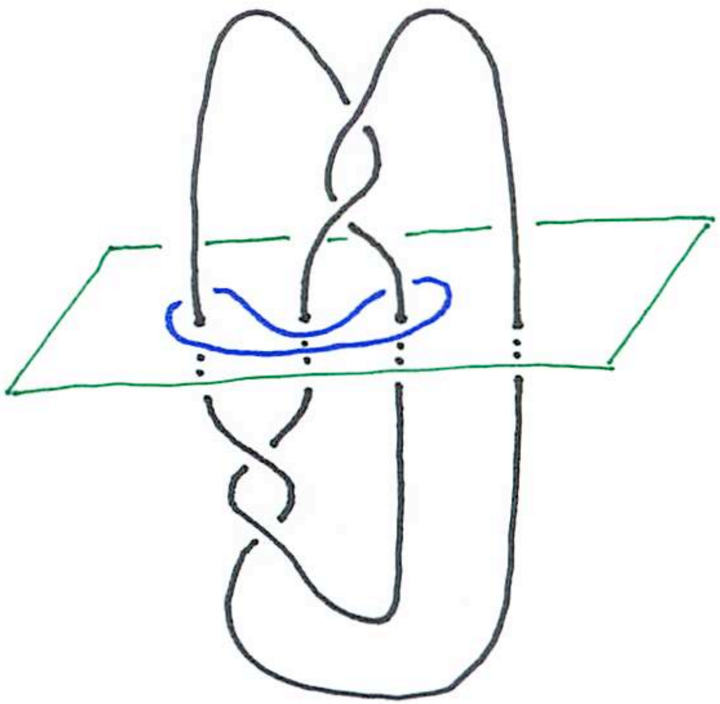
Epimorphisms between 2-bridge links:

Essential simple loops on 2-bridge spheres

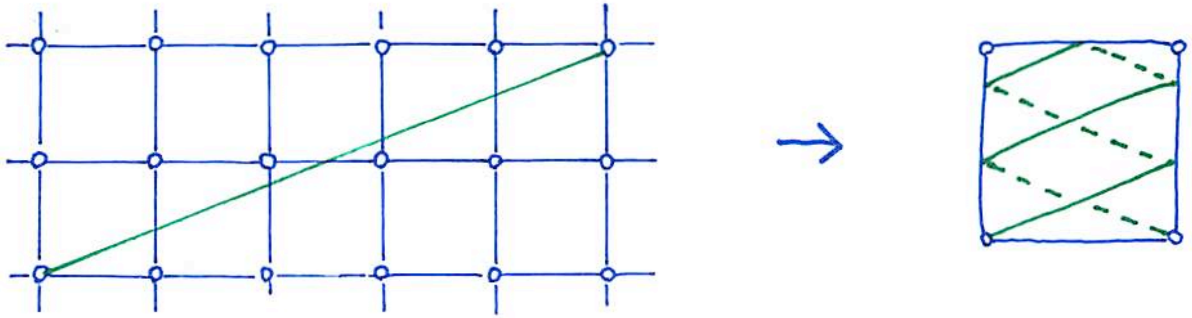


Donghi Lee (Pusan National Univ.)

