Tangle sums and factorization of A-polynomials

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Plan of this talk

§**1. Factorization of A-polynomials**

§**2. Alexander polynomials and epimorphisms**

§**3. Cyclic surgeries**

§1. Factorization of A-polynomials

K : a knot in S^3

M_K : the complement of K

$\imath^* : X(M_K) \to X(\partial M_K)$: induced by the $\imath_\# : \pi_1(\partial M_K) \to \pi_1(M_K)$

$\Lambda \subset R(\partial M_K)$: the set of diagonal representations of $\pi_1(\partial M_K)$

$t|_\Lambda : \Lambda \to X(\partial M_K)$

$p : \Lambda \to \mathbb{C}^* \times \mathbb{C}^*$: taking the left-top entries of $\rho(\mu)$ and $\rho(\lambda)$

$X_1, \cdots, X_k$: irreducible components of $X(M_K)$

$$X_i \xrightarrow{\imath^*} \imath^*(X_i) \xrightarrow{\text{alg. closure in } X(\partial M_K)} Y_i \xrightarrow{p\cdot t|_\Lambda^{-1}} D_i$$

$A_i(L, M)$: the defining equation of D_i

Definition

The ***A*-polynomial** of a knot K is defined as

$$A_K(L, M) = \prod_{i=1}^{k} A_i(L, M).$$

KnotInfo

http://www.indiana.edu/~knotinfo/

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Table of Knot Invariants

Please Cite KnotInfo	Knot Atlas	Knot Theory Links
Knot Theory Calculators	Unknown Values (last updated: 06 Feb 2009)	Acknowledgments

Build a Knot Table. Welcome to KnotInfo. Check the desired boxes in the sections below and then click SUBMIT on the page to produce your desired table of knots. If you do not know the name of a particular knot you are interested in, KnotFinder can help you.

Preferences (Select invariants to hide.)

New advanced search feature is now available! [Advanced Search]

Select knots you want tabulated. **[Advanced Search]** Submit

Specify crossing numbers. The letters a and n designate alternating and nonalternating knots. *12 crossing knots are grouped.*

3-6	7	8	9	10
11a	11n	12a (1-200)	12a (201-400)	12a (401-600)
12a (601-800)	12a (801-1000)	12a (1001-1200)	12a (1200-1288)	12n (1-200)
12n (201-400)	12n (401-600)	12n (601-800)	12n (801-888)	**All**

Names and descriptions. Please select the naming and notational descriptions desired. *Names are linked to diagrams.* Submit

Name	Name Rank	Alternating	DT Name
DT Notation	DT Rank	Classical Conway Name	Conway Notation
Gauss Notation	PD Notation	Braid Notation	Two-Bridge Notation
Fibered	Tetrahedral Census Name		

Three-Dimensional Invariants. Submit

Arc Index	Braid Index	Braid Length	Bridge Index
Crossing Number	Determinant	Nakanishi Index	Polygon Index
Seifert Matrix	Super Bridge Index	Symmetry Type	Three Genus
Crosscap Number	Thurston-Bennequin Number	Morse-Novikov Number	Tunnel Number
Turaev Genus	Unknotting Number		

Concordance and Four-Dimensional Invariants. Submit

Arf Invariant	Smooth Concordance Genus	Topological Concordance Genus	Smooth Concordance Order
Topological Concordance Order	Algebraic Concordance Order	Smooth Four Genus	Topological Four Genus
Smooth 4D Crosscap Number	Topological 4D Crosscap Number	Rasmussen Invariant	Ozsvath-Szabo Tau-Invariant
Signature	Signature Function	Smooth Concordance Crosscap Number	Topological Concordance Crosscap Number

Polynomial Invariants. Submit

A-Polynomial	Alexander Polynomial	Conway Polynomial	HOMFLY Polynomial
Jones Polynomial	Kauffman Polynomial	Khovanov Polynomial	Khovanov Torsion Polynomial

Show polynomials as coefficient vectors

Knot Table: A–Polynomial

http://www.indiana.edu/~knotinfo/descriptions/a_polynomial.htm

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The A-Polynomial

For more information about the A-poynomial we have posted a pdf version of a seminar presentation given by Marc Culler. The original source for the A-polynomial is the paper by Cooper, Culler, Gillet, Long, and P. Shalen, referenced below. We thank Abhijit Champanerkar for helping with the exposition on this page.

There is a map of the $SL_2(\mathbf{C})$ representation space of a knot complement to $\mathbf{C}^* \times \mathbf{C}^*$, given by evaluating the trace of the representation on the meridian and longitude. The closure of the image is a variety defined by a single polynomial, called the A-Polynomial. Jim Hoste gave us information on 2-bridge knots and Marc Culler provided us with further tables, based on glueing equations. These have not been proved to equal the A-polynomial; the issue is described next.

The set of isometry classes of ideal hyperbolic tetrahedra is paramaterized by the upper half complex plane. Thus, if the complement of a knot is decomposed into tetrahedra, the set of glueings that yield hyperbolic structures on the knot complement is determined by the solutions to glueing equations. The set of glueing equations defines an algebraic variety that maps to the $PSL_2(\mathbf{C})$ character variety of the knot. To the image variety there is associated an "A-polynomial", which is the $PSL_2(\mathbf{C})$ version of the classical A-polynomial. In many cases the $PSL_2(\mathbf{C})$ A-polynomial can be computed directly from the glueing and completeness equations by eliminating the tetrahedral parameters to get a 2-variable polynomial. However, the resulting polynomial depends on the choice of the triangulation and in general only divides the $PSL_2(\mathbf{C})$ A-polynomial. For an exposition of this alternative viewpoint of A-polynomials, see the appendix by N. Dunfield to Mahler's Measure and the Dilogarithm by Boyd, Rodrigues-Villegas, and Dunfield or "A-polynomial and Bloch invariants of hyperbolic 3-manifolds" by A. Champanerkar.

We have provided three tables of A-polynomials, all linked in Table of A-Polynomials: two-bridge knots. Jim Hoste has provided us with this table of values for 2-bridge knots of 9 crossings or less.

Table of A-Polynomials (Glueing equations approach). This data, based on glueing equaitons, was provided by Marc Culler.

Table of A-Polynomial: tetrahedral census (Glueing equations approach). This table, also provided by Marc Culler, lists the A-polynomials of knots in the tetrahedral enumeration. There is an overlap in the two tables. Warning: in the overlap, orientations changed for some knots, so one polynomial is related to the other by a change of variable (something like $L \to L^{-1}$).

