Lens space surgeries along certain 2-component links and Reidemeister-Turaev torsion

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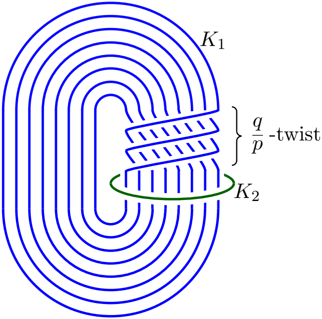
§1. Introduction and Main results

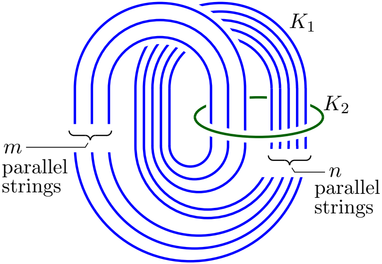
Overview. Alexander polynomial of a link restricts the coefficients of lens space surgery along the link.

We focus a pair of certain two-component links $B_{p,q}$ and $A_{m,n}$, whose one component is the zero-framed unknot (thus a surgery from $S^1 \times S^2$), and determine the coefficient of the other component to yield a lens space.

The link is related to a certain subfamily of lens space surgery of knots, and also to 4-dim. topology (the rational homology 4-ball, used in *Rational blow-down*).



$B_{p,q}$ (ex. $(8,3)$)



$A_{m,n}$ (ex. $(3,5)$)

In either link, K_2 is an unknot. We assume its coefficient is 0.
K_1 is a torus knot:

- In $B_{p,q}$, the standard $T(p,q)$.
- In $A_{m,n}$, K_1 is $T(m,n)$, but not in the standard position.

Lemma (Y)

Under the correspondence between (p, q) and (m, n) by the Algorithm below,

$$(A_{m,n}; mn, 0) \;\cong\; (B_{p,q}; pq-1, 0) \;\cong\; L(p^2, pq-1).$$

Algorithm A

For given pair (p, q) with $\gcd(p, q) = 1$ and $p > q$, starting with $(p-q, q)$, we get (m, n) as follows:

Ex. $(p, q) = (9, 2) \Rightarrow (m, n) = (4, 5)$

$$\begin{array}{ccccccccc} (7,2) & \to_L & (5,2) & \to_L & (3,2) & \to_L & (1,2) & \to_R & (1,1) \\ (1,1) & \to_L & (2,1) & \to_L & (3,1) & \to_L & (4,1) & \to_R & (4,5) \end{array}$$

Q. Does there exist another lens space surgery along $B_{p,q}$ or $A_{m,n}$?

Theorem (Main Theorem)

Assuming $r = \alpha/\beta \in \mathbb{Q}$.

(1) $(B_{p,q}; \alpha/\beta, 0)$ *is a lens space*

$\Leftrightarrow \; |\alpha - \beta pq| = 1$ *(ie,* $r = pq \pm \frac{1}{\beta}$*)* — $L(p^2\beta, \alpha)$

(2) $(A_{m,n}; r, 0)$ *is a lens space*

$\Leftrightarrow \cdot \; r = mn$ *(as Lemma)* *or* — $L((m+n)^2, m\overline{n})$

$\cdot \; (m, n) = (2, 3)$ *and* $r = 7$ *(Unexpected)* — $L(25, 7)$

where $\overline{n}n \equiv 1 \mod (m+n)^2$.

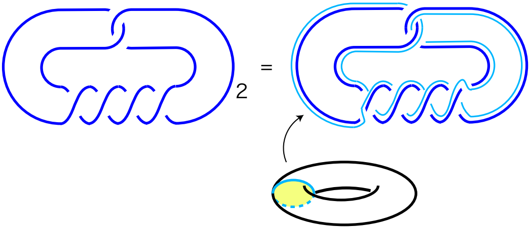
§2. Lens space surgery

Dehn surgery $=$ Cut and paste of a soliod torus.

$$(K; p) := \left(S^3 \setminus \text{open nbd} N(K)\right) \cup_\partial \text{Solid torus.}$$

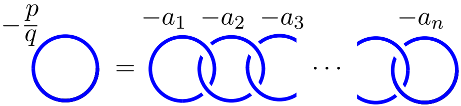
Coefficient (in $\mathbb{Z}$) "framing" $=$ a *parallel* curve ($\subset \partial N(K)$) of K,
or the linking number.