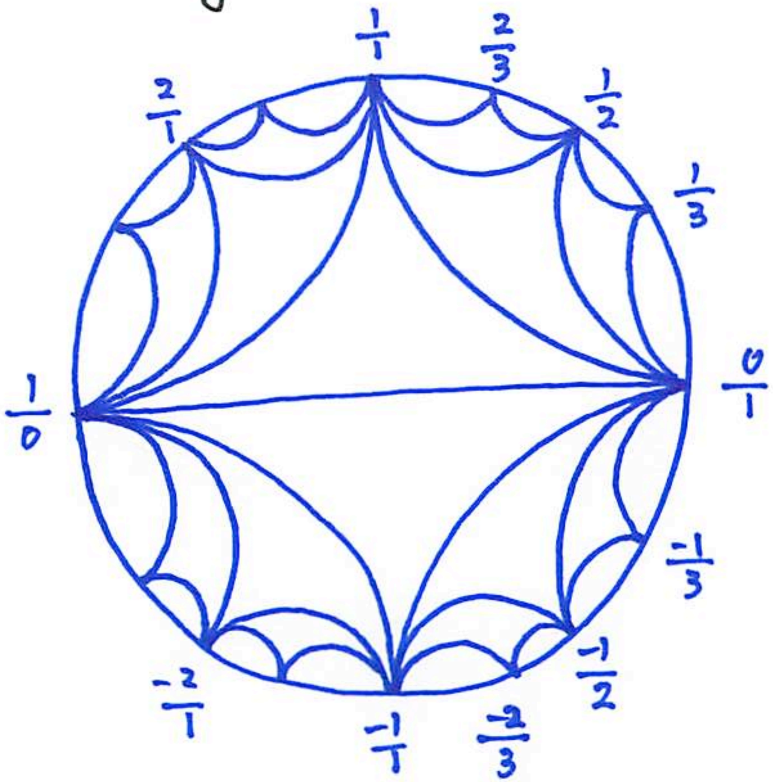
作間誠 (広島大学)

$S := \mathbb{R}^2 - \mathbb{Z}^2 / \langle \pi\text{-rotations around punctures} \rangle$: 4-punctured sphere

(Conway sphere)



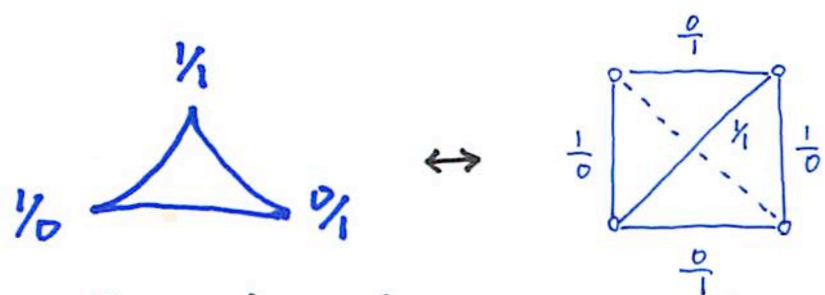
D : Farey tessellation



Vertex set of $D = \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\frac{1}{0}\} \ni r$

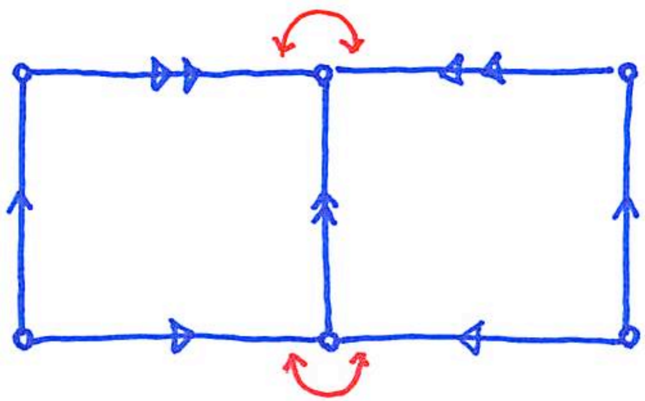
$\overset{1-1}{\leftrightarrow}$ {essential simple loops on S} $\ni \alpha_r$

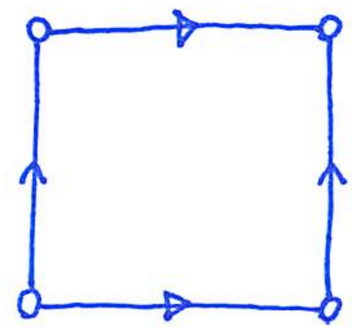
$\overset{1-2}{\leftrightarrow}$ {essential simple arcs on S} $\ni \delta_r$