Warning: a change of orientation, from a knot to its mirror image, changes the A-polynomial. The data in our tables has not be checked for its match to the choice of orientation in our diagrams. Also, the A-polynomial can be defined so that repeated factors are significant. In our table repeated factors have been removed.

Knot Table: List of A–Polynomials of knots

http://www.indiana.edu/~knotinfo/references/a_polys_table_glueing.html

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A-Polynomials-gluing Equations

This list of A-polynomials was compiled by Marc Culler. For a table using the labelings of the tetrahedral enumeration, go to A-Polynomials: tetrahedral enumeration, and for one produced by Jim Hoste for two-bridge knots, visit A-polynomials: two bridge enumeration.

Warning: See the description section for the A-polynomial for details regarding the various defintions of the A-polynomial, and possible distinctions between them.a change of orientation, from a knot to its mirror image, changes the A-polynomial. The data in our tables has not be checked for its match to the choice of orientation in our diagrams. Also, the A-polynomial can be defined so that repeated factors are significant. In our tables all repeated factors have been removed.

A_L103001 := (1*M^6) + (L^1)*(1);

A_L104001 := (1*M^4) + (L^1)*(-1 + 1*M^2 + 2*M^4 + 1*M^6 - 1*M^8) + (L^2)*(1*M^4);

A_L105001 := (1*M^10) + (L^1)*(1);

A_L105002 := (1) + (L^1)*(-1 + 2*M^2 + 2*M^4 - 1*M^8 + 1*M^10) + (L^2)*(1*M^4 - 1*M^6 + 2*M^10 + 2*M^12 - 1*M^14) + (L^3)*(1*M^14);

A_L106001 := (1*M^8) + (L^1)*(-2*M^4 + 3*M^6 + 3*M^8 + 1*M^14 - 1*M^16) + (L^2)*(1 - 3*M^2 - 1*M^4 + 3*M^6 + 6*M^8 + 3*M^10 - 1*M^12 - 3*M^14 + 1*M^16) + (L^3)*(-1 + 1*M^2 + 3*M^8 + 3*M^10 - 2*M^12) + (L^4)*(1*M^8);

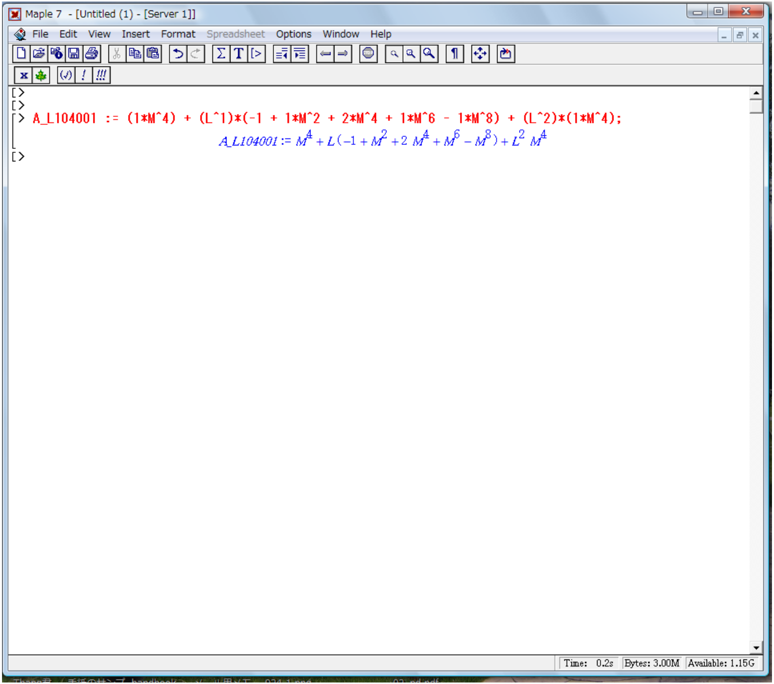
A_L106002 := (1*M^4) + (L^1)*(-1 + 2*M^2 - 1*M^4 - 2*M^6 + 5*M^8 + 5*M^10 - 3*M^12) + (L^2)*(-1*M^2 + 3*M^4 - 1*M^6 - 5*M^8 - 3*M^10 + 12*M^12 + 13*M^14 - 3*M^16 - 8*M^18 + 3*M^20) + (L^3)*(3*M^10 - 8*M^12 - 3*M^14 + 13*M^16 + 12*M^18 - 3*M^20 - 5*M^22 - 1*M^24 + 3*M^26 - 1*M^28) + (L^4)*(-3*M^18 + 5*M^20 + 5*M^22 - 2*M^24 - 1*M^26 + 2*M^28 - 1*M^30) + (L^5)*(1*M^26);

A_L106003 := (1*M^14) + (L^1)*(2*M^8 - 5*M^10 + 1*M^12 + 10*M^14 + 1*M^16 - 5*M^18 + 2*M^20) + (L^2)*(1*M^2 - 4*M^4 + 4*M^6 + 2*M^8 - 6*M^10 + 2*M^12 + 17*M^14 + 2*M^16 - 6*M^18 + 2*M^20 + 4*M^22 - 4*M^24 + 1*M^26) + (L^3)*(1 - 5*M^2 + 3*M^4 + 9*M^6 - 2*M^8 - 21*M^10 + 8*M^12 + 34*M^14 + 8*M^16 - 21*M^18 - 2*M^20 + 9*M^22 + 3*M^24 - 5*M^26 + 1*M^28) + (L^4)*(1*M^2 - 4*M^4 + 4*M^6 + 2*M^8 - 6*M^10 + 2*M^12 + 17*M^14 + 2*M^16 - 6*M^18 + 2*M^20 + 4*M^22 - 4*M^24 + 1*M^26) + (L^5)*(2*M^8 - 5*M^10 + 1*M^12 + 10*M^14 + 1*M^16 - 5*M^18 + 2*M^20) + (L^6)*(1*M^14);

A_L107001 := (1*M^14) + (L^1)*(1);

A_L107002 := (1*M^22) + (L^1)*(1*M^8 - 1*M^10 + 3*M^18 + 4*M^20 - 2*M^22) + (L^2)*(-2*M^4 + 5*M^6 + 1*M^8 - 4*M^10 + 6*M^14 + 5*M^16 + 2*M^18 - 4*M^20 + 1*M^22) + (L^3)*(1 - 4*M^2 + 2*M^4 + 5*M^6 + 6*M^8 - 4*M^12 + 1*M^14 + 5*M^16 - 2*M^18) + (L^4)*(-2 + 4*M^2 + 3*M^4 - 1*M^12 + 1*M^14) + (L^5)*(1);

