Graph manifolds have virtually positive Seifert volume.

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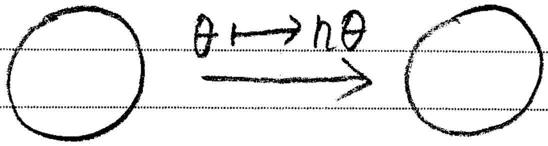
M, N closed oriented manifolds of $\dim = n$
Each pair (M, N) is associated with a set of integers $D(M, N) = \{d = \deg(f) \mid f: M \longrightarrow N\}$
Each N defines $D(N) = D(N, N)$.

Calculation of $D(M, N)$ is a classical topic.
Some people (not including me) though to determine $D(M, N)$ for all dim n is fundamental in topology
Some people (incuding me) are specially interested in the following easier question:

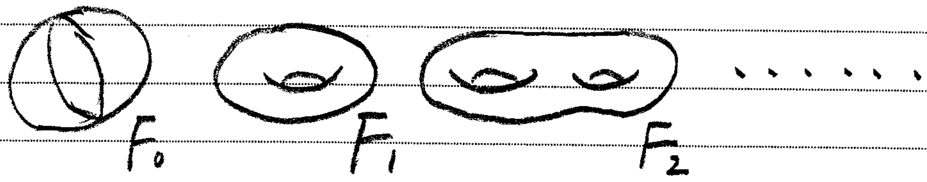
Q^*: For which given N, $|D(M, N)| < \infty$ for any M?

Note: $|D(N)| < \infty \iff D(N) \subset \{0, \pm 1\}$
$|D(N)| = \infty \Longrightarrow |D(M, N)| = \infty$ for some M

Ex 1: $n=1$ $D(S^1, S^1) = \mathbb{Z}$



Ex 2: $n=2$.



$$D(F_i, F_j) = \begin{cases} 0 & \text{if } i<j \\ \mathbb{Z} & \text{if } i \geq j, \text{ and } j=0,1 \\ \text{bounded by } \left|\chi(F_i)/\chi(F_j)\right| & \text{otherwise} \end{cases}$$

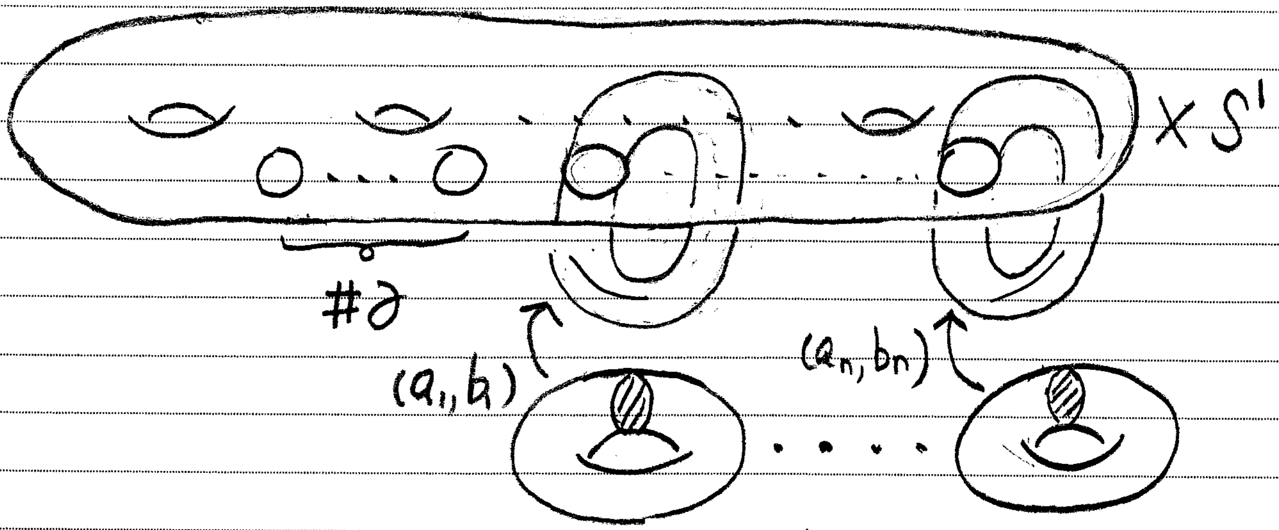
Ex 4: $n \geq 4$. Many interesting but special results. But hard to have general results Since no "reasonable" classification of mfds of dim ≥ 4