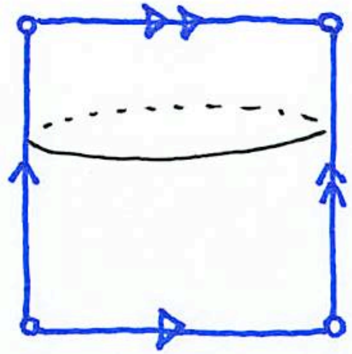
Farey triangle

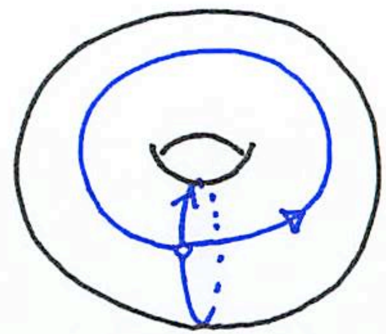
ideal triangulation of S





$\downarrow$ $\downarrow$

$S :=$  $\cong S^2 - 4$ points

 $=: T$

S and T are "commensurable"

[Schubert] $K(r) \cong K(r')$

$\Leftrightarrow$ The relative positions of $\{\infty, r\}$ and $\{\infty, r'\}$ in D are equivalent

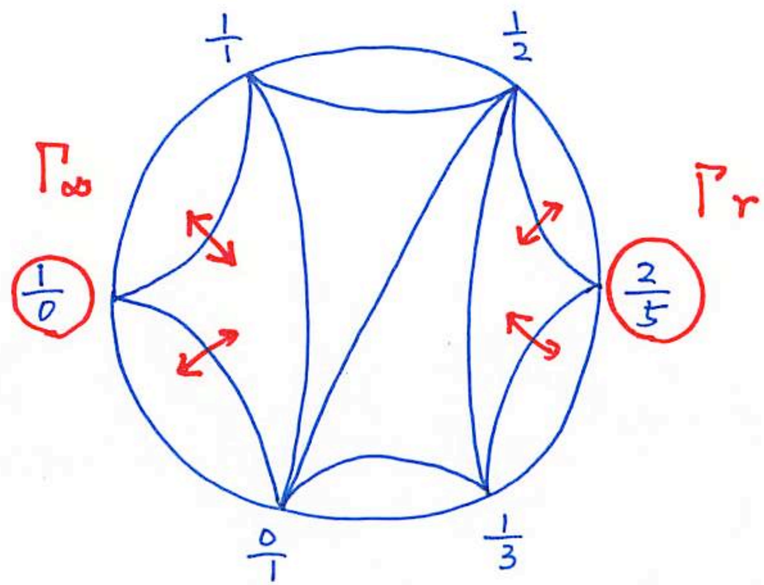
i.e. $\exists \varphi \in \mathrm{Aut}(D)$ st $\varphi\{\infty, r\} = \{\infty, r'\}$

<u>Def</u> The group $\hat{\Gamma}_r$ associated with $K(r)$

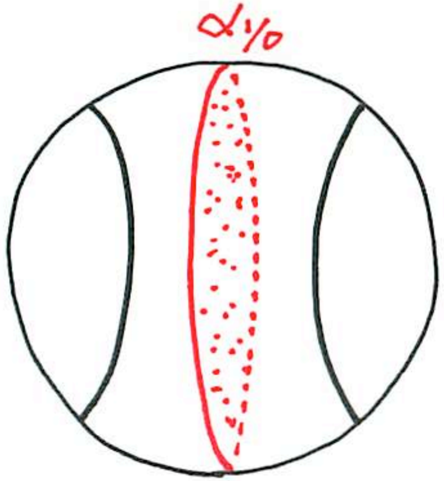
$\Gamma_r := \langle$ reflections in the edges of D with an endpoint $r \rangle \subset \mathrm{Aut}(D)$