A_L107003 := (1*M^52) + (L^1)*(2*M^38 - 3*M^40 + 2*M^42 + 5*M^44 - 1*M^48 + 2*M^50 - 1*M^52) + (L^2)*(1*M^24 - 3*M^26 + 3*M^28 + 2*M^30 + 4*M^32 - 4*M^34 + 2*M^36 + 14*M^38 + 2*M^40 - 9*M^42 + 3*M^44) + (L^3)*(-3*M^16 + 10*M^18 - 3*M^20 - 12*M^22 + 6*M^24 + 24*M^26 + 6*M^28 -


Maple 7 - [Untitled (1) - [Server 1]]
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> A_L104001 := (1*M^4) + (L^1)*(-1 + 1*M^2 + 2*M^4 + 1*M^6 - 1*M^8) + (L^2)*(1*M^4);
A_L104001 := M^4 + L (-1 + M^2 + 2 M^4 + M^6 - M^8) + L^2 M^4
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Available: 1.15G

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> A_L104001 := (1*M^4) + (L^1)*(-1 + 1*M^2 + 2*M^4 + 1*M^6 - 1*M^8) + (L^2)*(1*M^4);
```

$$A\_L104001 := M^4 + L(-1 + M^2 + 2M^4 + M^6 - M^8) + L^2 M^4$$

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> A_L109024 := (1*M^54) + (L^1)*(3*M^44 - 13*M^46 + 33*M^48 - 33*M^50 - 24*M^52 + 78*M^54 - 15*M^56 -
25*M^58 + 20*M^60 - 5*M^62) + (L^2)*(3*M^34 - 24*M^36 + 91*M^38 - 199*M^40 + 245*M^42 - 101*M^44 -
289*M^46 + 897*M^48 - 842*M^50 - 674*M^52 + 1835*M^54 - 543*M^56 - 907*M^58 + 1030*M^60 - 252*M^62
- 273*M^64 + 243*M^66 - 79*M^68 + 10*M^70) + (L^3)*(1*M^24 - 11*M^26 + 59*M^28 - 203*M^30 +
497*M^32 - 935*M^34 + 1234*M^36 - 422*M^38 - 2535*M^40 + 6387*M^42 - 5619*M^44 - 5847*M^46 +
19995*M^48 - 11757*M^50 - 18196*M^52 + 27784*M^54 - 2217*M^56 - 18376*M^58 + 15158*M^60 - 667*M^62
- 7947*M^64 + 6383*M^66 - 1259*M^68 - 1487*M^70 + 1410*M^72 - 567*M^74 + 116*M^76 - 10*M^78) +
(L^4)*(2*M^18 - 28*M^20 + 186*M^22 - 755*M^24 + 1969*M^26 - 3132*M^28 + 2018*M^30 + 3102*M^32 -
9493*M^34 + 11065*M^36 - 5142*M^38 - 12302*M^40 + 46515*M^42 - 55362*M^44 - 34299*M^46 +
147946*M^48 - 82077*M^50 - 124349*M^52 + 186668*M^54 - 30947*M^56 - 132273*M^58 + 146652*M^60 -
13022*M^62 - 101277*M^64 + 77765*M^66 + 4801*M^68 - 39260*M^70 + 23857*M^72 - 2320*M^74 - 5823*M^76
+ 4758*M^78 - 2012*M^80 + 516*M^82 - 76*M^84 + 5*M^86) + (L^5)*(1*M^12 - 19*M^14 + 178*M^16 -
1015*M^18 + 3694*M^20 - 8162*M^22 + 7635*M^24 + 11406*M^26 - 43599*M^28 + 32973*M^30 + 67477*M^32 -
147093*M^34 + 5587*M^36 + 247428*M^38 - 165305*M^40 - 270074*M^42 + 331996*M^44 + 281141*M^46 -
543051*M^48 - 317374*M^50 + 926222*M^52 + 186813*M^54 - 1321177*M^56 + 270071*M^58 + 1288151*M^60 -
733456*M^62 - 695881*M^64 + 775723*M^66 + 38320*M^68 - 402955*M^70 + 200725*M^72 + 46685*M^74 -
105368*M^76 + 52111*M^78 - 460*M^80 - 15048*M^82 + 10608*M^84 - 4205*M^86 + 1095*M^88 - 189*M^90 +
20*M^92 - 1*M^94) + (L^6)*(-3*M^8 + 56*M^10 - 488*M^12 + 2522*M^14 - 8112*M^16 + 14786*M^18 -
5515*M^20 - 41612*M^22 + 91245*M^24 - 10726*M^26 - 245488*M^28 + 329170*M^30 + 213227*M^32 -
906741*M^34 + 353261*M^36 + 1398486*M^38 - 1517702*M^40 - 1384323*M^42 + 2879991*M^44 + 848225*M^46
- 3972796*M^48 - 49986*M^50 + 4512594*M^52 - 894885*M^54 - 4159096*M^56 + 1877235*M^58 +
2712789*M^60 - 2361986*M^62 - 601374*M^64 + 1881180*M^66 - 948938*M^68 - 733330*M^70 + 1214651*M^72
- 201354*M^74 - 623707*M^76 + 412621*M^78 + 72895*M^80 - 192791*M^82 + 77022*M^84 + 12136*M^86 -
26750*M^88 + 14815*M^90 - 5124*M^92 + 1237*M^94 - 205*M^96 + 21*M^98 - 1*M^100) + (L^7)*(3*M^4 -
59*M^6 + 521*M^8 - 2662*M^10 + 8327*M^12 - 14504*M^14 + 4332*M^16 + 40110*M^18 - 78968*M^20 -
8053*M^22 + 239382*M^24 - 282143*M^26 - 250368*M^28 + 890978*M^30 - 321216*M^32 - 1476912*M^34 +
1740481*M^36 + 1272328*M^38 - 3471079*M^40 + 198703*M^42 + 4313576*M^44 - 2532416*M^46 -
3362836*M^48 + 4769453*M^50 + 768458*M^52 - 6205978*M^54 + 2578526*M^56 + 6623661*M^58 -
5661932*M^60 - 5627333*M^62 + 7525636*M^64 + 3302892*M^66 - 7290352*M^68 - 541914*M^70 +
5084759*M^72 - 1219685*M^74 - 2320862*M^76 + 1427687*M^78 + 433771*M^80 - 757899*M^82 + 207119*M^84
+ 156810*M^86 - 157343*M^88 + 41733*M^90 + 22526*M^92 - 26413*M^94 + 12540*M^96 - 3567*M^98 +
632*M^100 - 65*M^102 + 3*M^104) + (L^8)*(-1 + 20*M^2 - 181*M^4 + 974*M^6 - 3420*M^8 + 7898*M^10 -
10423*M^12 - 55*M^14 + 30410*M^16 - 48571*M^18 - 29475*M^20 + 207746*M^22 - 184259*M^24 -
382260*M^26 + 868332*M^28 + 176198*M^30 - 2166851*M^32 + 1301637*M^34 + 3518027*M^36 - 4949815*M^38
