An approach to classifying links up to link-homotopy using quandle colorings

Ayumu Inoue

(ayumu.inoue@math.titech.ac.jp)

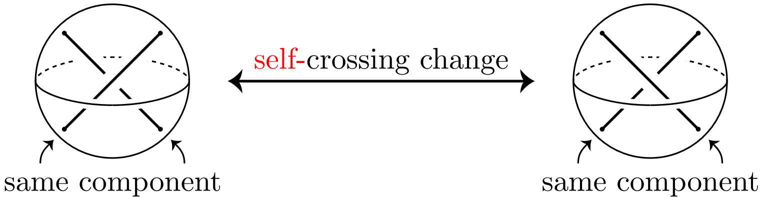
Tokyo Institute of Technology

May 28, 2012

1. Introduction

link-homotopy is ...

ambient isotopy $+$



Rough history

- J. Milnor (1954, 1957)
  - Defined the notion of link-homotopy
  - Defined Milnor invariants ($\overline{\mu}$ invariants)
  - Classified 3-component links up to link-homotopy completely
- J. P. Levine (1988)
  - Enhanced Milnor invariants
  - Classified 4-component links up to link-homotopy completely
- N. Habegger and X. S. Lin (1990)
  - Gave a necessary and sufficient condition for link-homotopic
  - Gave an algorithm judging two links are link-homotopic or not

Motivation

"Classify link-homotopy classes by invariants"

$$\text{numerical invariants} \Rightarrow \begin{cases} \text{easy to compute} \\ \text{easy to compare} \end{cases}$$

This talk

We have a lot of numerical invariants
if we modify the definition of a quandle cocycle invariant slightly.

Talk plan

2. Review of quandle cocycle invariant

Definition (quandle)

X : set ($\neq \emptyset$)

$* : X \times X \to X$: binary operation

$(X, *)$: quandle

$\overset{\text{def}}{\Leftrightarrow}$ $*$ satisfies the following axioms:

(Q1) ${}^\forall x \in X$, $x * x = x$.

(Q2) ${}^\forall x \in X$, $*x : X \to X$ ($\bullet \mapsto \bullet * x$) is bijective.

(Q3) ${}^\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

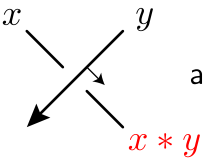
Definition (coloring)

X : quandle

D : oriented link diagram

$\mathscr{C} : \{\text{arcs of } D\} \to X$: X-coloring of D

$\overset{\text{def}}{\Leftrightarrow}$ $\mathscr{C}$ satisfies the condition



at each crossing.

Proposition

$\sharp\{X\text{-colorings of a diagram}\}$ is invariant under Reidemeister moves.

Definition (2-cocycle)

X : quandle

A : abelian group

$\theta : X \times X \to A$: 2-cocycle of X

$\overset{\text{def}}{\Leftrightarrow}$ θ satisfies the following conditions:

(C1) ${}^\forall x \in X$, $\theta(x, x) = 0$.

(C2) ${}^\forall x, y, z \in X$,

$$\theta(x, y) + \theta(x * y, z) = \theta(x, z) + \theta(x * z, y * z).$$

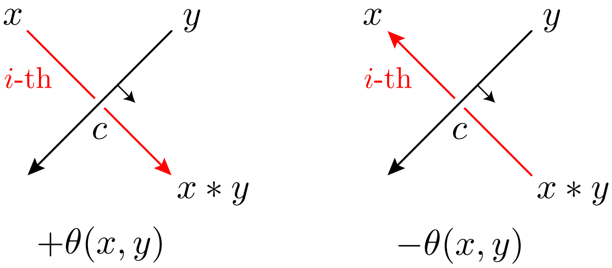
Definition (weight)

$\mathscr{C}$: X-coloring of a diagram

$\theta : X \times X \to A$: 2-cocycle

The i-th weight of $\mathscr{C}$ a.w. θ is a value

$$W(\mathscr{C}, \theta; i) = \sum_{c} \operatorname{sign}(c) \cdot \theta(x, y) \in A.$$



Theorem (J. S. Carter et al. 2003)

