Epimorphisms between knot groups: determination of the partial order

Masaaki Suzuki

Akita University

September 14, 2010

Twisted Alexander polynomial

G : a finitely presentable group

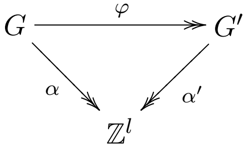
$\alpha : G \twoheadrightarrow \mathbb{Z}^l$: a surjective homomorphism of G

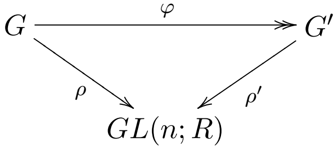
$\rho : G \longrightarrow GL(n; R)$: a representation of G $\qquad R$: UFD

$$\Longrightarrow \Delta_{G,\rho} = \frac{\Delta^N_{G,\rho}}{\Delta^D_{G,\rho}}$$: twisted Alexander polynomial

Theorem. (Kitano-S.-Wada)

G, G' : finitely presentable groups
$\alpha : G \twoheadrightarrow \mathbb{Z}^l$, $\alpha' : G' \twoheadrightarrow \mathbb{Z}^l$: surjective homomorphisms
${}^\exists \varphi : G \twoheadrightarrow G'$ s.t. $\alpha = \alpha' \circ \varphi$,
$\Longrightarrow \Delta^N_{G,\rho}$ can be divided by $\Delta^N_{G',\rho'}$ and $\Delta^D_{G,\rho} = \Delta^D_{G',\rho'}$
for any representation $\rho' : G' \longrightarrow GL(n;R)$, where $\rho = \rho' \circ \varphi$.





Example

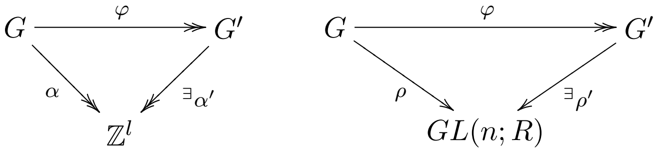
$G = \langle x_1, \ldots, x_u \mid r_1, \ldots, r_v \rangle$, $\alpha : G \twoheadrightarrow \mathbb{Z}^l$, $\rho : G \to GL(n, R)$

Example

$G = \langle x_1, \ldots, x_u \mid r_1, \ldots, r_v \rangle$, $\alpha : G \twoheadrightarrow \mathbb{Z}^l$, $\rho : G \longrightarrow GL(n, R)$
$G' = \langle x_1, \ldots, x_u \mid r_1, \ldots, r_v, s \rangle$
$\pi : G \twoheadrightarrow G'$: the projection

Example

$G = \langle x_1, \ldots, x_u \mid r_1, \ldots, r_v \rangle$, $\alpha : G \twoheadrightarrow \mathbb{Z}^l$, $\rho : G \longrightarrow GL(n, R)$
$G' = \langle x_1, \ldots, x_u \mid r_1, \ldots, r_v, s \rangle$
$\pi : G \twoheadrightarrow G'$: the projection
Suppose that $s \in \ker \alpha, s \in \ker \rho$



$\Longrightarrow \Delta^N_{G,\rho}$ can be divided by $\Delta^N_{G',\rho'}$ and $\Delta^D_{G,\rho} = \Delta^D_{G',\rho'}$

Twisted Alexander polynomial

G : a finitely presentable group
$\alpha : G \twoheadrightarrow \mathbb{Z}^l$: a surjective homomorphism
$\rho : G \longrightarrow GL(n; R)$: a representation of G $\qquad R$: UFD
$\Longrightarrow \Delta_{G,\rho}$: twisted Alexander polynomial

Twisted Alexander polynomial

G : a finitely presentable group
$\alpha : G \twoheadrightarrow \mathbb{Z}^l$: a surjective homomorphism
$\rho : G \longrightarrow GL(n; R)$: a representation of G $\qquad$ R : UFD
$\Longrightarrow \Delta_{G,\rho}$: twisted Alexander polynomial

Twisted Alexander polynomial for knots

$G(K)$: the knot group of a knot K
$\alpha : G(K) \twoheadrightarrow \mathbb{Z}$: the abelianization
$\rho : G(K) \longrightarrow SL(2; \mathbb{Z}/p\mathbb{Z})$: a representation $\qquad$ p : prime
$\Longrightarrow \Delta_{K,\rho}$: twisted Alexander polynomial for the knot K

K : a knot, $\rho : G(K) \longrightarrow SL(2;\mathbb{Z}/p\mathbb{Z})$
$\Delta_{K,\rho}$: the twisted Alexander polynomial of K
$\Delta^N_{K,\rho}$: the numerator of $\Delta_{K,\rho}$
$\Delta^D_{K,\rho}$: the denominator of $\Delta_{K,\rho}$

Corollary.