Solid torus is reglued such as "the meridian comes to the parallel"

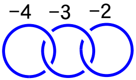


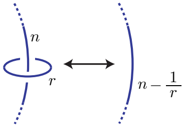
Lens space $L(p,q)$

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots - \cfrac{1}{a_n}}} \qquad (a_i > 1)$$



$L(18,5)$ $\quad \dfrac{18}{5} = 4 - \dfrac{1}{\frac{5}{2}} = 4 - \cfrac{1}{3 - \cfrac{1}{2}}$





For $n \in \mathbb{Z}$, $r \in \mathbb{Q}$

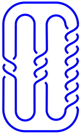
"Which $(K;r)$ is a lens space?" K : a knot.

ex.1 ['71 L. Moser] **Torus knots.**

$$\begin{array}{l}|\alpha-\beta pq|=1 \ \text{ie,}\\ \alpha/\beta = pq \pm \frac{1}{n}\ (n\in\mathbb{Z})\end{array} \quad \Rightarrow \quad (T(p,q);\alpha/\beta)\cong L(\alpha,-\beta p^2).$$

$K := T(3,5)$, then $(K;16)=L(16,7)$ and $(K;14)=L(14,5)$.

ex.2 ['80 R. Fintushel, R. Stern] **Hyperbolic** knot!
$K := P(-2,3,7)$, then $(K;19) = -L(19,7)$.
$(K;18) = -L(18,7)$.



— Cyclic surgery theorem ([CGLS]'87), Berge's list ('90), Heegaard Floer theory(200x), ...

Seifert v.s. Hyperbolic

Roughly speaking,
Torus knot (Seifert) — periodic — Zero's are on the unit circle

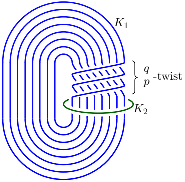
$$\Delta_{T(p,q)}(t) \doteq \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$$

$\Rightarrow$ A torus knot yields many lens spaces.

Hyperbolic knot — pseudo Anosov — out of the unit circle
$K = P(-2,3,7)$

$$\Delta_K(t) \doteq t^{10} - t^9 + t^7 - t^6 + t^5 - t^4 + t^3 - t + 1$$

(is known to be "Lehmer's polynomial")
$\Rightarrow$ A hyperbolic knot can yield lens space, but exceptionally.



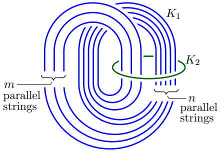
Alexander ploynomial of $B_{p,q}$
$\gcd(p, q) = 1$.
Ex. $(p, q) = (7, 3)$.

$$\Delta_{B_{7,3}}(t, x) = 1 + t^3x + t^6x^2 + t^9x^3 + t^{12}x^4 + t^{15}x^5 + t^{18}x^6$$

$$\Delta_{B_{p,q}}(t, x) = \frac{(t^q x)^p - 1}{t^q x - 1}$$

—— Periodic ——

Alexander ploynomial of $A_{m,n}$



$\gcd(m,n)=1.$

Ex. $(m,n)=(3,7)$.

$\Delta_{A_{3,7}}(t,x)$

$= 1+t^3x+t^6x^2+t^7x^3+t^9x^4+t^{12}x^5+t^{14}x^6+t^{15}x^7+t^{18}x^8+t^{21}x^9$

"less periodic"

Degrees (k_i) of t is "sorted sequence of multiples of m and n"

Lemma

$$\Delta_{A_{m,n}}(t,x) = \sum_{i=0}^{m+n-1} t^{k_i}x^i$$

$$(m\mathbb{Z}\cup n\mathbb{Z})\cap[0,mn] = \{\,0=k_0,\ k_1,k_2,\cdots,k_{m+n-1}=mn\,\}.$$

§3. Reidemeister-Turaev torsion

[Kadokami's Method]

Let X be a finite CW complex.