X : quandle

A : abelian group

$\theta : X \times X \to A$: 2-cocycle

For each link L, the multiset

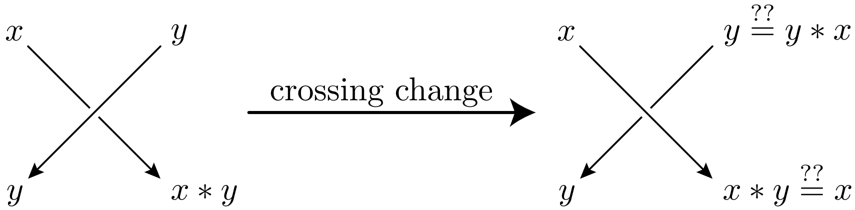
$$\Phi(L,\theta;i) = \{W(\mathscr{C},\theta;i) \in A \mid \mathscr{C} : X\text{-coloring of a diagram of } L\}$$

is invariant under Reidemeister moves.

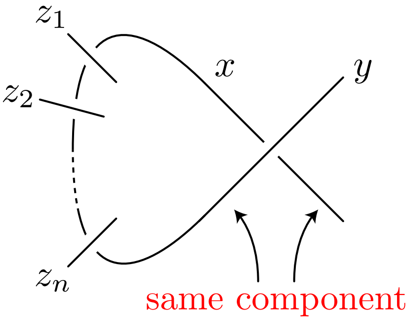
We call $\Phi(L,\theta;i)$ the i-th quandle cocycle invariant of L a.w. θ.

3. How do we ensure link-homotopy invariance?

Investigation for X-colorings



A crossing change does NOT relate X-colorings, in general.



$$
\begin{aligned}
y &= (*\, z_n)^{\varepsilon_n} \circ \cdots \circ (*\, z_2)^{\varepsilon_2} \circ (*\, z_1)^{\varepsilon_1}(x) \\
&= \varphi(x) \quad (\varphi \in \mathrm{Inn}(X)).
\end{aligned}
$$

- $\mathrm{Aut}(X) := \{\varphi : X \to X \text{ auto.}\}$: automorphism group of X
- $\mathrm{Inn}(X) := \langle\, *\, x : X \to X \ (x \in X) \,\rangle \lhd \mathrm{Aut}(X)$

  : inner automorphism group of X

Definition (quasi-trivial quandle)

X : quandle

X : quasi-trivial

$\overset{\text{def}}{\Leftrightarrow}$ ${}^{\forall}x \in X$, ${}^{\forall}\varphi \in \mathrm{Inn}(X)$, $x * \varphi(x) = x$.

Proposition

X : quasi-trivial quandle

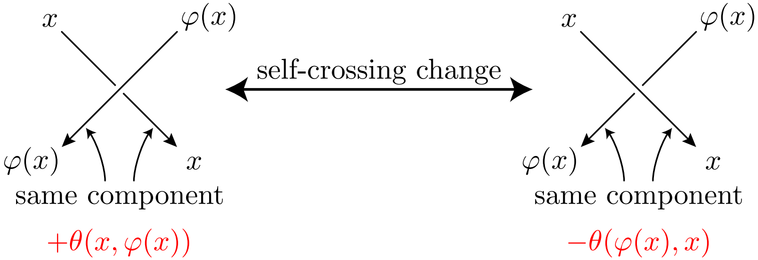
$\sharp\{X\text{-colorings of a diagram}\}$ is invariant under link-homotopy.

Investigation for weights

X : quasi-trivial quandle

$\mathscr{C}$: X-coloring of a diagram

$\theta : X \times X \to A$: 2-cocycle



Consider the following condition:

(C3) $^{\forall}x \in X, {}^{\forall}\varphi \in \mathrm{Inn}(X), \ \theta(x, \varphi(x)) = 0$

Theorem

X : quasi-trivial quandle

A : abelian group

$\theta : X \times X \to A$: 2-cocycle satisfying the condition (C3)

For a link L, the i-th quandle cocycle invariant $\Phi(L, \theta; i)$ is invariant under link-homotopy.