Ex 3: The most attractive dim $=3$ in the topic, where Thurston's geometrization conj., which has been verified, provides a "reasonable" Classification of 3-mfds

Picture of 3-mfds (below dim = 3)

- Call N prime if $N = N_1 \# N_2$ implies $N_i = S^3$ for $i = 1$ or 2. Each N has a prime decomp. which is unique up to order & homeo.
- For each prime N, $\exists!$ (up to isotopy) geometric splitting tori & Klein bottles $\mathcal{T}$ (a variation of JSJ tori) s.t. each piece of $N \setminus \mathcal{T}$ support one of the below eight geometries $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}$, $\widetilde{PSL_2\mathbb{R}}$, Sol, Nil, $\mathbb{E}^3$, S^3, $S^2 \times \mathbb{E}$
- If $\mathcal{T} \neq \phi$, then each piece of $N \setminus \mathcal{T}$ supports either $\mathbb{H}^3$- or $\mathbb{H}^2 \times \mathbb{E}^1$ geom. In partic. mfds supporting the remaining 6 geom. are closed.
- Call N a non-trivial graph mfd if $\mathcal{T} \neq \phi$ and each piece of $N \setminus \mathcal{T}$ supports $\mathbb{H}^2 \times \mathbb{E}^1$ geom.
- Except $\mathbb{H}^3$ and Sol, each N supporting one of remaining geom. is Seifert mfds.

• Seifert mfd $\Sigma = \{F_g, \#\partial\, ; a_1, b_1 ; \cdots a_n, b_n\}$



define $\chi(O_\Sigma) = 2 - 2g - \sum_1^n \left(1 - \frac{1}{a_i}\right) - \#\partial$

and $e(\Sigma) = \begin{cases} \sum \frac{b_i}{a_i} & \text{if } \#\partial = 0 \\ 0 & \text{otherwise} \end{cases}$

$e \backslash \chi$	<0	$=0$	>0
$\neq 0$	$\widetilde{PSL_2\mathbb{R}}$	Nil	S^3
$=0$	$\mathbb{H}^2 \times E^1$	E^3	$S^2 \times E^1$

• Nil, Sol, E^3-mfds covered by torus bundles
$\mathbb{H}^2 \times E^1$, E^3, $S^2 \times E^1$-mfds covered by trivial circle bundles.

D(N) is "known".

It is known no later than early 1990's that $|D(N)| = \infty$ if and only if either

(i) N is covered by a torus bundle, or

(ii) N is covered by a trivial S^1-bundle, or

(iii) Each prime factor of N is covered by either S^3 or $S^2 \times S^1$.

Indeed D(N) is completely determined when $|D(N)| = \infty$ quite recently. People involved this determination in last 10 years includes:

X. Du, C. Hayat, E. Kudryavseva, S. Matveev, A. Perfiliev, H. Sun, S. Wang, J. Wu, H. Zheng, H. Zieschang.

Ex. (i) $D(N) = \{1, 49 \bmod 120\}$, N = Poincare H. S.

(ii) $D(N) = \{k^4 \mid k \equiv 1 \bmod 6\}$ $N = \{S^2, (2,1)(3,1),(6,1)\}$ ← Nil

(iii) $-1 \notin D(N)$, $5 \in D(N)$

$$N = T^2 \times I \Big/ \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$ ← sol

Q^* is answered.