- 3579780*M^40 + 10838404*M^42 + 1102322*M^44 - 17664776*M^46 + 3673531*M^48 + 23250770*M^50 -
8795064*M^52 - 25668578*M^54 + 12216889*M^56 + 23975844*M^58 - 13018204*M^60 - 18416468*M^62 +
11731741*M^64 + 10771196*M^66 - 8843572*M^68 - 3701319*M^70 + 5118642*M^72 - 611534*M^74 -
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$$+1969 M^{88} - 755 M^{90} + 186 M^{92} - 28 M^{94} + 2 M^{96}) + L^{16} (15158 M^{54} - 1487 M^{44} - 1259 M^{46} + 6383 M^{48} - 7947 M^{50}$$
$$- 667 M^{52} - 18376 M^{56} - 2217 M^{58} + 27784 M^{60} - 18196 M^{62} - 10 M^{36} + 116 M^{38} - 567 M^{40} + 1410 M^{42} - 11757 M^{64}$$
$$+ 19995 M^{66} - 5847 M^{68} - 5619 M^{70} + 6387 M^{72} - 2535 M^{74} - 422 M^{76} + 1234 M^{78} - 935 M^{80} + 497 M^{82} - 203 M^{84} + 59 M^{86}$$
$$- 11 M^{88} + M^{90}) + L^{17} (10 M^{44} - 79 M^{46} + 243 M^{48} - 273 M^{50} - 252 M^{52} + 1030 M^{54} - 907 M^{56} - 543 M^{58} + 1835 M^{60}$$
$$- 674 M^{62} - 842 M^{64} + 897 M^{66} - 289 M^{68} - 101 M^{70} + 245 M^{72} - 199 M^{74} + 91 M^{76} - 24 M^{78} + 3 M^{80})$$
$$+ L^{18} (-5 M^{52} + 20 M^{54} - 25 M^{56} - 15 M^{58} + 78 M^{60} - 24 M^{62} - 33 M^{64} + 33 M^{66} - 13 M^{68} + 3 M^{70}) + L^{19} M^{60}$$

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> factor(A_L109024);
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$$-(-L^2 M^4 + L - M^2 L - 2 M^4 L - M^4 - M^6 L + M^8 L) (91 L^2 M^{34} - 11 L^3 M^{22} + L^5 M^8 + 38225 L^9 M^{84} + 11260 L^9 M^{12}$$
$$- 163416 L^9 M^{86} - 2546365 L^9 M^{72} + 1180137 L^9 M^{76} + 919361 L^9 M^{74} + 381620 L^6 M^{54} - 324 L^6 M^8 - 226162 L^9 M^{22}$$
$$+ 399147 L^9 M^{82} - 372070 L^9 M^{80} - 2 L^6 M^4 + 10 L^{15} M^{40} - 723 L^{15} M^{60} + 854 L^{15} M^{62} - 24 L^2 M^{32} - 25 L M^{48}$$
$$- 27 L^4 M^{16} - L^5 M^{90} - 4643448 L^9 M^{56} + 5742849 L^9 M^{60} + 101956 L^9 M^{58} - 743671 L^{12} M^{48} + 479673 L^{12} M^{52}$$
$$+ 1013983 L^{12} M^{50} + 38705 L^8 M^{16} - 36729 L^8 M^{90} + 65056 L^8 M^{88} - 285561 L^9 M^{48} + 2574796 L^9 M^{52} + 289729 L^9 M^{50}$$
$$+ 238 L^{15} M^{44} - 568 L^{15} M^{54} - 283959 L^6 M^{58} - 28380 L^6 M^{62} + 1421081 L^6 M^{60} + 76 L M^{50} - 1035745 L^6 M^{44}$$
$$+ 614294 L^6 M^{48} + 147612 L^6 M^{46} + 20 L M^{56} - 231924 L^9 M^{54} - 117297 L^9 M^{46} - 1459341 L^9 M^{44} - 13 L M^{42}$$
$$- 2590 L^8 M^{92} + 7743 L^8 M^{10} + 11260 L^8 M^{94} - 5504707 L^9 M^{64} + 4203183 L^9 M^{68} + 508221 L^9 M^{66} - 792 L^{15} M^{52}$$
$$- 651 L^{15} M^{58} + 1667 L^{15} M^{56} + 38 L^6 M^6 - 32 L M^{46} - 196 L^2 M^{36} - 17 L^5 M^{10} - 850218 L^9 M^{70} + 439896 L^9 M^{26}$$
$$+ 132933 L^9 M^{24} - 5 L M^{58} - 707007 L^9 M^{78} + 32205 L^9 M^{20} + 65056 L^9 M^{18} + 33 L M^{44} + 26 L^9 M^4 + 1699 L^9 M^8 - L^9 M^2$$
$$- 165867 L^9 M^{62} - 2065094 L^9 M^{36} + 431569 L^9 M^{34} + 229 L^2 M^{38} + 26 L^8 M^{102} - L^8 M^{104} - 79 L^{15} M^{42} - 61 L^{15} M^{66}$$
$$- 331 L^{15} M^{64} - 594897 L^9 M^{28} + 1343929 L^9 M^{32} - 556611 L^9 M^{30} + 2 L^4 M^{14} - 24 L M^{54} - 323127 L^6 M^{50}$$
$$- 866134 L^6 M^{56} + 101394 L^6 M^{52} - 16 L M^{52} - 61 L^2 M^{40} - 19 M^2 L^7 - 5997 L^8 M^{96} - 283 L^8 M^{100} + 1699 L^8 M^{98}$$
$$- 283 L^9 M^6 - 1240986 L^{12} M^{54} - 445954 L^{12} M^{46} + 652324 L^{12} M^{44} - 248 L^{15} M^{46} + 1003 L^{15} M^{50} - 286 L^{15} M^{48}$$
$$+ 3 L M^{40} + L^3 M^{20} + 3 L^2 M^{30} - 147934 L^9 M^{38} - 362 L^9 M^{42} + 2235115 L^9 M^{40} + M^{50} - 331 L^2 M^{42} + 854 L^2 M^{44}$$
$$- 723 L^2 M^{46} - 651 L^2 M^{48} + 1667 L^2 M^{50} - 568 L^2 M^{52} - 792 L^2 M^{54} + 1003 L^2 M^{56} - 286 L^2 M^{58} - 248 L^2 M^{60}$$

Time: 2.1s Bytes: 4.00M Available: 986M

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$$- 45 M^{30} + 125 M^{32} - 104 M^{34} - 235 M^{36} + 683 M^{38} - 491 M^{40} - 649 M^{42} + 1321 M^{44} - 159 M^{46} - 787 M^{48} + 523 M^{50}$$
$$- 148 M^{52} - 57 M^{54} + 198 M^{56} - 181 M^{58} + 86 M^{60} - 24 M^{62} + 3 M^{64})$$
$$+ L^{13} (-4 M^{36} + 15 M^{38} - 15 M^{40} - 28 M^{42} + 73 M^{44} - 10 M^{46} - 41 M^{48} + 33 M^{50} - 14 M^{52} + 3 M^{54}) + L^{14} M^{44}$$

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> factor(A_L109037);
```