4. Example (non-triviality of the Borromean rings)

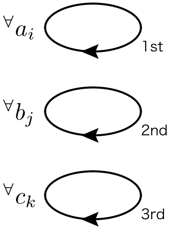
X : quasi-trivial quandle

$*$	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	c_1	c_2	c_3	c_4
a_1	a_1	a_1	a_1	a_1	a_2	a_2	a_2	a_2	a_3	a_3	a_3	a_3
a_2	a_2	a_2	a_2	a_2	a_1	a_1	a_1	a_1	a_4	a_4	a_4	a_4
a_3	a_3	a_3	a_3	a_3	a_4	a_4	a_4	a_4	a_1	a_1	a_1	a_1
a_4	a_4	a_4	a_4	a_4	a_3	a_3	a_3	a_3	a_2	a_2	a_2	a_2
b_1	b_3	b_3	b_3	b_3	b_1	b_1	b_1	b_1	b_2	b_2	b_2	b_2
b_2	b_4	b_4	b_4	b_4	b_2	b_2	b_2	b_2	b_1	b_1	b_1	b_1
b_3	b_1	b_1	b_1	b_1	b_3	b_3	b_3	b_3	b_4	b_4	b_4	b_4
b_4	b_2	b_2	b_2	b_2	b_4	b_4	b_4	b_4	b_3	b_3	b_3	b_3
c_1	c_2	c_2	c_2	c_2	c_3	c_3	c_3	c_3	c_1	c_1	c_1	c_1
c_2	c_1	c_1	c_1	c_1	c_4	c_4	c_4	c_4	c_2	c_2	c_2	c_2
c_3	c_4	c_4	c_4	c_4	c_1	c_1	c_1	c_1	c_3	c_3	c_3	c_3
c_4	c_3	c_3	c_3	c_3	c_2	c_2	c_2	c_2	c_4	c_4	c_4	c_4

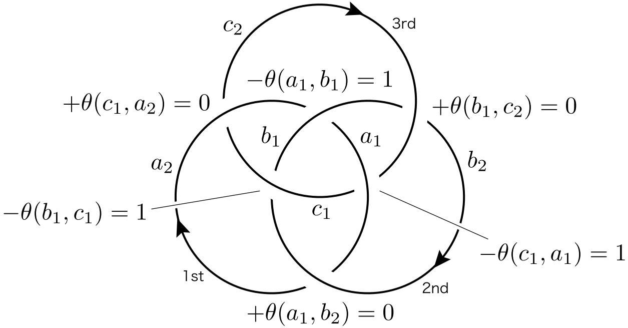
$\theta : X \times X \to \mathbb{Z}_2$: 2-cocycle satisfying the condition (C3)

θ	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	c_1	c_2	c_3	c_4
a_1	0	0	0	0	1	0	1	0	1	1	0	0
a_2	0	0	0	0	0	1	0	1	0	0	1	1
a_3	0	0	0	0	1	0	1	0	1	1	0	0
a_4	0	0	0	0	0	1	0	1	0	0	1	1
b_1	1	1	0	0	0	0	0	0	1	0	1	0
b_2	0	0	1	1	0	0	0	0	0	1	0	1
b_3	1	1	0	0	0	0	0	0	1	0	1	0
b_4	0	0	1	1	0	0	0	0	0	1	0	1
c_1	1	0	1	0	1	1	0	0	0	0	0	0
c_2	0	1	0	1	0	0	1	1	0	0	0	0
c_3	1	0	1	0	1	1	0	0	0	0	0	0
c_4	0	1	0	1	0	0	1	1	0	0	0	0

L_1



L_2 (Borromean rings)



$W(\mathscr{C}, \theta; i) = 0$

$W(\mathscr{C}, \theta; i) = 1$

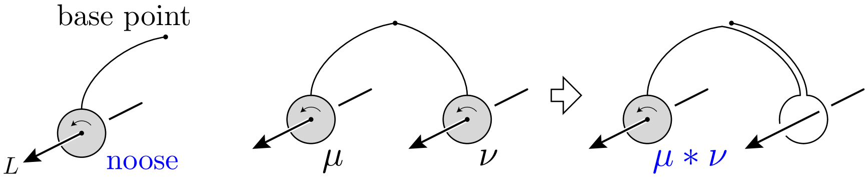
$$\therefore\ L_1 \not\sim L_2.$$

Remark

$$\sharp\{X\text{-colorings of } L_1\} = \sharp\{X\text{-colorings of } L_2\}.$$

4. Backstage

L : link



$Q(L) := \{\text{nooses of } L\}/\text{homotopy}.$

$(Q(L), *)$: knot quandle of L (D. Joyce 1982, S. V. Matveev 1982)