Thm 1. For each given N, $|D(M,N)| = \infty$ for some $M \Longleftrightarrow |D(R)| = \infty$ for each prime factor of N. ($\Longleftrightarrow$ each prime factor R of N is covered by a torus bundle, or a trivial S^1-bundle, or S^3).

Before a brief recall of the development of Thm 1, we give some definitions.

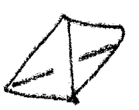
Call a non-negative invariant μ of 3-mfds has property D if $\forall$ map $f: M \longrightarrow N$, $\mu(M) \geqslant |\deg(f)|\,\mu(N)$ (resp. Property C, if $\forall$ covering $f: M \to N$, $\mu(M) = |\deg(f)|\,\mu(N)$).

Cor.* μ has property D and $\mu(N) \neq 0$ $\Longrightarrow D(M,N) < \infty$ for any M

Lemma*: μ has property D and $\mu(\widetilde{N}) \neq 0$ for some finite cover $\widetilde{N}$ of N $\Longrightarrow$ $D(M,N) < \infty$ for any M.

Devolopment of Thm 1.1

Thm 0.1 (Milnor–Thurston 1978)
Given closed hyperbolic 3-mfd N, $|D(M,N)| < \infty$ for any M.
Proved by using minimum integer # of  to build N

Gromov introduced simplicial volume $\|\cdot\|$.
- $\|N\|$ is minimum real # of  to build N
- $\|N\|$ measures exactly the volume of hyper. piece of N

Gromov–Thurston for hyper 3-mfd, Soma for $T \neq 0$.
- $\|\cdot\|$ has both properties D and C

Thm 0.2 $|D(M,N)| < \infty$ for any M if a prime factor of N contains hyperbolic piece

Brooks-Goldman introduced Seifert Volume $SV(\cdot)$
- $SV(\cdot)$ nonvanish for N supporting $\widetilde{PSL_2 R}$-geom
- $SV(\cdot)$ has property D.

Thm 0.3 (Brooks-Goldman early 1980s)
$|D(M,N)| < \infty$ for any M if R supports the geometry of $\widetilde{PSL_2 R}$ for some prime factor R of N.

Derbez-W proved in 2009 that each non-trivial graph mfd N has a finite cover $\widetilde{N}$ s.t. $SV(\widetilde{N}) \neq 0$. By Lemma*

Thm 0.4 (Derbez-W 2009) $|D(M,N)| < \infty$ for any M if N is a nontrivial graph mfd.

Rk: (August of 2010. Derbez-W proved $\exists$ non-trivial graph mfd N (similar to that given by Reid-W before ... for other purpose) s.t. $SV(N) = 0$. Hence $SV(*)$ has no property C.

Thm 0.5 (Derbez-Sun-W 2010)

If $|D(R)| = \infty$ for each prime factor R of N, then $\exists M$ s.t. $|D(M,N)| = \infty$.

Rk: $|D(T^3)| = \infty$, but $|D(T^3 \# T^3)| < \infty$.

This indicates that Thm 0.5 may be not too easy.

Now Thms 0.1, 0.2, 0.3, 0.4, 0.5 $\Longrightarrow$ Thm 1

The remaining of this talk will outline the proof of

Thm* (Derbez-W) Each non-trivial graph mfd N has a finite cover $\widetilde{N}$ s.t. $SV(\widetilde{N}) \neq 0$

- relations on foliation and representation

G (semi-simple) Lie group acts effectively on X.

G-foliation (flat, connection) $\longleftrightarrow$ representation to G

$$\left\{\begin{array}{l} M \times_\varphi X = \left(\dfrac{\widetilde{M} \times X}{\pi_1 M}\right) \text{ (via deck, via } \varphi) \\ \widetilde{M} \times * \text{ proj. horizontal} \\ G\text{-foliations on } M \times_\varphi X \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \varphi : \pi_1 M \longrightarrow G \\ \text{lifting loops in } M \\ \text{to leaf } p(\widetilde{M} \times *) \subset M \times_\varphi X \end{array}\right\}$$

Let $\mathcal{F}$ be a co-dim 1 G-foliation defined by 1-form ω. Then $d\omega = \omega \wedge \delta$ for some 1-form δ. Godbillon-Vey observed $\delta \wedge d\delta$ is a closed 3-form and $[\delta \wedge d\delta] \in H^3(M, \mathbb{R})$ depends only on $\mathcal{F}$, denoted as $GV(\mathcal{F})$.