If there exists a representation $\rho' : G(K') \to SL(2;\mathbb{Z}/p\mathbb{Z})$ such that for any representation $\rho : G(K) \to SL(2;\mathbb{Z}/p\mathbb{Z})$,

$$\Delta^N_{K,\rho} \text{ can not be divided by } \Delta^N_{K',\rho'} \text{ or } \Delta^D_{K,\rho} \neq \Delta^D_{K',\rho'},$$

$\Longrightarrow$ there exists no epimorphism $G(K) \twoheadrightarrow G(K')$.

Corollary.

If there exists a representation $\rho' : G(K') \to SL(2; \mathbb{Z}/p\mathbb{Z})$ such that for any representation $\rho : G(K) \to SL(2; \mathbb{Z}/p\mathbb{Z})$,

$$\Delta^N_{K,\rho} \text{ can not be divided by } \Delta^N_{K',\rho'} \text{ or } \Delta^D_{K,\rho} \neq \Delta^D_{K',\rho'},$$

$\Longrightarrow$ there exists no epimorphism $G(K) \twoheadrightarrow G(K')$.

K : a knot, Δ_K : the Alexander polynomial of K

Fact.

K, K' : two knots

If Δ_K can not be divided by $\Delta_{K'}$,

$\Longrightarrow$ there exists no epimorphism $G(K) \twoheadrightarrow G(K')$.

K : a *prime* knot in S^3
$G(K)$: the knot group of K i.e. $G(K) = \pi_1(S^3 - K)$

Definition.

$$K \geq K' \iff {}^{\exists}\varphi : G(K) \twoheadrightarrow G(K')$$

K : a *prime* knot in S^3
$G(K)$: the knot group of K i.e. $G(K) = \pi_1(S^3 - K)$

Definition.

$$K \geq K' \iff {}^{\exists}\varphi : G(K) \twoheadrightarrow G(K')$$

Fact.

The relation "$\geq$" is a partial order on the set of prime knots.

- $K \geq K$
- $K \geq K',\ K' \geq K \implies K = K'$
- $K \geq K',\ K' \geq K'' \implies K \geq K''$

Theorem (Horie-Kitano-Matsumoto-S.)

The partial order "$\geq$" on the set of prime knots with up to 11 crossings is given by

$$\left.\begin{array}{l}
8_5, 8_{10}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}, 9_1, 9_6, 9_{16}, 9_{23}, 9_{24}, 9_{28}, 9_{40}, \\
10_5, 10_9, 10_{32}, 10_{40}, 10_{61}, 10_{62}, 10_{63}, 10_{64}, 10_{65}, 10_{66}, 10_{76}, \\
10_{77}, 10_{78}, 10_{82}, 10_{84}, 10_{85}, 10_{87}, 10_{98}, 10_{99}, 10_{103}, 10_{106}, \\
10_{112}, 10_{114}, 10_{139}, 10_{140}, 10_{141}, 10_{142}, 10_{143}, 10_{144}, \\
10_{159}, 10_{164}, \\
11a_{43}, 11a_{44}, 11a_{46}, 11a_{47}, 11a_{57}, 11a_{58}, 11a_{71}, 11a_{72}, 11a_{73}, \\
11a_{100}, 11a_{106}, 11a_{107}, 11a_{108}, 11a_{109}, 11a_{117}, 11a_{134}, 11a_{139}, \\
11a_{157}, 11a_{165}, 11a_{171}, 11a_{175}, 11a_{176}, 11a_{194}, 11a_{196}, 11a_{203}, \\
11a_{212}, 11a_{216}, 11a_{223}, 11a_{231}, 11a_{232}, 11a_{236}, 11a_{244}, 11a_{245}, \\
11a_{261}, 11a_{263}, 11a_{264}, 11a_{286}, 11a_{305}, 11a_{306}, 11a_{318}, 11a_{332}, \\
11a_{338}, 11a_{340}, 11a_{351}, 11a_{352}, 11a_{355}, 11n_{71}, 11n_{72}, 11n_{73}, \\
11n_{74}, 11n_{75}, 11n_{76}, 11n_{77}, 11n_{78}, 11n_{81}, 11n_{85}, 11n_{86}, 11n_{87}, \\
11n_{94}, 11n_{104}, 11n_{105}, 11n_{106}, 11n_{107}, 11n_{136}, 11n_{164}, 11n_{183}, \\
11n_{184}, 11n_{185}
\end{array}\right\} \geq 3_1$$