$\pi : \tilde{X} \to X$ its maximal abelian covering.

Then $\tilde{X}$ has a CW structure induced by that of X and π, the cell chain complex $\mathbf{C}_*$ of $\tilde{X}$ has a $\mathbb{Z}[H]$-module structure, where $H = H_1(X; \mathbb{Z})$ is the first homology of X.

For an integral domain R (We use $\mathbb{Q}(\zeta_d)$) and a ring homomorphism $\psi : \mathbb{Z}[H] \to R$,
"the chain complex of $\tilde{X}$ related with ψ",

$$\mathbf{C}_*^{\psi} := \mathbf{C}_* \otimes_{\mathbb{Z}[H]} Q(R),$$

where $Q(R)$ is the quotient field of R.

The Reidemeister torsion of X related with ψ is defined

$$\mathbf{C}_*^{\psi} \text{ acyclic} \Rightarrow \qquad \tau^{\psi}(X) \in Q(R),$$

up to multiplication of $\pm\psi(h)$ $(h \in H)$.

$$\tau^{\psi}(X) := \text{“} \prod_{q=0}^{\dim} \det\left[\mathbf{b}_q, \overline{\mathbf{b}_{q-1}}/\mathbf{c}_q\right]^{(-1)^{q+1}} \text{”}$$

In the case $R = \mathbb{Z}[H]$ and $\psi = \mathrm{id}$, we omit ψ as $\tau(X)$.

History: Reidemeister torsion (Franz, de Rham, Reidemeister, Whitehead, ...) gave us classification of lens spaces $L(p, q)$.

Notations: We will use the usual ones.

E_L	the complement of L.
m_i, l_i	a meridian and a longitude of the i-th component.
$[m_i], [l_i]$	their homology classes.
$\Delta_L(t_1, \ldots, t_\mu)$	the Alexander polynomial of L, where t_i is represented by $[m_i]$.
$(L; r_1, \ldots, r_\mu)$	the result of Dehn surgery along L, where $r_i = p_i/q_i \in \mathbb{Q} \cup \{\infty, \emptyset\}$ is the surgery coefficient of K_i.
V_i	the solid torus attached along K_i
m'_i, $[m'_i]$	a meridian of V_i, and its homology class.
l'_i, $[l'_i]$	an oriented core curve of V_i, its homology class.

Lemma 1. (Surgery formula I) [Turaev '70s]
Suppose that ∂E consists of μ tori.
For $M = E \cup V_1 \cup V_2 \cup \cdots \cup V_\mu$,
if $\psi([l_i']) \neq 1$ $(i = 1, \ldots, n)$, then

$$\tau^{\psi}(M) \doteq \tau^{\psi'}(E) \prod_{i=1}^{\mu} (\psi([l_i']) - 1)^{-1},$$

where $\psi' = \psi \circ \iota_*$, ($\iota_*$ is a ring homomorphism induced by the inclusion).

Each solid torus $\cup V_i$ contributes as $(\psi([l_i']) - 1)^{-1}$.

Reidemeister torsion is closely related to Alexander polynomial.
· Link case is slightly different from Knot case.

Lemma 2. [Milnor '62]
Let $\Delta_L(t_1, \ldots, t_\mu)$ be the Alexander polynomial of a μ-component link $L = K_1 \cup \ldots \cup K_\mu$ in S^3, where $t_i = [m_i]$, the meridian of K_i $(i = 1, \ldots, \mu)$. Then

$$\tau(E_L) \doteq \begin{cases} \Delta_L(t_1)(t_1 - 1)^{-1} & (\mu = 1), \\ \Delta_L(t_1, \ldots, t_\mu) & (\mu \geq 2). \end{cases}$$

Lemma 3. (Surgery formula II) [Sakai '84, Turaev '86]

(1) In the case $M = (K; p/q)$ $(|p| \geq 2)$, *we have* $H = H_1(M) \cong \langle T \mid T^p = 1 \rangle \cong \mathbb{Z}/|p|\mathbb{Z}$, *where* $T = [m]$.