$$-(-L + M^4)(-L^2 M^4 + L - M^2 L - 2 M^4 L - M^4 - M^6 L + M^8 L)(3 M^4 L + L^2 M^{18} + 2 L^2 M^8 - 2 M^{16} L^2 - 2 M^{14} L + 3 L^2 M^{14}$$
$$+ M^8 L^3 + 2 M^6 L + 2 M^{10} L + 6 M^{12} L + 2 M^{12} L^2 + 6 L^2 M^6 - 2 M^2 L - 7 M^8 L - 2 L^2 M^4 + L + M^{10} - 7 L^2 M^{10})(31 L^2 M^{34}$$
$$+ 34 L^2 M^{18} + 12 L M^{28} + 1091 L^3 M^{22} + 4 L^5 M^8 - 30 M^{18} L - 26 L^2 M^8 - 89 L^4 M^4 + 105 L^3 M^4 - 254 M^{16} L^2 - 2 M^{14} L$$
$$+ 85 L^2 M^{14} - 15 L M^{30} + 829 L^2 M^{20} + 41 L M^{20} + 363 L^2 M^{32} - 1196 L^4 M^{16} - 69 L M^{24} + 841 L^4 M^{12} + 164 M^8 L^3$$
$$- 37 L^6 M^{44} - 26 L^6 M^{48} + 42 L^6 M^{46} + 1604 L^3 M^{16} + 12 M^{16} L - 113 L^2 M^{36} - 45 L^5 M^{10} - 442 M^{14} L^3 - 980 M^{12} L^3$$
$$+ 35 M^{26} L + 45 L^2 M^{38} - 376 L^3 M^{18} - 37 M^{12} L^2 + 1085 L^2 M^{26} - 96 L^2 M^{28} + 4 L M^{32} - 742 L^2 M^{22} - 687 L^2 M^{24}$$
$$+ 449 L^4 M^{14} + 625 M^{10} L^3 + 242 L^4 M^6 - 277 L^3 M^6 + 8 L^2 M^6 + 15 L^4 M^2 - 17 L^3 M^2 + 8 L^6 M^{50} - L^6 M^{52} - 6 L^2 M^{40}$$
$$- 186 M^8 L^4 - 1208 L^3 M^{20} - 588 L^2 M^{30} - L^2 M^4 - M^{24} + L^3 + 190 L^3 M^{44} - 45 L^3 M^{46} + 4 L^3 M^{48} - 1236 L^3 M^{34}$$
$$- 751 L^3 M^{36} + 1073 L^3 M^{38} - 140 L^3 M^{40} - 308 L^3 M^{42} + 583 L^3 M^{24} - 1091 L^3 M^{26} - 1005 L^3 M^{28} + 947 L^3 M^{30}$$
$$+ 1433 L^3 M^{32} - L^4 + 15 L^4 M^{54} + 841 L^4 M^{44} - 471 L^4 M^{46} - 186 L^4 M^{48} + 242 L^4 M^{50} - 89 L^4 M^{52} - L^4 M^{56}$$
$$+ 1728 L^4 M^{34} + 1290 L^4 M^{36} - 919 L^4 M^{38} - 1196 L^4 M^{40} + 449 L^4 M^{42} - 1614 L^4 M^{24} - 1044 L^4 M^{26} + 1840 L^4 M^{28}$$
$$- 1044 L^4 M^{30} - 1614 L^4 M^{32} - 919 L^4 M^{18} + 1290 L^4 M^{20} + 1728 L^4 M^{22} - 17 L^5 M^{54} - 980 L^5 M^{44} + 625 L^5 M^{46}$$
$$+ 164 L^5 M^{48} - 277 L^5 M^{50} + 105 L^5 M^{52} + L^5 M^{56} + 1091 L^5 M^{34} - 1208 L^5 M^{36} - 376 L^5 M^{38} + 1604 L^5 M^{40} - 442 L^5 M^{42}$$
$$+ 1433 L^5 M^{24} + 947 L^5 M^{26} - 1005 L^5 M^{28} - 1091 L^5 M^{30} + 583 L^5 M^{32} + 1073 L^5 M^{18} - 751 L^5 M^{20} - 1236 L^5 M^{22}$$
$$+ 190 L^5 M^{12} - 308 L^5 M^{14} - 140 L^5 M^{16} - 742 L^6 M^{34} + 829 L^6 M^{36} + 34 L^6 M^{38} - 254 L^6 M^{40} + 85 L^6 M^{42} + 363 L^6 M^{24}$$
$$- 588 L^6 M^{26} - 96 L^6 M^{28} + 1085 L^6 M^{30} - 687 L^6 M^{32} + 45 L^6 M^{18} - 113 L^6 M^{20} + 31 L^6 M^{22} - 6 L^6 M^{16} + 4 L^7 M^{34}$$
$$+ 41 L^7 M^{36} - 30 L^7 M^{38} + 12 L^7 M^{40} - 2 L^7 M^{42} + 4 L^7 M^{24} - 15 L^7 M^{26} + 12 L^7 M^{28} + 35 L^7 M^{30} - 69 L^7 M^{32} - L^8 M^{32}$$
$$+ 4 L M^{22} - 471 M^{10} L^4 + 42 L^2 M^{10})$$