$$\left.\begin{array}{l} 8_{18}, 9_{37}, 9_{40}, \\ 10_{58}, 10_{59}, 10_{60}, 10_{122}, 10_{136}, 10_{137}, 10_{138}, \\ 11a_5, 11a_6, 11a_{51}, 11a_{132}, 11a_{239}, 11a_{297}, 11a_{348}, 11a_{349}, \\ 11n_{100}, 11n_{148}, 11n_{157}, 11n_{165} \end{array}\right\} \geq 4_1$$

$$11n_{78}, 11n_{148} \geq 5_1$$

$$10_{74}, 10_{120}, 10_{122}, 11n_{71}, 11n_{185} \geq 5_2$$

$$11a_{352} \geq 6_1$$

$$11a_{351} \geq 6_2$$

$$11a_{47}, 11a_{239} \geq 6_3$$

To determine the partial order on the set of prime knots

For each pair of two prime knots K, K',
determine whether there exists an epimorphism

$$\varphi : G(K) \twoheadrightarrow G(K')$$

To determine the partial order on the set of prime knots

For each pair of two prime knots K, K',
determine whether there exists an epimorphism

$$\varphi : G(K) \twoheadrightarrow G(K')$$

The number of prime knots with up to 11 crossings is $\mathbf{801}$.
Then the number of cases to consider is ${}_{801}\mathrm{P}_2 = \mathbf{640,800}$.

To determine the partial order on the set of prime knots

For each pair of two prime knots K, K',
determine whether there exists an epimorphism

$$\varphi : G(K) \twoheadrightarrow G(K')$$

The number of prime knots with up to 11 crossings is $\mathbf{801}$.
Then the number of cases to consider is ${}_{801}\mathrm{P}_2 = \mathbf{640}, \mathbf{800}$.

The number of prime knots with up to 12 crossings is $\mathbf{2{,}977}$.
Then the number of cases to consider is ${}_{2977}\mathrm{P}_2 = \mathbf{4{,}429{,}776}$.

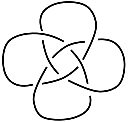
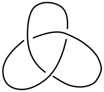
To prove the existence of an epimorphism

Constructing an epimorphism explicitly

To prove the existence of an epimorphism

Constructing an epimorphism explicitly

Example. $8_{18} \geq 3_1$? i.e. ${}^{?\exists}\varphi : G(8_{18}) \twoheadrightarrow G(3_1)$

$8_{18} =$  , $3_1 =$ 

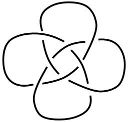
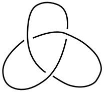
$$G(8_{18}) = \left\langle \begin{array}{l} x_1, x_2, x_3, \\ x_4, x_5, x_6, \\ x_7, x_8 \end{array} \middle| \begin{array}{l} x_4 x_1 \bar{x}_4 \bar{x}_2, x_5 x_3 \bar{x}_5 \bar{x}_2, x_6 x_3 \bar{x}_6 \bar{x}_4, \\ x_7 x_5 \bar{x}_7 \bar{x}_4, x_8 x_5 \bar{x}_8 \bar{x}_6, x_1 x_7 \bar{x}_1 \bar{x}_6, \\ x_2 x_7 \bar{x}_2 \bar{x}_8 \end{array} \right\rangle$$

$$G(3_1) = \langle y_1, y_2, y_3 \,|\, y_3 y_1 \bar{y}_3 \bar{y}_2,\ y_1 y_2 \bar{y}_1 \bar{y}_3 \rangle$$

To prove the existence of an epimorphism

Constructing an epimorphism explicitly

Example. $8_{18} \geq 3_1$? i.e. ${}^{?\exists}\varphi : G(8_{18}) \twoheadrightarrow G(3_1)$

$8_{18} =$  , $3_1 =$ 

$$G(8_{18}) = \left\langle \begin{array}{l} x_1, x_2, x_3, \\ x_4, x_5, x_6, \\ x_7, x_8 \end{array} \middle| \begin{array}{l} x_4 x_1 \bar{x}_4 \bar{x}_2, x_5 x_3 \bar{x}_5 \bar{x}_2, x_6 x_3 \bar{x}_6 \bar{x}_4, \\ x_7 x_5 \bar{x}_7 \bar{x}_4, x_8 x_5 \bar{x}_8 \bar{x}_6, x_1 x_7 \bar{x}_1 \bar{x}_6, \\ x_2 x_7 \bar{x}_2 \bar{x}_8 \end{array} \right\rangle$$

$$G(3_1) = \langle y_1, y_2, y_3 \mid y_3 y_1 \bar{y}_3 \bar{y}_2,\ y_1 y_2 \bar{y}_1 \bar{y}_3 \rangle$$

$$\begin{array}{llll} \varphi(x_1) = y_1, & \varphi(x_2) = y_2, & \varphi(x_3) = y_1, & \varphi(x_4) = y_3, \\ \varphi(x_5) = y_3, & \varphi(x_6) = y_1 y_3 \bar{y}_1, & \varphi(x_7) = y_3, & \varphi(x_8) = y_1 \end{array}$$

$$8_{18} \geq 3_1$$

To prove the non-existence of any epimorphism

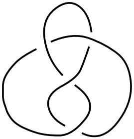
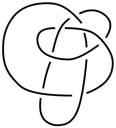
(1) By the (classical) Alexander polynomial

K : a knot
Δ_K : the Alexander polynomial of K

Fact.