For a divisor d (≥ 2) *of* p, *and* $\psi_d : \mathbb{Z}[H] \to \mathbb{Q}(\zeta_d)$ *by* $\psi_d(T) = \zeta_d$, *We have*

$$\tau^{\psi_d}(M) \doteq \Delta_K(\zeta_d)(\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}} - 1)^{-1}$$

where $q\bar{q} \equiv 1 \pmod{p}$.

Since $L(p,q)$ is $(\mathrm{unknot}; -p/q)$, we have

$$\boxed{\tau^{\psi_d}(L(p,q)) \doteq (\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}} - 1)^{-1}}$$

· Link case is slightly different from Knot case.

(2) In the case $M = (L; p_1/q_1, \ldots, p_\mu/q_\mu)$ ($\mu \geq 2$).
We take integers r_i and s_i satisfying $p_i s_i - q_i r_i = -1$.

Let F be a field and $\psi : \mathbb{Z}[H_1(M)] \to F$ a ring homomorphism. If $\psi([m_i]^{r_i}[l_i]^{s_i}) \neq 1$ ($i = 1, \ldots, \mu$), then we have

$$\tau^{\psi}(M) \doteq \Delta_L(\psi([m_1]), \ldots, \psi([m_\mu])) \prod_{i=1}^{\mu} (\psi([m_i]^{r_i}[l_i]^{s_i}) - 1)^{-1}.$$

We use $\mathbb{Q}(\zeta_d)$ as the field F.

§4. Cyclotomic fields and Polynomials

Definition (the d-th cyclotomic field)

$$\mathbb{Q}(\zeta_d) := \mathbb{Q}[\zeta_d] \quad \subset \mathbb{C} \qquad {\zeta_d}^d = 1$$

$$\begin{aligned}
\mathbb{Q}(\zeta_3) &= \mathbb{Q}\langle 1, \zeta_3 \rangle &&\cong \mathbb{Q}[t]/(t^2+t+1) \\
\mathbb{Q}(\zeta_4) &= \mathbb{Q}\langle 1, \zeta_4 \rangle &&\cong \mathbb{Q}[t]/(t^2+1) \\
\mathbb{Q}(\zeta_{12}) &= \mathbb{Q}\langle 1, \zeta_{12}, \zeta_{12}^2, \zeta_{12}^3 \rangle &&\cong \mathbb{Q}[t]/(t^4-t^2+1) \\
\mathbb{Q}(\zeta_d) & &&\cong \mathbb{Q}[t]/(\Phi_d(t))
\end{aligned}$$

where $\Phi_d(t)$ is "the d-th cyclotomic polynomial", whose degree is $\sharp(\mathbb{Z}/d\mathbb{Z})^\times = \sharp\{\text{coprime integers to } d \text{ in } \mathbb{Z}/d\mathbb{Z}\}$.

Its Galois group is $\begin{array}{ccc} (\mathbb{Z}/d\mathbb{Z})^\times & \cong & \mathrm{Gal}\,(\mathbb{Q}(\zeta_d)/\mathbb{Q}) \\ j & \mapsto & (\sigma_j : \zeta \mapsto \zeta^j) \end{array}$

Definition (d-norm)

The *d-norm* of x in $\mathbb{Q}(\zeta_d)$ is defined as

$$N_d(x) = \prod_{\sigma \in \mathrm{Gal}\,(\mathbb{Q}(\zeta_d)/\mathbb{Q})} \sigma(x), \qquad \in \mathbb{Q}$$