$\cong$ infinite dihedral group D_∞

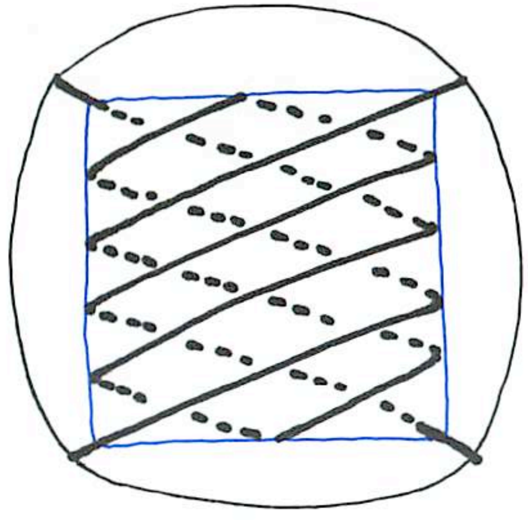
$\hat{\Gamma}_r := \langle \Gamma_\infty, \Gamma_r \rangle \cong \Gamma_\infty * \Gamma_r$ if $d(\infty, r) \geq 2$



Rational tangle $(B^3, t(r))$ of slope r :



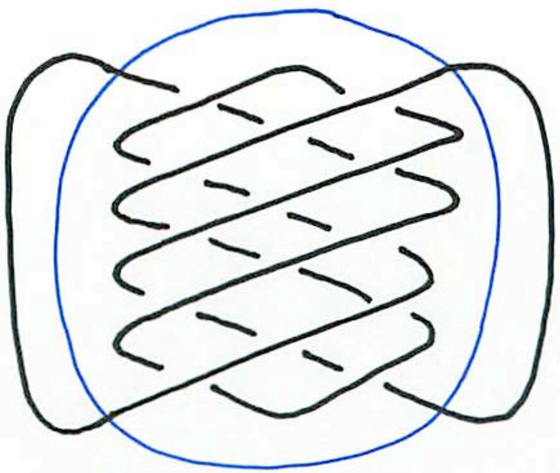
$(B^3, t(1/0))$



$(B^3, t(2/5))$

$$\pi_1(B^3 - t(r)) \cong \pi_1(S) / \langle\langle \alpha_r \rangle\rangle$$

$(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$: 2-bridge link of slope r



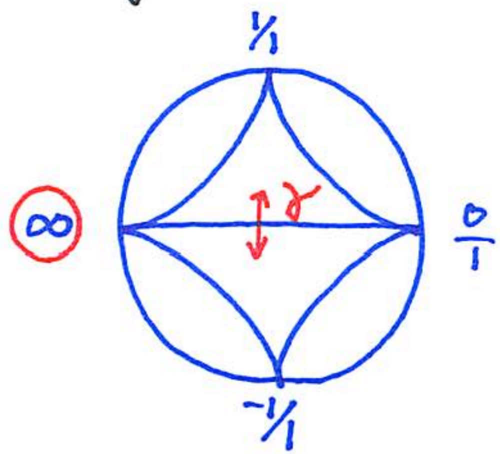
$$G(K(r)) := \pi_1(S^3 - K(r)) \cong \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle$$

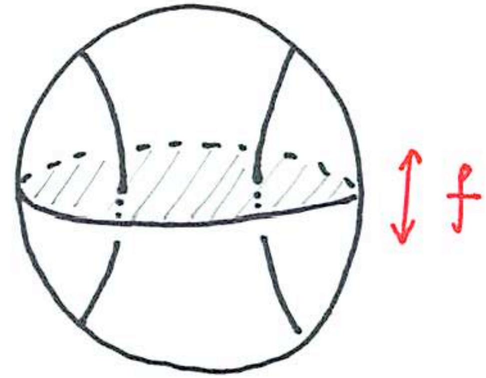
[Komori - Series]

$\alpha_s \sim \alpha_{s'}$ in $B^3 - t(r)$ $\Leftrightarrow$ $s' = \gamma(s)$ for some $\gamma \in \Gamma_r$

(proof)

($\Leftarrow$) We may assume $r = \infty$ and γ is a reflection in $\langle \infty, 0 \rangle$





Then γ is induced by $f : (B^3, t(\infty)) \xrightarrow[\cong]{} (B^3, t(\infty))$

i.e. $f(\alpha_s) = \alpha_{-s} = \alpha_{\gamma(s)}$

On the other hand $f_* = id : \pi_1(B^3 - t(\infty)) \to \pi_1(B^3 - t(\infty))$.