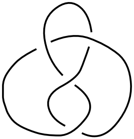
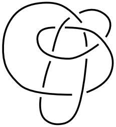
K, K' : two knots
If Δ_K can not be divided by $\Delta_{K'}$,
$\Longrightarrow$ there exists no epimorphism $G(K) \twoheadrightarrow G(K')$.

Example. $4_1 \geq 8_{21}$? i.e. $^{?\exists}\varphi : G(4_1) \twoheadrightarrow G(8_{21})$

$4_1 =$  , $8_{21} =$ 

$$\Delta_{4_1} = t^2 - 3t + 1, \qquad \Delta_{8_{21}} = t^4 - 4t^3 + 5t^2 - 4t + 1$$

Example. $4_1 \geq 8_{21}$? i.e. ${}^{?\exists}\varphi : G(4_1) \twoheadrightarrow G(8_{21})$

$4_1 =$  , $8_{21} =$ 

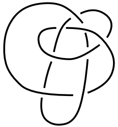
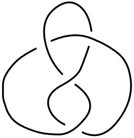
$$\Delta_{4_1} = t^2 - 3t + 1, \qquad \Delta_{8_{21}} = t^4 - 4t^3 + 5t^2 - 4t + 1$$

$$\frac{\Delta_{4_1}}{\Delta_{8_{21}}} = \frac{t^2 - 3t + 1}{t^4 - 4t^3 + 5t^2 - 4t + 1}$$

$\Delta_{8_{21}}$ can not divide Δ_{4_1}

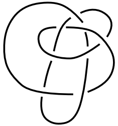
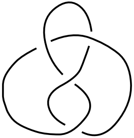
$$4_1 \not\geq 8_{21}$$

Example. $8_{21} \geq 4_1$? i.e. ${}^{?\exists}\varphi : G(8_{21}) \twoheadrightarrow G(4_1)$

$8_{21} =$  , $4_1 =$ 

$$\Delta_{8_{21}} = t^4 - 4t^3 + 5t^2 - 4t + 1, \qquad \Delta_{4_1} = t^2 - 3t + 1$$

Example. $8_{21} \geq 4_1$? i.e. ${}^{?\exists}\varphi : G(8_{21}) \twoheadrightarrow G(4_1)$

$8_{21} =$  , $4_1 =$ 

$\Delta_{8_{21}} = t^4 - 4t^3 + 5t^2 - 4t + 1,\qquad \Delta_{4_1} = t^2 - 3t + 1$

$$\frac{\Delta_{8_{21}}}{\Delta_{4_1}} = \frac{t^4 - 4t^3 + 5t^2 - 4t + 1}{t^2 - 3t + 1} = t^2 - t + 1$$

Δ_{4_1} can divide $\Delta_{8_{21}}$!

We cannot determine the existence of an epimorphism from $G(8_{21})$ onto $G(4_1)$ by the Alexander polynomial.

To prove the non-existence of any epimorphism

(1) By the (classical) Alexander polynomial
(2) By the twisted Alexander polynomial

$\Delta_{K,\rho}$: the twisted Alexander polynomial of K
$\Delta^N_{K,\rho}, \Delta^D_{K,\rho}$: the numerator and denominator of $\Delta_{K,\rho}$

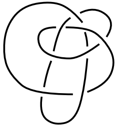
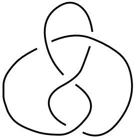
Theorem. (Kitano-S.-Wada)

If there exists a representation $\rho' : G(K') \to SL(2; \mathbb{Z}/p\mathbb{Z})$ such that for any representation $\rho : G(K) \to SL(2; \mathbb{Z}/p\mathbb{Z})$,

$$\Delta^N_{K,\rho} \text{ can not be divided by } \Delta^N_{K',\rho'} \text{ or } \Delta^D_{K,\rho} \neq \Delta^D_{K',\rho'},$$

$\Longrightarrow$ there exists no epimorphism $G(K) \twoheadrightarrow G(K')$.