• • Below we will restrict our attention to

$PSL_2\mathbb{R} \cong Iso_+ \mathbb{H}^2$ acts on S^1 and

$\widetilde{SL_2}\mathbb{R} \cong \widetilde{PSL_2}\mathbb{R}$ acts on $\mathbb{R}^1$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} = \{sh(n)\} & \longrightarrow & \widetilde{SL_2} & \longrightarrow & \widetilde{PSL_2} & \longrightarrow & 1 \\
 & & \downarrow \; \cap & & \downarrow \; \cap & & \wr\| & & \\
1 & \longrightarrow & \mathbb{R} = \{sh(\alpha)\} & \longrightarrow & Iso_e\widetilde{SL_2} & \longrightarrow & PSL_2 & \longrightarrow & 1
\end{array}$$

$Sh(\alpha): \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto x + 2\pi\alpha$, $\ Center(\widetilde{SL}) = \{sh(n) \mid n \in \mathbb{Z}\}$

For $\varphi: \pi_1 M \longrightarrow PSL_2$, define

$$e(\varphi) = e(M \times_\varphi S^1).$$

Fact:

$$\begin{array}{ccc}
 & & \widetilde{SL} \\
 & \nearrow \tilde{\tilde{\phi}} & \downarrow \\
 & & Iso_e\widetilde{SL} \\
 & \nearrow \tilde{\phi} & \downarrow \\
\pi_1 M & \xrightarrow{\ \phi\ } & PSL_2
\end{array}$$

$\phi: \pi_1 M \longrightarrow PSL_2$

ϕ lifts to $\tilde{\phi}: \pi_1 M \longrightarrow Iso_e\widetilde{SL}$

$\iff e(\varphi)$ torsion

ϕ lifts to $\tilde{\phi}: \pi_1 M \longrightarrow \widetilde{SL_2}$

$\iff e(\varphi) = 0$

Suppose $\phi: \pi_1 M^3 \longrightarrow PSL_2\mathbb{R}$ with $e(\phi)=0$.
Then the flat S^1-bundle $M\times_\varphi S^1$ has a horiz. G-foliation $\mathcal{F}_\phi$ and $GV(\mathcal{F}_\phi)$. Since $e(\phi)=0$ $\exists$ section $s: M \longrightarrow M\times_\varphi S^1$ and the pullback $s^*GV(\mathcal{F}_\phi)$ depends only on ϕ, and define

$$\underline{GV(\phi) = \int_M s^*GV(\mathcal{F}_\phi)}$$

Rk: If M has a G-foliation $\mathcal{F}$, $\exists$ an canonical way to construct $\varphi: \pi_1 M \to G$ and $s: M \to M\times_\varphi S^1$ s.t. $s^*\mathcal{F}_\phi = \mathcal{F}$. Then $GV(\phi) = \int_M GV(\mathcal{F})$.

On the other hand, since $e(\phi)=0$, φ has a lift $\tilde\varphi: \pi_1 M \longrightarrow \widetilde{SL_2}$. Now we have

$$\begin{array}{ccccc} f^*\omega & M\times\widetilde{SL_2} & \xrightarrow[\text{proj}]{f} & \widetilde{SL_2} & \omega \leftarrow \text{volume form, G-invar.} \\ & \downarrow & & & \\ \bar\omega & M\times_{\tilde\varphi}\widetilde{SL} & \xleftarrow{s} & M & s^*\bar\omega \end{array}$$

$\widetilde{SL}$ contractible, section s exists

define $\underline{SV(\tilde\varphi) = \int_M s^*\bar\omega}$

Thm (Brook-Goldman)

For $\pi_1 M \xrightarrow{\varphi} PSL_2\mathbb{R}$ with $e(\phi) = 0$.