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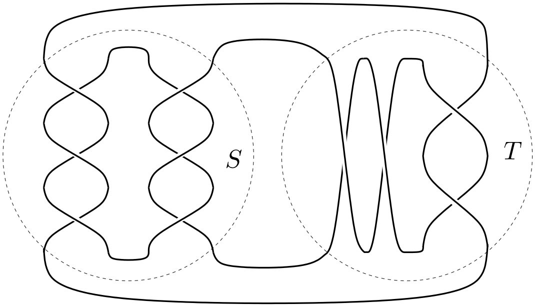
- $\mathbf{9_{24}}$ **is given by the sum of tangles** $1/3 + (-1/3)$ **and** $5/2$**.**



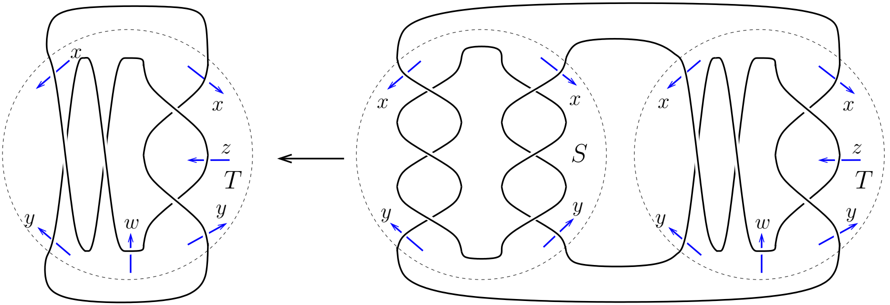
- $\mathbf{9_{37}}$ **is given by the sum of tangles** $1/3 + (-1/3)$ **and** $5/3$**.**

Theorem (Mattman-Shimokawa-I., 2011)

Suppose that $N(T)$ and $N(S+T)$ are knots and $N(S)$ is a split link in S^3. Then

$$A^{\circ}_{N(T)}(L,M) \mid A_{N(S+T)}(L,M)$$

Here $A^{\circ}_K(L,M)$ is the product of factors of $A_K(L,M)$ containing the variable L.



§2. Alexander polynomials and epimorphisms

	RTЯ	A_K fac.	type	Alex. poly.	epi.
8_{10}	$1/3, 3/2, -1/3$	3_1	A	$(3_1)^3$	$\to 3_1$
8_{11}	$[2, -2, 3, 2, -2]$	3_1	B	$(3_1)(6_1)$	No
9_{24}	$1/3, 5/2, -1/3$	4_1	A	$(3_1)^2(4_1)$	$\to 3_1$
9_{37}	$1/3, 5/3, -1/3$	4_1	B	$(4_1)(6_1)$	$\to 4_1$
10_{21}	$[2, -2, 5, 2, -2]$	5_1	B	$(5_1)(6_1)$	No
10_{40}	$[2, 2, 3, -2, -2]$	3_1	B	$(3_1)(8_8)$	$\to 3_1$

(continued)

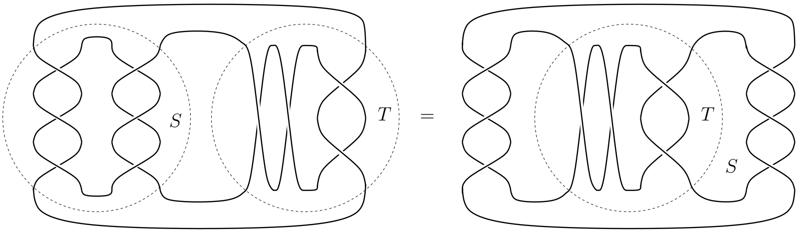


Table: Factorizations of RTЯ knots (2nd page)

	RTЯ	A_K fac.	type	Alex. poly.	epi.
10_{59}	$2/5, 3/2, -2/5$	3_1	A	$(3_1)(4_1)^2$	$\to 4_1$
10_{62}	$1/3, 5/4, -1/3$	5_1	A	$(3_1)^2(5_1)$	$\to 3_1$
10_{65}	$1/3, 7/4, -1/3$	5_2	A	$(3_1)^2(5_2)$	$\to 3_1$
10_{67}	$1/3, 7/5, -1/3$	5_2	B	$(5_2)(6_1)$	No
10_{74}	$1/3, 7/3, -1/3$	5_2	B	$(5_2)(6_1)$	$\to 5_2$
10_{77}	$1/3, 7/2, -1/3$	5_2	A	$(3_1)^2(5_2)$	$\to 3_1$
10_{98}	$1/3,\ T_0,\ -1/3$	$3_1\#3_1$	B	$(3_1)^2(6_1)$	$\to 3_1$
10_{99}	$1/3,\ T_1,\ -1/3$	$3_1\#3_1^{\text{mir}}$	A	$(3_1)^4$	$\to 3_1$
10_{143}	$1/3, 3/4, -1/3$	3_1	A	$(3_1)^3$	$\to 3_1$
10_{147}	$1/3, 3/5, -1/3$	3_1	B	$(3_1)(6_1)$	No

Lemma.

Let $K = N(R + T + Я)$ be an RTЯ knot with $R = R(p/q)$ and $q > 0$. Then

(i) $q > 1$.

(ii) **If K is of type A then $\Delta_K(t) = \Delta_{N(T)}(t)\Delta_{D(R)}(t)^2$.**

(iii) **If K is of type B then $\Delta_K(t) = \Delta_{N(T)}(t)\Delta_{N(\mathrm{R}+\mathrm{R}(1/1)+Я)}(t)$.**

(iv) **The knot determinant of K is divisible by q^2.**

Proposition

Let K be a prime knot of 10 or fewer crossings. Suppose that K is not 8_{18}, 9_{40}, 10_{82}, 10_{87}, or 10_{103}. Then K is RTЯ with $N(T)$ a non-trivial knot of 10 or fewer crossings if and only if it is in the above table.