Fact

· *The map $N_d : \mathbb{Q}(\zeta_d) \setminus \{0\} \to \mathbb{Q} \setminus \{0\}$ is a group homomorphism.*
· *If $x \in \mathbb{Z}[\zeta_d]$, then $N_d(x) \in \mathbb{Z}$.*

ex. · $N_d(\pm\zeta_d) = \begin{cases} \pm 1 & (d = 2), \\ 1 & (d \geq 3). \end{cases}$

· $N_d(1 - \zeta_d) = \begin{cases} \ell & (d \text{ is a power of a prime } \ell \geq 2), \\ 1 & (\text{otherwise}). \end{cases}$

· If $(j, d) = 1$, then $N_d(1 - \zeta_d^j) = N_d(\sigma_j(1 - \zeta_d)) = N_d(1 - \zeta_d)$.

Lemma (Norm and Lens space surgery [Kad])

If $(K;p)$ is a lens space, for any divisor d (≥ 2) of p,

$$N_d(\Delta_K(\zeta_d)) = \pm 1$$

ex.1 $(T(p,q); pq \pm 1) = -L(pq \pm 1, p^2)$

$$\Delta_{T(p,q)}(\zeta) = \frac{(\zeta^{pq}-1)(\zeta-1)}{(\zeta^p-1)(\zeta^q-1)} \qquad d|(pq \pm 1)$$

ex.2 ([KY]) $(P(-2,3,7);19) = -L(19,7) = -L(19,11)$

$$\Delta_{P(-2,3,7)}(t) \equiv \frac{(t^{7\cdot 11}-1)(t-1)}{(t^7-1)(t^{11}-1)} \mod (t^{19}-1)$$

Combinatorial Euler structure

For a homology lens space M with $H = H_1(M) \cong \mathbb{Z}/P\mathbb{Z}[T]$, we consider

$$\tau^{\psi_d}(M) \in \mathbb{Q}(\zeta_d)$$

w.r.t the homomorphism $\psi_d(T) = \zeta_d$,

$$\boxed{\text{for any divisors } d \text{ of } P},$$

and every primitive d-th root ζ_d of unity, except 1 itself.

In the process, we fix *Combinatorial Euler structure*, the choice of the basis of the complex.

(Another reason is to fix the ambiguity $\pm\zeta^m$.)

Chinese Remainder theorem

$$\begin{array}{l} x \equiv 2 \mod 3 \\ x \equiv 3 \mod 5 \\ x \equiv 2 \mod 7 \end{array} \Rightarrow x \equiv {}^{\exists!}23 \mod 105 (= 3 \cdot 5 \cdot 7)$$

We use it.
By $\mathbb{Q}(\zeta_d) \cong \mathbb{Q}[t]/(\Phi_d(t))$ and $\prod_{d|N} \Phi_d(t) = (t^N - 1)$,

we have

$$\bigoplus_{d|N, d\geq 2} \mathbb{Q}(\zeta_d) \cong \mathbb{Q}[t, t^{-1}]/(t^{N-1} + \cdots + t^2 + t + 1)$$

$$\{\tau^{\psi_d}(M)\}_{d\geq 2, d|p}$$

Lemma (Identity of symmetric Laurent polynomials)

If two symmetric Laurent polynomials

$$F(t) = a_0 + \sum_{i=1}^{[\frac{N}{2}]-1} a_i(t^i + t^{-i}), \quad G(t) = b_0 + \sum_{i=1}^{[\frac{N}{2}]-1} b_i(t^i + t^{-i}),$$

satistfy $F(\zeta_d) = G(\zeta_d)$ *for every divisor* $\underline{d \geq 2}$ *of* N,

then, we have $F(t) = G(t)$, *ie,* $a_i = b_i$ $(\forall i)$.

Because the range of the degrees is restricted:

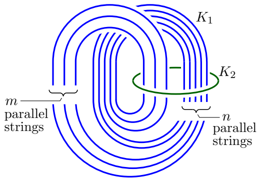
$$\text{deg-span}(t^{N-1} + \cdots + t^2 + t + 1) = N - 1 \; > 2\left(\left[\frac{N}{2}\right] - 1\right).$$

§5. Outline of the proof.