(1). $GV(\phi) = SV(\widehat{\phi})$.

(2) $SV(\widehat{\phi})$ take only finitely many values.

Then they extended the Thm to $e(\phi)$ torsion case and define $SV(M) \overset{\max}{=} \{SV(\widehat{\phi}) \mid \pi_1 M \longrightarrow \mathrm{Iso}_e \widetilde{SL_2}\}$.

We still need some facts to prove Thm*. First recall

Thm (Milnor-Wood) For S^1-bundle over surface $S^1(h) \longrightarrow M \longrightarrow F_g$. below are "$\Longleftrightarrow$"

(1) the bundle is induced from $\varphi: \pi_1 F_g \longrightarrow PSL_2$,

(2) $\exists\ \widetilde{\varphi}: \pi_1 M \longrightarrow \widetilde{SL_2}$ s.t. $\widetilde{\varphi}(h) = sh(1)$,

(3) $\exists\ PSL_2\mathbb{R}$ horizontal foliation on M

(4). $e(M) = e(\varphi) \leq |\chi(F_g)|$.

Prop (Eisenbud-Hirsch-Neumann's ext. of Milnor-Wood thm)

For Seifert manifold M over F_g, with $\mathbb{R}$-fiber h

- ⊙ $\exists PSL_2\mathbb{R}$ horizontal-$\mathcal{F}$ on $M \Longleftrightarrow$

  $\exists \widetilde{\Phi}: \pi_1 M \longrightarrow \widetilde{SL_2\mathbb{R}}$ with $\widetilde{\Phi}(h) = sh(1)$

- ⊙⊙ $\exists PSL_2\mathbb{R}$ horizontal-$\mathcal{F}$ on $M \Longleftrightarrow |\chi(F_g)| \geq$

  $|e(M)| + |\#\text{ singular fibers}|$

- ⊙⊙⊙ For $x_1, \cdots, x_r \in \widetilde{SL_2}$, x_i conj. $sh(\alpha_i)$, then

  $x_1 \cdots x_r = \prod_{i=1}^{g} [v_i, w_i] \Longleftrightarrow |\alpha_1 + \cdots + \alpha_r| < 2g-1$.

Prop (Bott, B-G)