Definition

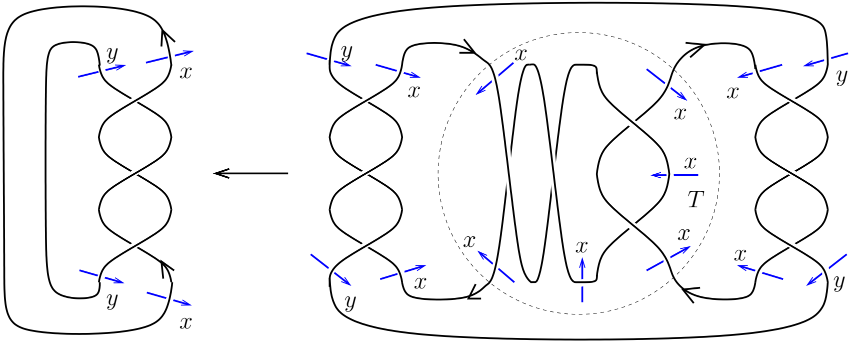
An epimorphism $\phi : \pi_1(M_{K_1}) \to \pi_1(M_{K_2})$ is said to be preserving peripheral structures if $\phi(\pi_1(\partial M_{K_1})) \subset \pi_1(\partial M_{K_2})$.

Theorem (Hoste-Shanahan, 2010)

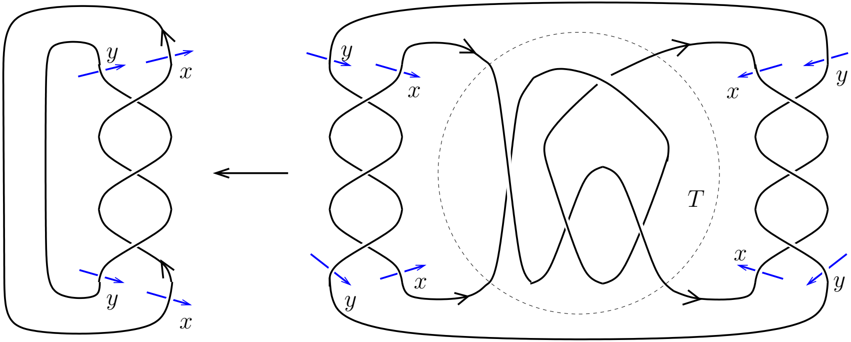
Suppose that there exists an epimorphism $\phi : \pi_1(M_{K_1}) \to \pi_1(M_{K_2})$ preserving peripheral structures. Then

- $\phi(\mu_1) = \mu_2$ **and** $\phi(\lambda_1) = \lambda_2^d$ **for some** $d \in \mathbb{Z}$.
- $A_{K_2}(L, M) \mid (L^d - 1)A_{K_1}(L^d, M)$.

	RTR	A_K fac.	type	Alex. poly.	epi.
9_{24}	$1/3, 5/2, -1/3$	4_1	A	$(3_1)^2(4_1)$	$\to 3_1$



	RTЯ	A_K fac.	type	Alex. poly.	epi.
8_{11}	$[2, -2, 3, 2, -2]$	3_1	B	$(3_1)(6_1)$	No
9_{37}	$1/3, 5/3, -1/3$	4_1	B	$(4_1)(6_1)$	$\to 4_1$



	RTЯ	A_K fac.	type	Alex. poly.	epi.
9_{37}	$1/3, 5/3, -1/3$	4_1	B	$(4_1)(6_1)$	$\to 4_1$

Fact (Kitano-Suzuki, 2008)

There exists an epimorphism $\phi : \pi_1(M_{9_{37}}) \to \pi_1(M_{4_1})$ such that

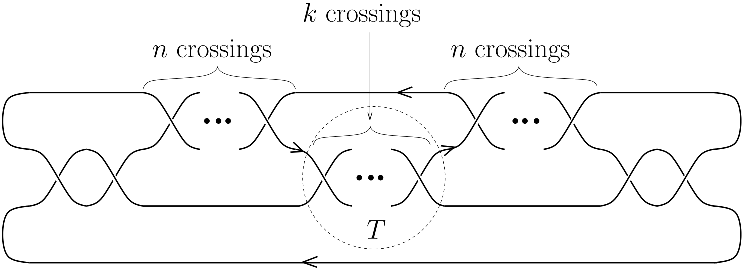
$\pi_1(M_{9_{37}})$	$81\bar{8}\bar{2}, 72\bar{8}\bar{3}, 94\bar{9}\bar{3}, 34\bar{3}\bar{5}, 15\bar{1}\bar{5}, 56\bar{5}\bar{7}, 27\bar{2}\bar{8}, 49\bar{4}\bar{8}$
(μ_1, λ_1)	$(1, \bar{8}\bar{7}9\bar{3}1\bar{5}\bar{2}461)$
(μ_2, λ_2)	$(2, \bar{1}2\bar{3}4)$
ϕ	$1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 14\bar{1}, 4 \mapsto 3, 5 \mapsto 1,$ $6 \mapsto \bar{1}41, 7 \mapsto 4, 8 \mapsto 1, 9 \mapsto 4$
$\phi(\lambda)$	$\bar{4}3\bar{2}1 = -\lambda$

By Hoste-Shanahan, $A_{4_1}(L, M) \mid A_{9_{37}}(L, M)$.

$K_{2,k}$: $(2,k)$-torus knot.

Corollary (Mattman-Shimokawa-I., 2011)

Let K be the 2-bridge knot described below, where $k > 2$ is odd and $n > 1$. Then $\pi_1(M_K)$ admits no epimorphism onto $\pi_1(M_{K_{2,k}})$ preserving peripheral structure, although $A_{K_{2,k}}(L,M) \mid A_K(L,M)$.



- **First assertion follows from González-Acuña - Ramírez.**
- **Second assertion is a corollary of our factorization.**

§3. Cyclic surgeries

M : a compact, connected, irreducible and ∂-irreducible 3-manifold whose boundary ∂M is a torus.

$R(M) = \mathrm{Hom}(\pi_1(M), SL(2, \mathbb{C}))$

$X(M)$: the character variety of M

- $R(M) \ni \rho \mapsto \chi_\rho \in X(M)$: the character of ρ

$\gamma \in \pi_1(M)$

- $I_\gamma : X(M) \to \mathbb{C}$: the regular function defined by

$$I_\gamma(\chi_\rho) = \chi_\rho(\gamma)$$

- $f_\gamma : X(M) \to \mathbb{C}$: defined by $f_\gamma = I_\gamma^2 - 4$.