The 1st half : Calculus.

1-1. Calculate $\Delta_{A_{m,n}}(t,x) = \det(I - xM(m,n))$
By Brau-rep. of the braids. (Thanks to Prof. Morifuji)

$$M(3,5) = \begin{bmatrix} 0 & 0 & -t & 1 & 0 & 0 & 0 \\ 0 & 0 & -t^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -t^3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -t^4 & 0 & 0 & 0 & 1 \\ 0 & 0 & -t^5 & 0 & 0 & 0 & 0 \\ t^5 & 0 & -t^5 & 0 & 0 & 0 & 0 \\ 0 & t^5 & -t^5 & 0 & 0 & 0 & 0 \end{bmatrix}$$



$$\Delta_{A_{m,n}}(t,x) \doteq \sum_{i=0}^{m+n-1} t^{k_i} x^i,$$

$(\{\, 0 = k_0,\ k_1, k_2, \cdots, k_{m+n-1} = mn \,\} = (m\mathbb{Z} \cup n\mathbb{Z}) \cap [0, mn].)$

1-2. Study the homology generators and relations, to use the surgery formula II (Link case)

$$\tau^{\psi}(M) \doteq \Delta_L(\psi([m_1]), \dots, \psi([m_\mu])) \prod_{i=1}^{\mu} (\psi([m_i]^{r_i}[l_i]^{s_i}) - 1)^{-1}.$$

$M = (K_1 \cup K_2; \alpha/\beta, 0)$ $\qquad H_1(E_L;\mathbb{Z}) = \langle [m_1], [m_2] \rangle$

Under $H_1(M) \cong \langle T \,|\, T^{(m+n)^2\beta} = 1 \rangle \cong \mathbb{Z}/(m+n)^2\beta\mathbb{Z}$, we have

$$[m_1] = T^{(m+n)\beta}, \quad [m_2] = T^{-\alpha}, \quad [l_1'] = T^{m+n}.$$

Thus, for $M_1 = E_L \cup V_1 = (K_1 \cup K_2; \alpha/\beta, \emptyset)$

$$\begin{aligned} \tau(M_1) &\doteq \Delta_{A_{m,n}}(T^{(m+n)\beta}, T^{-\alpha})(T^{m+n} - 1)^{-1} \\ &= \left(\sum_{i=0}^{m+n-1} T^{k_i(m+n)\beta - i\alpha} \right) (T^{m+n} - 1)^{-1} \end{aligned}$$

It looks as if $\psi(T) = \zeta_d$ with a divisor d of $m+n$ is impossible...

But, deforming

$$= \quad T^{-i\alpha} \sum_{i=0}^{m+n-1} \frac{T^{k_i(m+n)\beta} - 1}{T^{m+n} - 1} + \frac{\sum_{i=0}^{m+n-1} T^{-i\alpha}}{T^{m+n} - 1}$$

and $\displaystyle\sum_{i=0}^{m+n-1} T^{-i\alpha} = \frac{T^{-(m+n)\alpha} - 1}{T^{-\alpha} - 1}$, we can take $\psi(T) = \zeta_d$.

A kind of l'Hopital's rule.

Lemma (R-T Torsion)

Finally, we get the Reidemeister-Turaev torsion of M

$$\tau^{\psi_d}(M) \doteq \frac{\beta R(m,n) - \alpha}{(\zeta - 1)^2}$$

with "magic element" $R(m,n) = (\zeta - 1) \displaystyle\sum_{i=0}^{m+n-1} k_i \zeta^i$

1-3. Calculus on $R(m,n)$ in $\mathbb{Q}(\zeta_d)$

Lemma (on the magic element $R(m,n)$)

(1) *$R(m,n)$ is a real number.*

(2) $$R(m,n) = mn + \frac{1}{2}\sum_{i=1}^{m+n-1} (k_{i-1} - k_i)(\zeta^i + \zeta^{-i}).$$