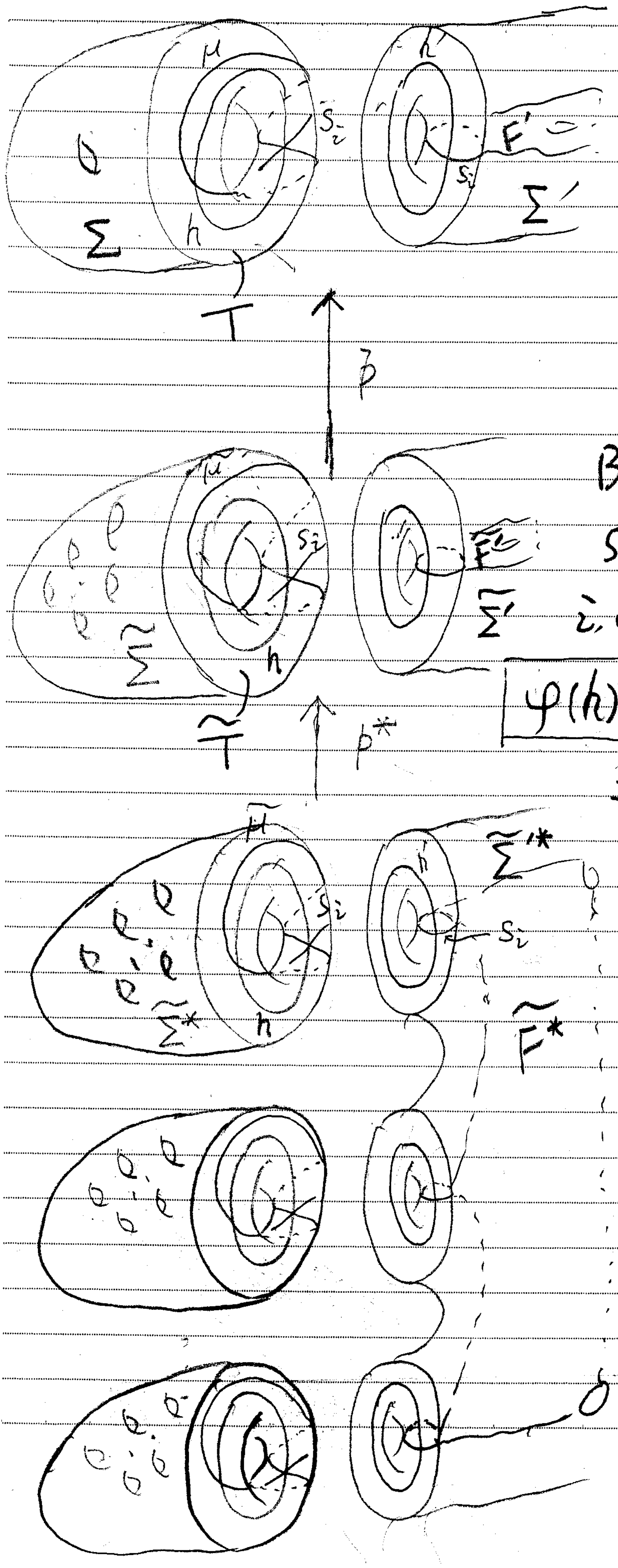
⊙⊙⊙⊙ $\int_M GV(\mathcal{F}) = 4\pi e(M)$ for $PSL_2\mathbb{R}$ horizontal foliation $\mathcal{F}$ on $S^1 \to M \to F$.

Non-trivial graph mfd N: Up to finite cover, Each JSJ piece of N^* is a S^1-bundle over F with $g(F) > 1$. For each Σ_i of N^*, $\widehat{\Sigma}_i$ denote the closed Seifert mfd obtained by Dehn filling on Σ_i along S^1-fiber of adjacent piece.

Define $|e|(N) = \sum_{i=1}^{n} |e(\widehat{\Sigma}_i)|$.

Now we prove Thm* for an easy case $|e|(N) \neq 0$.

Then $\exists\, \Sigma \in N \setminus T$ s.t. $e(\hat{\Sigma}) = e(\Sigma(\mu)) \neq 0$ (P14)



$\exists$ good cover $\tilde{N} \xrightarrow{p} N$ s.t.

(1) $\tilde{\Sigma} = p^{-1}(\Sigma)$, $\tilde{T} = p^{-1}(T)$ connected

(2) $e(\tilde{\Sigma}(\hat{\mu})) = e(\Sigma(\mu))$

(3) $g(\tilde{F})$ very large

By (1) (2) (3) ⊙⊙ and ⊙ $\exists\, \varphi : \pi_1 \tilde{\Sigma}(\hat{\mu}) \to \widetilde{SL_2}$

s.t. $\varphi(h) = sh(1)$

i.e. $\exists\, \varphi : \pi_1 \tilde{\Sigma} \to \widetilde{SL_2}$ s.t.

$\varphi(h) = sh(1)$, $\varphi(\hat{\mu}) = sh(0)$, $\varphi(s_i) = sh(\alpha_i)$

$\exists$ good cover $\tilde{N}^* \xrightarrow{p^*} N$ s.t.