X : quandle

$\mathscr{C}$: X-coloring of a diagram of L $\overset{1:1}{\longleftrightarrow}$ $f_{\mathscr{C}} : Q(L) \to X$: homo.

$L = K_1 \cup K_2 \cup \cdots \cup K_n$

$[K_i] \in H_2^Q(Q(L);\mathbb{Z})$: i-th fundamental class

X : quandle

$\theta : X \times X \to A$: 2-cocycle ($\theta \in Z_Q^2(X;A)$)

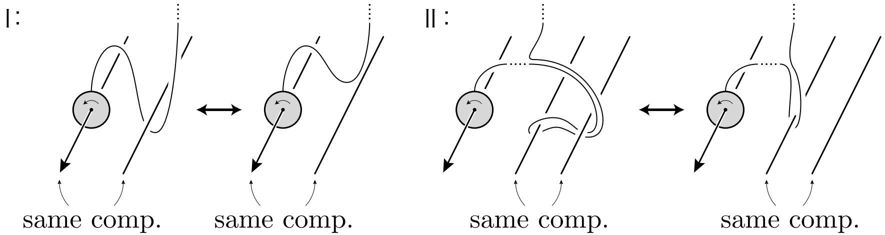
$f_{\mathscr{C}} : Q(L) \to X$: homo. ($\leftrightarrow \mathscr{C}$: X-coloring of L)

$W(\mathscr{C},\theta;i) = \langle [\theta], f_{\mathscr{C}}^*([K_i]) \rangle$.

Theorem (M. Eisermann 2003)

$K_1,\ldots,K_m$: non-trivial, $K_{m+1},\ldots,K_n$: trivial

$H_2^Q(Q(L);\mathbb{Z}) = \mathrm{span}_{\mathbb{Z}}\{[K_1],\ldots,[K_m]\} \cong \mathbb{Z}^m$.



$RQ(L) := Q(L)/(\text{the above moves})$

$(RQ(L), *)$: reduced knot quandle of L (J. R. Hughes 2011)

Theorem (J. R. Hughes 2011)

$RQ(L)$ is invariant under link-homotopy.

X : quasi-trivial quandle

$\mathscr{C}$: X-coloring of a diagram of L $\overset{1:1}{\longleftrightarrow}$ $f_{\mathscr{C}} : RQ(L) \to X$: homo.

X : quasi-trivial quandle

A : abelian group

$H_n^{Q,qt}(X;A)$ ($H^n_{Q,qt}(X;A)$)

: quasi-trivial quandle (co)homology group

$[K_i] \in H_2^{Q,qt}(RQ(L);\mathbb{Z})$: i-th fundamental class

(well-defined up to link-homotopy)

Remark

$\theta : X \times X \to A$: 2-cocycle

θ satisfies the condition (C3) $\Leftrightarrow$ $\theta \in Z^2_{Q,qt}(X;A)$.

X : quasi-trivial quandle

$\theta : X \times X \to A$: 2-cocycle satisfying (C3) ($\theta \in Z^2_{Q,qt}(X; A)$)

$f_{\mathscr{C}} : RQ(L) \to X$: homo. ($\leftrightarrow \mathscr{C}$: X-coloring of L)

$W(L, \theta; i) = \langle [\theta], f^*_{\mathscr{C}}([K_i]) \rangle$.

Theorem

$(L = K_1 \cup K_2 \cup \cdots \cup K_n)$

$K_1, \ldots, K_m$: non-trivial up to link-homotopy

$K_{m+1}, \ldots, K_n$: trivial up to link-homotopy

$H_2^{Q,qt}(RQ(L); \mathbb{Z})$ is generated by $[K_1], [K_2], \ldots, [K_m]$.