Hence $\alpha_{\gamma(s)} \sim f(\alpha_s) \sim \alpha_s$ in $B^3 - t(\infty)$. $\square$

Prop A (Ohtsuki - Riley - S)

$\alpha_s \sim \alpha_{s'}$ in $S^3 - K(r)$, if $s' = \gamma(s)$ for some $\gamma \in \hat{\Gamma}_r = \Gamma_\infty * \Gamma_r$.

(proof)

If $\gamma \in \Gamma_\infty$, then $\alpha_s \sim \alpha_{\gamma(s)}$ in $B^3 - t(\infty)$ and hence in $S^3 - K(r)$

because $S^3 - K(r) = (B^3 - t(\infty)) \cup (B^3 - t(r))$.

Similarly, if $\gamma \in \Gamma_r$, then $\alpha_s \sim \alpha_{\gamma(s)}$ in $B^3 - t(r)$, hence in $S^3 - K(r)$.

$\square$

Cor A (ORS)

$\alpha_s \sim 1$ in $S^3 - K(r)$, if $s \in \hat{\Gamma}_r \cdot \infty \cup \hat{\Gamma}_r \cdot r$.

(proof)

If $s = \gamma(\infty)$ for some $\gamma \in \hat{\Gamma}_r$, then $\alpha_s \sim \alpha_\infty = 1$ in $S^3 - K(r)$

$(s = \gamma(r))$ $(\alpha_s \sim \alpha_r = 1)$

$\square$

Cor B (ORS)

If $\tilde{r} \in \hat{P}_r \{r, \infty\}$, then there is an epimorphism $G(K(\tilde{r})) \twoheadrightarrow G(K(r))$

(proof)

Suppose $\tilde{r} \in \hat{P}_r \{r, \infty\}$.

Then $\alpha_{\tilde{r}} = 1$ in $G(K(r))$ by Cor A.

On the other hand, $\alpha_\infty = 1$ in $G(K(r)) = \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle$.

Hence, the identity map $\pi_1(S) \to \pi_1(S)$

descends to an epimorphism

$$
\begin{array}{ccc}
G(K(\tilde{r})) & & G(K(r)) \\
\| & & \| \\
\pi_1(S) / \langle\langle \alpha_\infty, \alpha_{\tilde{r}} \rangle\rangle & \twoheadrightarrow & \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle
\end{array}
$$

$\square$

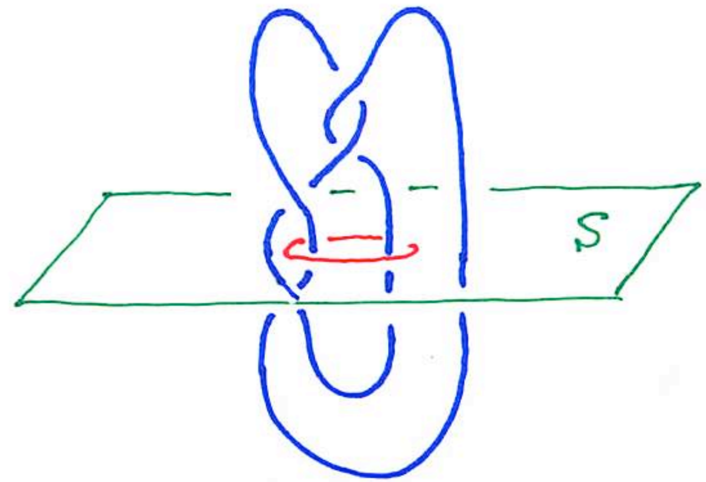
Question Does the converse to Cor A hold?