Example. $8_{21} \geq 4_1$? i.e. $^{?\exists}\varphi : G(8_{21}) \twoheadrightarrow G(4_1)$

$8_{21} =$  , $4_1 =$ 

$$\Delta_{8_{21}} = t^4 - 4t^3 + 5t^2 - 4t + 1, \qquad \Delta_{4_1} = t^2 - 3t + 1$$

$$\frac{\Delta_{8_{21}}}{\Delta_{4_1}} = \frac{t^4 - 4t^3 + 5t^2 - 4t + 1}{t^2 - 3t + 1} = t^2 - t + 1$$

Δ_{4_1} divides $\Delta_{8_{21}}$!

We cannot determine the existence of an epimorphism from $G(8_{21})$ onto $G(4_1)$ by the Alexander polynomial.

For a certain representation $\rho' : G(4_1) \longrightarrow SL(2;\mathbb{Z}/3\mathbb{Z})$,

$$\Delta^N_{4_1,\rho'} = t^4 + t^2 + 1, \qquad \Delta^D_{4_1,\rho'} = t^2 + t + 1$$

For a certain representation $\rho' : G(4_1) \longrightarrow SL(2;\mathbb{Z}/3\mathbb{Z})$,

$$\Delta^N_{4_1,\rho'} = t^4 + t^2 + 1, \qquad \Delta^D_{4_1,\rho'} = t^2 + t + 1$$

Table of the twisted Alexander polynomials of $G(8_{21})$ for all representations $\rho : G(8_{21}) \longrightarrow SL(2;\mathbb{Z}/3\mathbb{Z})$

	$\Delta^N_{8_{21},\rho_i}$	$\Delta^D_{8_{21},\rho_i}$
ρ_1	$t^8 + t^4 + 1$	$t^2 + 1$
ρ_2	$t^8 + t^7 + 2t^6 + 2t^4 + 2t^2 + t + 1$	$t^2 + t + 1$
ρ_3	$t^8 + t^7 + 2t^6 + 2t^4 + 2t^2 + t + 1$	$t^2 + 2t + 1$
ρ_4	$t^8 + 2t^7 + 2t^6 + 2t^4 + 2t^2 + 2t + 1$	$t^2 + t + 1$
ρ_5	$t^8 + 2t^7 + 2t^6 + 2t^4 + 2t^2 + 2t + 1$	$t^2 + 2t + 1$

For a certain representation $\rho' : G(4_1) \longrightarrow SL(2;\mathbb{Z}/3\mathbb{Z})$,

$$\Delta^N_{4_1,\rho'} = t^4 + t^2 + 1, \qquad \Delta^D_{4_1,\rho'} = t^2 + t + 1$$

Table of the twisted Alexander polynomials of $G(8_{21})$ for all representations $\rho : G(8_{21}) \longrightarrow SL(2;\mathbb{Z}/3\mathbb{Z})$

	$\Delta^N_{8_{21},\rho_i}$	$\Delta^D_{8_{21},\rho_i}$
ρ_1	$t^8 + t^4 + 1$	$t^2 + 1$
ρ_2	$t^8 + t^7 + 2t^6 + 2t^4 + 2t^2 + t + 1$	$t^2 + t + 1$
ρ_3	$t^8 + t^7 + 2t^6 + 2t^4 + 2t^2 + t + 1$	$t^2 + 2t + 1$
ρ_4	$t^8 + 2t^7 + 2t^6 + 2t^4 + 2t^2 + 2t + 1$	$t^2 + t + 1$
ρ_5	$t^8 + 2t^7 + 2t^6 + 2t^4 + 2t^2 + 2t + 1$	$t^2 + 2t + 1$

$$8_{21} \not\geq 4_1$$

To determine the partial order on the set of prime knots

For each pair of two prime knots K, K',
determine whether there exists an epimorphism

$$\varphi : G(K) \twoheadrightarrow G(K')$$

The number of prime knots with up to 11 crossings is $\mathbf{801}$.
Then the number of cases to consider is ${}_{801}\mathrm{P}_2 = \mathbf{640,800}$.

$\mathbf{146}$ cases: existence of an epimorphism
$\mathbf{637,501}$ cases : non-existence by the Alexander polynomial
$\mathbf{3,153}$ cases : non-existence by the twisted Alexander poly.

Theorem (Horie-Kitano-Matsumoto-S.)

