>$10_{141}$</td>
<td>$(L - M^4)(1 + LM^2)$</td>
</tr>
<tr>
<td>$10_{142}$</td>
<td>$1 - LM^{12}$</td>
</tr>
<tr>
<td>$10_{143}$</td>
<td>$L - M^8$</td>
</tr>
<tr>
<td>$10_{144}$</td>
<td>$L - M^{12}$</td>
</tr>
<tr>
<td>$10_{152}$</td>
<td>$(L + M^{11})(L - M^{11})$</td>
</tr>
<tr>
<td>$10_{155}$</td>
<td>$L + M^2$</td>
</tr>
</tbody>
</table>

Remark. Among the RT$R$ knots with torus knot factor, $8_{10}, 8_{11}, 10_{21}, 10_{143}$ and $10_{147}$ are calculated by Culler, though we could not find the $r$-curves in his calculation except $8_{11}$.
Theorem (Boyer-Zhang, 1998)

Suppose that an $r$-curve in $X^{PSL}(M)$ contains the character of an irreducible representation and that $r$ is not a boundary slope of an essential surface in $M$. If $\pi_1(M(\alpha))$ is cyclic then $\Delta(r, \alpha) \leq 1$.

Here $\Delta(p_1/q_1, p_2/q_2) = |p_1q_2 - p_2q_1|$.

Corollary ($SL_2(\mathbb{C})$)-version of Boyer-Zhang, 1998

Let $K$ be a knot in $S^3$. Suppose that the meridian is not a boundary slope of an essential surface (for instance when $K$ is small). If $X(M_K)$ has an $r$-curve then $r \in \mathbb{Z}$.

Proof. Let $Y$ be an $r$-curve in $X(M_K)$. If $Y$ consists of the characters of reducible representations, then $r = 0 \in \mathbb{Z}$. Suppose that $Y$ contains the character of an irreducible representation. There exists an $r$-curve in $X^{PSL}(M_K)$ with the same $r$. Since $1/0$ is not a boundary slope, $r \neq \infty$. Since $M(1/0) = S^3$, $\alpha = 1/0$ is a cyclic slope. Hence $\Delta(p/q, 1/0) \leq 1$ only when $q = 1$. \qed
The following result also follows from Boyer-Zhang.

**Corollary**

Let $K$ be a small knot in $S^3$. Suppose that $X(M_K)$ has an $r_1$-curve and $r_2$-curve with $r_i \neq 0$ for $i = 1, 2$ and $|r_1 - r_2| > 2$. Then $K$ has no cyclic slope.

**Example.** $10_{141} = N(1/4 + 2/3 + (-1/3))$ is small and have two $r$-curves $(L - M^4)(1 + LM^2)$, whose slopes are $-4$ and $2$. Hence $10_{141}$ has no cyclic slope.
References


Thank you for your attention!