Which simple loop on the level 4-punctured sphere S in a 2-bridge knot complement $E(K(r)) = S^3 - K(r)$ is null-homotopic in $E(K(r))$?

Main Theorem 1 [Donghi Lee - S]

The converse to Cor A holds.

Namely, $\alpha_s = 1$ in $G(K(r))$ iff $s \in \hat{\Gamma}_r\{\infty, r\}$.



(Idea)

Small cancellation theory

Corollary

There is an "upper meridian-pair preserving" epimorphism $G(K(\tilde{r})) \to G(K(r))$,

iff $\tilde{r} \in \hat{\Gamma}_r \{\infty, r\}$.

Question

Does every epimorphism $G(K(\tilde{r})) \to G(K(r))$ send the upper meridian pair of $K(\tilde{r})$ to the upper or lower meridian pair of $K(r)$?

Minsky's Question (Geom. Top. Monograph 12)

- V : handlebody

$$
\begin{aligned}
\mathcal{M}(\partial V) &:= \pi_0 \mathrm{Diff}(\partial V) \\
\cup\quad & \\
\mathcal{M}(V) &:= \pi_0 \mathrm{Diff}(V) \\
\cup\quad & \\
\mathcal{M}_0(V) &:= \{ f \in \mathcal{M}(V) \mid f_* = id \in \mathrm{Out}(\pi_1(V)) \}
\end{aligned}
$$

- $M = V_+ \cup_S V_-$ Heegaard splitting

$$
\begin{aligned}
\Gamma_\pm &:= \mathcal{M}_0(V_\pm) \subset \mathcal{M}(S) && \leftrightarrow \Gamma_\infty, \Gamma_r \\
\Gamma &:= \langle \Gamma_+, \Gamma_- \rangle \subset \mathcal{M}(S) && \leftrightarrow \hat{\Gamma}_r := \langle \Gamma_\infty, \Gamma_r \rangle \\
\Delta_\pm &:= \{ \text{meridians of } V_\pm \} \subset \mathcal{C}(S) && \leftrightarrow \{\infty\}, \{r\}
\end{aligned}
$$

<u>Fact</u> If $\alpha \in \Gamma(\Delta_+ \cup \Delta_-)$, then $\alpha = 1$ in $\pi_1(M)$. $\leftrightarrow$ [ORS]

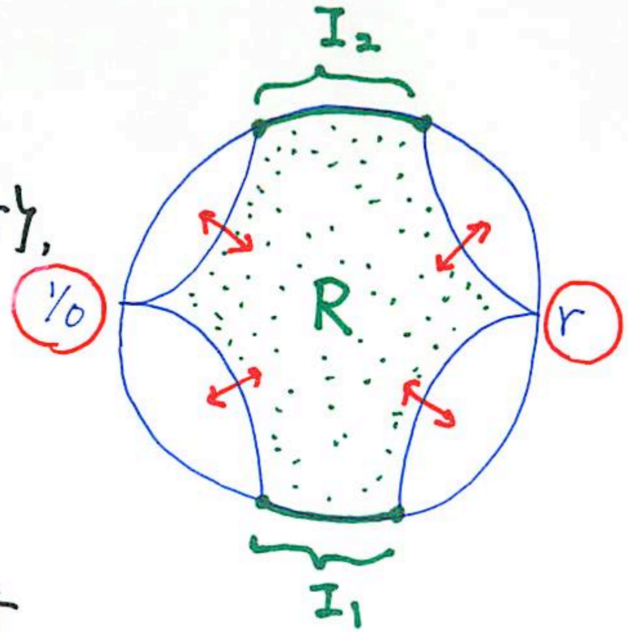
<u>Question</u> Does the converse hold when M is hyperbolic ?

Main Theorem 1 may be regarded as an answer to a special variation of the above question.