Definition

A 1-dimensional algebraic subset X_1 of $X(M)$ is called a norm curve if f_α is not constant for any $\alpha \in H_1(\partial M, \mathbb{Z}) \setminus \{0\}$.

X_1 **: a norm curve**
$\tilde{X}_1$ **: the smooth model of the projective completion of** X_1
$\alpha \in \pi_1(\partial M)$
$\|\alpha\|_{X_1}$ **: the degree of** f_α **on** $\tilde{X}_1$

Lemma

$\|\cdot\|_{X_1}$ **is a norm.**

$X_1^{(i)}$: irreducible component of X_1
$d_1^{(i)}$: the degree of the map $\imath^*|_{X_1^{(i)}} : X_1^{(i)} \to X(\partial M)$

Definition

The A-polynomial of X_1 with multiplicity is defined as

$$A_1^{\mathrm{d}}(L, M) = \prod_{i=1}^{k} A_1^{(i)}(L, M)^{d_1^{(i)}}.$$

Theorem (Boyer-Zhang, 2001)

$\| \cdot \|_{X_1} = \| \cdot \|_{A_1^{\mathrm{d}}}$.

Example:

$A_{4_1}(L, M) = M^4 + L(-1 + M^2 + 2M^4 + M^6 - M^8) + L^2 M^4$

The next results follow immediately from CGLS.

Theorem

Suppose that $N(S+T)$ is a knot, $N(T)$ is a hyperbolic knot and $N(S)$ is a split link in S^3. Let X_0 be the irreducible component of $X(M_{N(T)})$ containing the character of a discrete faithful representation of $\pi_1(M_{N(T)})$. If α is not a strict boundary class of $N(T)$ associated with an ideal point of X_0 and satisfies $\|\alpha\|_{X_0} > \|\mu\|_{X_0}$ then $\pi_1(M_{N(S+T)}(\alpha))$ is not cyclic as well as $\pi_1(M_{N(T)}(\alpha))$ is not.

Corollary

Suppose further that $N(S+T)$ is a small knot. If every $\alpha \in H_1(\partial M_{N(T)}; \mathbb{Z}) \setminus \{0\}$ except strict boundary classes of $N(T)$ satisfies $\|\alpha\|_{X_0} > \|\mu\|_{X_0}$ then $N(S+T)$ has no non-trivial cyclic slope.

Definition

A 1-dimensional algebraic subset Y of $X(M)$ is called an r-curve if $\|\cdot\|_Y$ is non-zero, not a norm curve and $\|\alpha\|_Y = 0$ only when $\alpha = r$.

Proposition (CCGLS, 1994)

The A-polynomial of the (p, q)-torus knot has the factor $1 + LM^{pq}$ or $L + M^{pq}$.

Hence the (p, q)-torus knot has the r-curve with slope $r = pq$.

K : a knot in S^3

M_K : the complement of K.

Proposition

Suppose that $X(M)$ contains an algebraic curve X_1 consisting of two r-curves, with different slopes, containing the characters of irreducible representations. If α is not the slopes of these curves and satisfies $\|\alpha\|_{X_1} > \|\mu\|_{X_1}$ then $\pi_1(M(\alpha))$ is not cyclic.

- Y consists of reducible representations

  $\Rightarrow A_Y(L, M) = L - 1$ (mentioned in CCGLS).

- If K is small and $\|\alpha\|_{X_1} > \|\mu\|_{X_1}$ for any $\alpha \neq \mu$ then K has no non-trivial cyclic slope.

The list of r-curves (*The torus knots are removed from the list*)

$$8_{11} : L + M^6$$
$$8_{21} : L + M^2$$
$$9_{23} : L + M^{18}$$
$$9_{37} : L - M^4$$
$$9_{38} : (1 - M)^2(1 + M)^2$$
$$9_{41} : 1 + LM^2$$
$$9_{46} : 1 + LM^2$$
$$9_{48} : L - M^4$$
$$10_{61} : 1 - LM^{12}$$
$$10_{139} : 1 - LM^{20}$$
$$10_{140} : 1 - L$$
$$10_{141} : (L - M^4)(1 + LM^2)$$
$$10_{142} : 1 - LM^{12}$$
$$10_{143} : L - M^8$$
$$10_{144} : L - M^{12}$$
$$10_{152} : (L + M^{11})(L - M^{11})$$
$$10_{155} : L + M^2$$

Remark. Among the RTЯ knots with torus knot factor, 8_{10}, 8_{11}, 10_{21}, 10_{143} and 10_{147} are calculated by Culler, though we could not find the r-curves in his calculation except 8_{11}.

Theorem (Boyer-Zhang, 1998)

Suppose that an r-curve in $X^{PSL}(M)$ contains the character of an irreducible representation and that r is not a boundary slope of an essential surface in M. If $\pi_1(M(\alpha))$ is cyclic then $\Delta(r, \alpha) \leq 1$.

Here $\Delta(p_1/q_1, p_2/q_2) = |p_1q_2 - p_2q_1|$.

Corollary ($SL_2(\mathbb{C})$-version of Boyer-Zhang, 1998)

Let K be a knot in S^3. Suppose that the meridian is not a boundary slope of an essential surface (for instance when K is small). If $X(M_K)$ has an r-curve then $r \in \mathbb{Z}$.

Proof. **Let Y be an r-curve in $X(M_K)$. If Y consists of the characters of reducible representations, then $r = 0 \in \mathbb{Z}$. Suppose that Y contains the character of an irreducible representation. There exists an r-curve in $X^{PSL}(M_K)$ with the same r. Since $1/0$ is not a boundary slope, $r \neq \infty$. Since $M(1/0) = S^3$, $\alpha = 1/0$ is a cyclic slope. Hence $\Delta(p/q, 1/0) \leq 1$ only when $q = 1$.** □

The following result also follows from Boyer-Zhang.

Corollary

Let K be a small knot in S^3. Suppose that $X(M_K)$ has an r_1-curve and r_2-curve with $r_i \neq 0$ for $i = 1, 2$ and $|r_1 - r_2| > 2$. Then K has no cyclic slope.

Example. $10_{141} = N(1/4 + 2/3 + (-1/3))$ is small and have two r-curves $(L - M^4)(1 + LM^2)$, whose slopes are -4 and 2. Hence 10_{141} has no cyclic slope.

Refereces

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- [Hoste-Shanahan] *Epimorphisms and boundary slopes of 2-bridge knots*, Alge. & Geom. Topology, 2010.
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Thank you for your attention!