The partial order "$\geq$" on the set of prime knots with up to 11 crossings is given by

$$\left.\begin{array}{l}
8_5, 8_{10}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}, 9_1, 9_6, 9_{16}, 9_{23}, 9_{24}, 9_{28}, 9_{40}, \\
10_5, 10_9, 10_{32}, 10_{40}, 10_{61}, 10_{62}, 10_{63}, 10_{64}, 10_{65}, 10_{66}, 10_{76}, \\
10_{77}, 10_{78}, 10_{82}, 10_{84}, 10_{85}, 10_{87}, 10_{98}, 10_{99}, 10_{103}, 10_{106}, \\
10_{112}, 10_{114}, 10_{139}, 10_{140}, 10_{141}, 10_{142}, 10_{143}, 10_{144}, \\
10_{159}, 10_{164}, \\
11a_{43}, 11a_{44}, 11a_{46}, 11a_{47}, 11a_{57}, 11a_{58}, 11a_{71}, 11a_{72}, 11a_{73}, \\
11a_{100}, 11a_{106}, 11a_{107}, 11a_{108}, 11a_{109}, 11a_{117}, 11a_{134}, 11a_{139}, \\
11a_{157}, 11a_{165}, 11a_{171}, 11a_{175}, 11a_{176}, 11a_{194}, 11a_{196}, 11a_{203}, \\
11a_{212}, 11a_{216}, 11a_{223}, 11a_{231}, 11a_{232}, 11a_{236}, 11a_{244}, 11a_{245}, \\
11a_{261}, 11a_{263}, 11a_{264}, 11a_{286}, 11a_{305}, 11a_{306}, 11a_{318}, 11a_{332}, \\
11a_{338}, 11a_{340}, 11a_{351}, 11a_{352}, 11a_{355}, 11n_{71}, 11n_{72}, 11n_{73}, \\
11n_{74}, 11n_{75}, 11n_{76}, 11n_{77}, 11n_{78}, 11n_{81}, 11n_{85}, 11n_{86}, 11n_{87}, \\
11n_{94}, 11n_{104}, 11n_{105}, 11n_{106}, 11n_{107}, 11n_{136}, 11n_{164}, 11n_{183}, \\
11n_{184}, 11n_{185}
\end{array}\right\} \geq 3_1$$

$$\left.\begin{array}{l} 8_{18}, 9_{37}, 9_{40}, \\ 10_{58}, 10_{59}, 10_{60}, 10_{122}, 10_{136}, 10_{137}, 10_{138}, \\ 11a_5, 11a_6, 11a_{51}, 11a_{132}, 11a_{239}, 11a_{297}, 11a_{348}, 11a_{349}, \\ 11n_{100}, 11n_{148}, 11n_{157}, 11n_{165} \end{array}\right\} \geq 4_1$$

$$11n_{78}, 11n_{148} \geq 5_1$$

$$10_{74}, 10_{120}, 10_{122}, 11n_{71}, 11n_{185} \geq 5_2$$

$$11a_{352} \geq 6_1$$

$$11a_{351} \geq 6_2$$

$$11a_{47}, 11a_{239} \geq 6_3$$

Definition

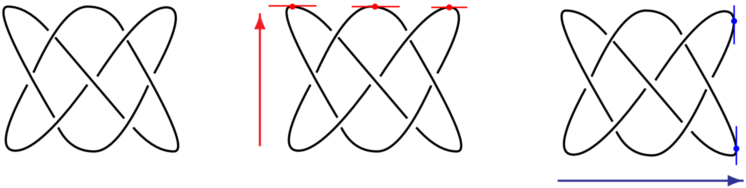
K : a knot

D : a regular diagram of K

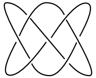
$\vec{v}(D)$: the minimal number of local maximal points of D

$$\mathrm{br}(K) = \min_{D} \vec{v}(D)$$

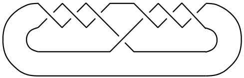
Example. 2-bridge knot 7_4



Example. 2-bridge knot 7_4



=



$[3,1,3]$

$$\Longrightarrow \quad [3,1,3] = \cfrac{1}{3+\cfrac{1}{1+\frac{1}{3}}} = \frac{4}{15}$$

$$[a_1, a_2, \ldots, a_{2k+1}] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_{2k+1}}}}} = r \qquad \in \mathbb{Q}$$

2-bridge knot $\Longrightarrow$ $\dfrac{q}{p} \in \mathbb{Q}$

$$\frac{q}{p} \in \mathbb{Q} \quad \Longleftrightarrow \quad \text{2-bridge knot } K(q/p)$$

Theorem (Schubert)

2-bridge knots $K(q/p)$ and $K(q'/p')$ are equivalent, if and only if the following conditions hold.
(1) $p = p'$.
(2) Either $q \equiv q' \pmod{p}$ or $qq' \equiv \pm 1 \pmod{p}$.

Theorem (Kitano-S.)

The partial order "$\geq$" on the set of 2-bridge knots with up to 12 crossings is given by

$$\left.\begin{array}{l} 9_1, 9_6, 9_{23}, \\ 10_5, 10_9, 10_{32}, 10_{40}, \\ 11a117, 11a175, 11a176, 11a203, \\ 11a236, 11a306, 11a355 \\ 12a302, 12a528, 12a579, 12a580, \\ 12a718, 12a736, 12a1136, 12a1276 \end{array}\right\} \geq 3_1,$$

$$12a259, 12a471, 12a506 \quad \geq 4_1$$

Remark

There exist 361 2-bridge knots with up to 12 crossings.