(1*) $p^*|$ on each cpt of $p^{*-1}(\tilde{\Sigma})$ is trivial

(2*) $\tilde{\Sigma}'^* = p^{-1}(\tilde{\Sigma}')$ connected

$\tilde{\Sigma}'^* = \tilde{F}^* \times S^1$, $g(\tilde{F}^*) \gg 1$

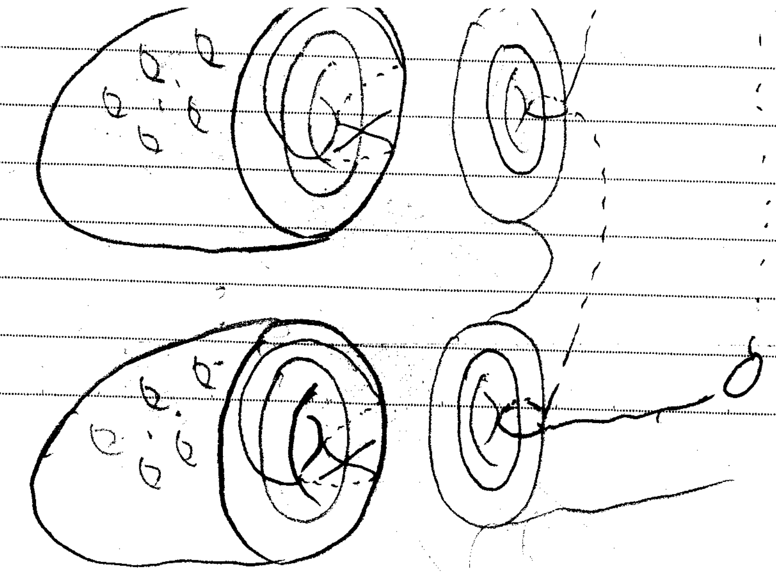
Now pick cpt $\tilde{\Sigma}^*$ of $p^{*-1}(\tilde{\Sigma})$.

By (1*), $\exists\, \varphi : \pi_1 \tilde{\Sigma}^* \to \widetilde{SL_2}$

s.t. [ ]

By (2*) and ⊙⊙⊙,

$\exists\, \varphi' : \pi_1 \tilde{\Sigma}'^* \to \widetilde{SL_2}$



Now pick cpt $\widetilde{\Sigma}^*$ of $\widetilde{p^{*-1}(\Sigma)}$.

By (1*), $\exists\ \varphi : \pi_1 \widetilde{\Sigma}^* \longrightarrow \widetilde{SL_2}$

s.t. $\boxed{}$

By (2*) and ⊙⊙⊙,

$\exists\ \varphi' : \pi_1 \widetilde{\Sigma}'^* \longrightarrow \widetilde{SL_2}$

$\varphi'(s_i) = sh(\alpha_i)$, $\Sigma sh(\alpha_i) = \varphi'(\prod_1^{g(F'^*)} [x_i, y_i])$, $\varphi'(h') = sh(0)$

and $\varphi'|_T$ is trivial for each cpt T of $\partial \widetilde{\Sigma}'^*$ not adjacent to $\widetilde{\Sigma}^*$. clearly φ' extends φ.

Finally extends φ to the remaing via trivial presentation to get $\varphi : \pi_1 \widetilde{N}^* \longrightarrow \widetilde{SL_2}$

Claim: $SV(\varphi) > 0$, why

(i) By ⊙, $\widetilde{\Sigma}^*(\hat{\mu})$ admits $PSL_2\mathbb{R}$ horiz. foliation.

By ⊙⊙⊙⊙ $SV(\varphi|\widetilde{\Sigma}^*) = SV(\varphi|\widetilde{\Sigma}^*(\mu)) = GV(\Phi|\widetilde{\Sigma}^*(\mu)) > 0$

(ii) $SV(\varphi|\widetilde{\Sigma}'^*) = 0$ since $\varphi|\widetilde{\Sigma}'^*$ factor through gp of 2-cx

(iii) $SV(\varphi|\widetilde{N} \setminus \widetilde{\Sigma}^* \cup \widetilde{\Sigma}'^*) = 0$ since φ trivial on this part

More precise proof use $\int_M GV(\varphi) = * \int CS(A)$ in $\widetilde{SL_2}$