(3) — *Omitted (combinatorial)*

(4) $$R(m,n) = m(n+1) + \sum_{j=1}^{m-1} (m-j)\left(\xi^j + \xi^{-j}\right),$$ *and*

(5) $$R(m,n) = \left|\frac{\xi^m - 1}{\xi - 1}\right|^2 + mn,$$
where $\xi = \zeta^{\overline{m}}$ *(ie,* $\zeta = \xi^m$*) with* $m\overline{m} \equiv 1 \mod (m+n)$.

ex. $(3,7)$ 

The 2nd half : Estimate M to be a lens space.

If $M = (K_1 \cup K_2; \alpha/\beta, 0)$ is a lens space $L(p, q)$ (for $\exists q$), ...

$$\begin{array}{rcl} \tau^{\psi_d}(M) & \doteq & \tau^{\phi_d}(L(p,q)) \\ \dfrac{\beta R(m,n) - \alpha}{(\xi^m - 1)^2} & \doteq & \dfrac{1}{(\xi^i - 1)(\xi^j - 1)} \quad (\text{for } \exists i, j) \end{array}$$

2-1. Norm condition

Lemma

$$N_d(\beta R(m,n) - \alpha) = \pm 1$$

$$\beta R(m,n) - \alpha = \beta \left| \frac{\xi^m - 1}{\xi - 1} \right|^2 - (\alpha - mn\beta) \qquad \Rightarrow \alpha - mn\beta \geq 0$$

2-2. To the identity of polynomials

$$\frac{\beta R(m,n)-\alpha}{(\xi^m-1)^2} \doteq \frac{1}{(\xi^i-1)(\xi^j-1)} \in \mathbb{Q}(\zeta_d) \quad (\text{for } \exists i,j),$$

for every primitive d-th root $\xi = \xi_d$ of unity,

$\boxed{\text{for any divisors } d \text{ of } m+n}$ Target is $L((m+n)^2, m\bar{n})$

By Lemma 4 (Chinese Remainder Thm.), the equalities above lift to an identity of symmetric Laurent polynomials.
We regard it as an equation of i, j.
We can assume $1 \leq i+j \leq m+n-1$ and $i+j$ is even.

2-3. The final part is divided into 3 cases :

(1) $m = 2$

(2) $n = m + 1$ (1)(2) includes $(m, n) = (2, 3)$ and $\alpha/\beta = 7$

(3) Otherwise. Only $\alpha/\beta = mn$.

(1) Case $m = 2$ and $i + j < n + 1$. Set $\alpha'' = \alpha - 2(n+1)\beta$

The identity is

$$t^{-\frac{i+j-2}{2}} \cdot \left\{\beta\left(t + t^{-1}\right) - \alpha''\right\} \frac{(t^i - 1)(t^j - 1)}{(t-1)^2} = t^{-1} \cdot \frac{(t^2-1)^2}{(t-1)^2}.$$

Note. the degree $(i + j)/2 \leq [\frac{m+n}{2}] - 1$.

$\Rightarrow \beta\left(t + t^{-1}\right) - \alpha'' = t + 2 + t^{-1}$ $\Rightarrow \beta = 1, \alpha = 2n$.

(1') Case $m = 2$ and $i + j = n + 1$. Set $e := \frac{j-i}{2} < \frac{n+1}{2}$

The identity is deformed to Set $\alpha'' = \alpha - 2(n+1)\beta$.

$$\begin{aligned} & (\beta - \alpha'')\left(t^{\frac{n+1}{2}} + t^{-\frac{n+1}{2}}\right) + \beta\left(t^{\frac{n-1}{2}} + t^{-\frac{n-1}{2}}\right) \\ & \quad -\beta\left(t^{e+1} + t^{-(e+1)}\right) + \alpha''\left(t^{e} + t^{-e}\right) - \beta\left(t^{e-1} + t^{-(e-1)}\right) \\ = & \quad (t^2 + t^{-2}) - 2. \end{aligned}$$

Note. the degree $\frac{n+1}{2} = [\frac{m+n}{2}] - 1$ and "$0 = 0$" at $t = 1$.