For $r = [m_1, m_2, \ldots, m_k] \in \mathbb{Q}$ and $\epsilon \in \{+, -\}$, put

$$\begin{aligned} a &= (m_1, m_2, \ldots, m_k), & \epsilon a &= (\epsilon m_1, \epsilon m_2, \ldots, \epsilon m_k) \\ a^{-1} &= (m_k, m_{k-1}, \ldots, m_1), & \epsilon a^{-1} &= (\epsilon m_k, \epsilon m_{k-1}, \ldots, \epsilon m_1) \end{aligned}$$

Theorem (Ohtsuki-Riley-Sakuma)

If a rational number $\tilde{r}$ has a continued fraction expansion

$$\tilde{r} = 2c + [\epsilon_1 a,\ 2c_1,\ \epsilon_2 a^{-1},\ 2c_2, \ldots, 2c_{n-1},\ \epsilon_n a^{(-1)^{n-1}}]$$

where $\epsilon_i \in \{+, -\}$ and $c, c_i \in \mathbb{Z}$, then there exists an (upper-meridian-pair-preserving) epimorphism $G(K(\tilde{r})) \twoheadrightarrow G(K(r))$.

Case 1. Onto $3_1 = K(1/3)$ $\frac{1}{3} = [3]$,

Theorem (Ohtsuki-Riley-Sakuma)

If a 2-bridge knot $K(\tilde{r})$ admits a continued fraction expansion

$$\tilde{r} = [\pm 3,\, 2a_1,\, \pm 3,\, 2a_2,\, \pm 3,\, \ldots,\, \pm 3,\, 2a_n,\, \pm 3], \qquad a_i \in \mathbb{Z},$$

then there exists an epimorphism $G(K(\tilde{r})) \twoheadrightarrow G(3_1)$.

Case 1. Onto $3_1 = K(1/3)$ $\frac{1}{3} = [3]$,

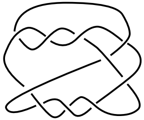
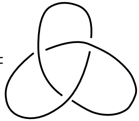
Theorem (Ohtsuki-Riley-Sakuma)

If a 2-bridge knot $K(\tilde{r})$ admits a continued fraction expansion

$$\tilde{r} = [\pm 3,\ 2a_1,\ \pm 3,\ 2a_2,\ \pm 3,\ \ldots,\ \pm 3,\ 2a_n,\ \pm 3], \qquad a_i \in \mathbb{Z},$$

then there exists an epimorphism $G(K(\tilde{r})) \twoheadrightarrow G(3_1)$.

Example. $10_9 = K(7/39)$,

$10_9 =$  $3_1 =$ 

$$\frac{7}{39} = [3,\ 0,\ 3,\ -2,\ -3] \implies G(10_9) \twoheadrightarrow G(3_1)$$

Case 2. Onto $4_1 = K(2/5)$ $\frac{2}{5} = [2, 2]$,

Theorem (Ohtsuki-Riley-Sakuma)

If a 2-bridge knot $K(\tilde{r})$ admits a continued fraction expansion

$$\tilde{r} = [\pm 2,\ \pm 2,\ 2a_1,\ \pm 2,\ \pm 2,\ 2a_2,\ \dots,\ \pm 2,\ \pm 2,\ 2a_n,\ \pm 2,\ \pm 2],$$

then there exists an epimorphism $G(K(\tilde{r})) \twoheadrightarrow G(4_1)$.

Case 2. Onto $4_1 = K(2/5)$ $\frac{2}{5} = [2, 2]$,

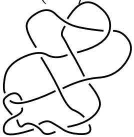
Theorem (Ohtsuki-Riley-Sakuma)

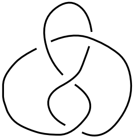
If a 2-bridge knot $K(\tilde{r})$ admits a continued fraction expansion

$$\tilde{r} = [\pm 2,\ \pm 2,\ 2a_1,\ \pm 2,\ \pm 2,\ 2a_2,\ \ldots,\ \pm 2,\ \pm 2,\ 2a_n,\ \pm 2,\ \pm 2],$$

then there exists an epimorphism $G(K(\tilde{r})) \twoheadrightarrow G(4_1)$.