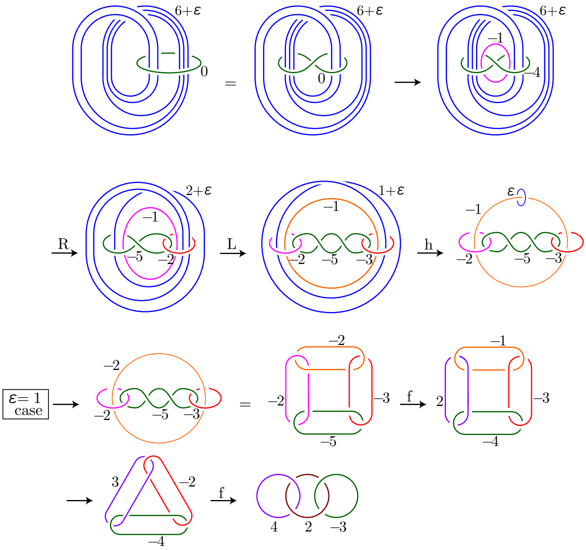
$\Rightarrow e = 1, \beta = 1, n = 3$

$$-\alpha''(t^2 + t^{-2}) + (1 + \alpha'')(t + t^{-1}) - 2 = (t^2 + t^{-2}) - 2$$

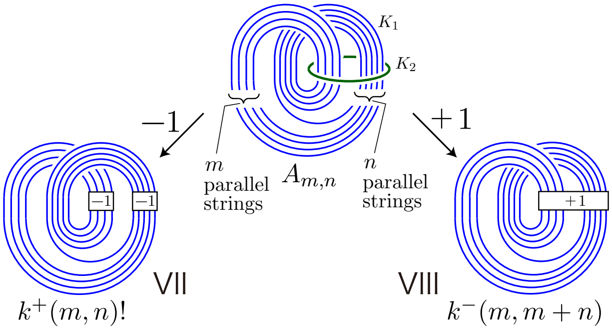
$\Rightarrow \alpha'' = -1 \ \Rightarrow \alpha = 7$.

Case (2) is similar. (3) is very hard (7pages). □

Proof of $(A_{2,3}; 6, 0) \cong L(25, 9)$, $(A_{2,3}; 7, 0) \cong L(25, 7)$



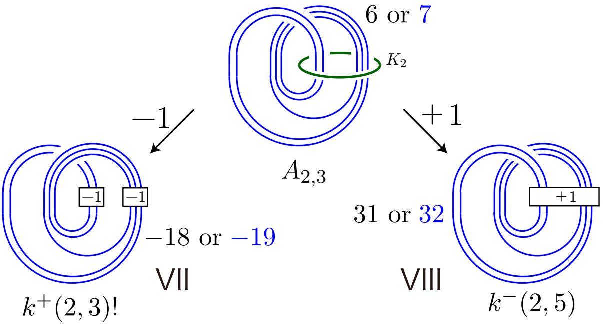
Remark. -1 (or $+1$) full-twist of K_1 along K_2 is *directly* related to knots of lens space surgery "Type" **VII** (or **VIII**). [Baker]



From $A_{m,n}$, we get

$k^+(m,n)$	m^2+mn+n^2 -surgey	**VII**
$k^-(m,m+n)$	$-m^2+mn+n^2$ -surgey	**VIII**

Furthermore, $(m, n) = (2, 3)$ induces the "famous examples"



From $(A_{2,3}; 6)$, $(A_{2,3}; 7)$, we get
$k^+(2,3) = P(-2,3,7)$: 19- and 18-surgeries are lens spaces.
$k^-(2,5)$: 31- and 32-surgeries are lens spaces.

$$(7 - 25 = -18, \ \ 7 + 25 = 32).$$

Thank you very much !