Example. $12a259 = K(52/115)$,

$12a259 =$ 

$4_1 =$ 

$$\frac{52}{115} = [2,\ 2,\ 0,\ 2,\ 2,\ -2,\ 2,\ 2] \implies G(12a259) \twoheadrightarrow G(4_1)$$

Theorem (Ohtsuki-Riley-Sakuma)

If a rational number $\tilde{r}$ has a continued fraction expansion

$$\tilde{r} = 2c + [\epsilon_1 a,\, 2c_1,\, \epsilon_2 a^{-1},\, 2c_2, \ldots, 2c_{n-1},\, \epsilon_n a^{(-1)^{n-1}}]$$

where $\epsilon_i \in \{+,-\}$ and $c, c_i \in \mathbb{Z}$, then there exists an (upper-meridian-pair-preserving) epimorphism $G(K(\tilde{r})) \twoheadrightarrow G(K(r))$.

Problem

Is every pair of 2-bridge knots $(K(\tilde{r}), K(r))$ with $G(K(\tilde{r})) \twoheadrightarrow G(K(r))$ given by the Ohtsuki-Riley-Sakuma construction?

Theorem (Ohtsuki-Riley-Sakuma)

If a rational number $\tilde{r}$ has a continued fraction expansion

$$\tilde{r} = 2c + [\epsilon_1 a,\ 2c_1,\ \epsilon_2 a^{-1},\ 2c_2, \ldots, 2c_{n-1},\ \epsilon_n a^{(-1)^{n-1}}]$$

where $\epsilon_i \in \{+, -\}$ and $c, c_i \in \mathbb{Z}$, then there exists an (upper-meridian-pair-preserving) epimorphism $G(K(\tilde{r})) \twoheadrightarrow G(K(r))$.

Theorem (Lee-Sakuma)

If there exists an upper-meridian-pair-preserving epimorphism $G(K(\tilde{r})) \twoheadrightarrow G(K(r))$, then a rational number $\tilde{r}$ has a continued fraction expansion

$$\tilde{r} = 2c + [\epsilon_1 a,\ 2c_1,\ \epsilon_2 a^{-1},\ 2c_2, \ldots, 2c_{n-1},\ \epsilon_n a^{(-1)^{n-1}}]$$

Theorem (Lee-Sakuma)

If there exists an upper-meridian-pair-preserving epimorphism $G(K(\tilde{r})) \twoheadrightarrow G(K(r))$, then a rational number $\tilde{r}$ has a continued fraction expansion

$$\tilde{r} = 2c + [\epsilon_1 a,\, 2c_1,\, \epsilon_2 a^{-1},\, 2c_2, \ldots, 2c_{n-1},\, \epsilon_n a^{(-1)^{n-1}}]$$

Problem

How about non-upper-meridian-pair-preserving epimorphism?

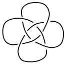
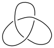
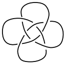
Problem

Does there eixst a 2-bridge knot which surjects onto $G(3_1)$ and onto $G(4_1)$?

Problem

Does there eixst a 2-bridge knot which surjects onto $G(3_1)$ and onto $G(4_1)$?

c.f. 3-bridge knot 8_{18}

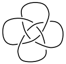
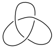
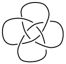
 $\geq$  ,  $\geq$ 

$8_{18} \geq 3_1$, $8_{18} \geq 4_1$

Problem

Does there eixst a 2-bridge knot which surjects onto $G(3_1)$ and onto $G(4_1)$?

c.f. 3-bridge knot 8_{18}

 $\geq$  ,  $\geq$ 

$8_{18} \geq 3_1$, $8_{18} \geq 4_1$

c.f. 2-bridge link $\frac{11}{30}$

$$\frac{11}{30} = [3,\ -4,\ 3] = [2,\ 2,\ -2,\ 2,\ 2]$$

Problem

Does there eixst a 2-bridge knot which surjects onto $G(3_1)$ and onto $G(4_1)$?

Problem

For given two rational numbers r, r', determine whether there eixst a 2-bridge knot $K(\tilde{r})$ such that $K(\tilde{r}) \geq K(r)$ and $K(\tilde{r}) \geq K(r')$.

Example. $r = 1/3, \quad r' = 2/5$

$$^{?}\exists\, \tilde{r} \in \mathbb{Q} \ \text{ s.t. } \begin{array}{l} K(\tilde{r}) \geq K(1/3) = 3_1 \\ K(\tilde{r}) \geq K(2/5) = 4_1 \end{array}$$

Problem

For given two rational numbers r, r', determine whether there eixst a 2-bridge knot $K(\tilde{r})$ such that $K(\tilde{r}) \geq K(r)$ and $K(\tilde{r}) \geq K(r')$.

Theorem. (Hoste-Shanahan)

We can determine it for the *ORS construction*.

Example.

There does not eixst a 2-bridge knot which surjects onto $G(3_1)$ and onto $G(4_1)$ with respect to the ORS construction.