Lemma For any $s \in \hat{\mathbb{Q}}$,

there is a unique $s_0 \in I_1 \cup I_2 \cup \{\infty, r\}$,

st $s \in \hat{\Gamma}_r s_0$



(Proof)

This follows from the fact that $R \subset \mathbb{H}^2$ is a fundamental domain for the action $\hat{\Gamma}_r \curvearrowright \mathbb{H}^2$.

Thus Main Theorem I is equivalent to:

Theorem 1′

If $s \in I_1 \cup I_2$, then $\alpha_s \neq 1$ in $G(K(r))$.

$$G(K(r)) = \pi_1(S) / \langle\langle \alpha_{1/0}, \alpha_r \rangle\rangle$$

Small Cancellation Theory

[Dehn] Consider :

length $4g$

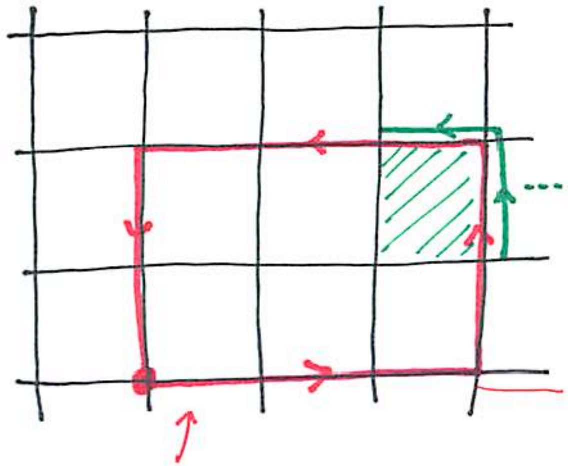
$$\pi_1(F_g) = \langle x_1, y_1, \cdots, x_g, y_g \mid \prod_{i=1}^{g} [x_i, y_i] \rangle \qquad (g \geqq 2)$$

If a cyclically reduced word $w \in \langle x_1, y_1, \cdots, x_g, y_g \rangle$ represents a trivial element in $\pi_1(F_g)$, then the cyclic word w contains a subword of the cyclic word represented by the relator $\prod_{i=1}^{g} [x_i, y_i]$ (or its inverse) of length $\geqq 2g+1 = \frac{1}{2}(\text{length of } \prod_{i=1}^{g} [x_i, y_i]) + 1$.

(Thus we can find a shorter cyclic word w' st $[w] = [w']$ in $\pi_1(F_g)$.)

(Idea of Dehn's theorem)

If $g=1$, $\pi_1(F_1) = \langle x, y \mid x y \bar{x} \bar{y} \rangle$



subword of length 2 of the relator

$x\,x\,x\,y\,\boxed{y\,\bar{x}}\,\bar{x}\,\bar{x}\,\bar{y}\,\bar{y}$

lift of a cyclic word representing the trivial element

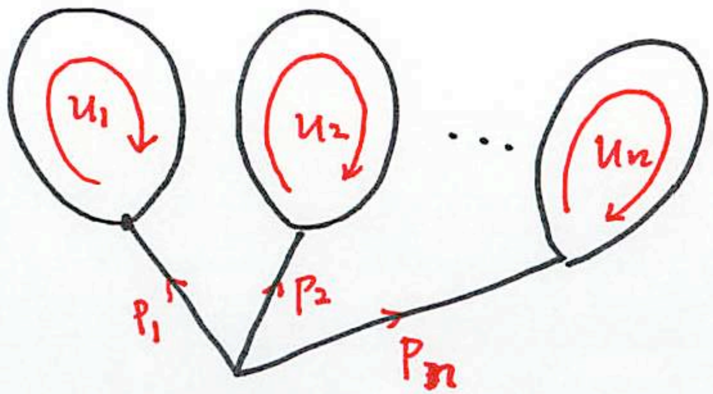
- In small cancellation theory "van-Kampen diagram" plays the role of the "region" in the universal cover bounded by the lift of a word representing trivial element.

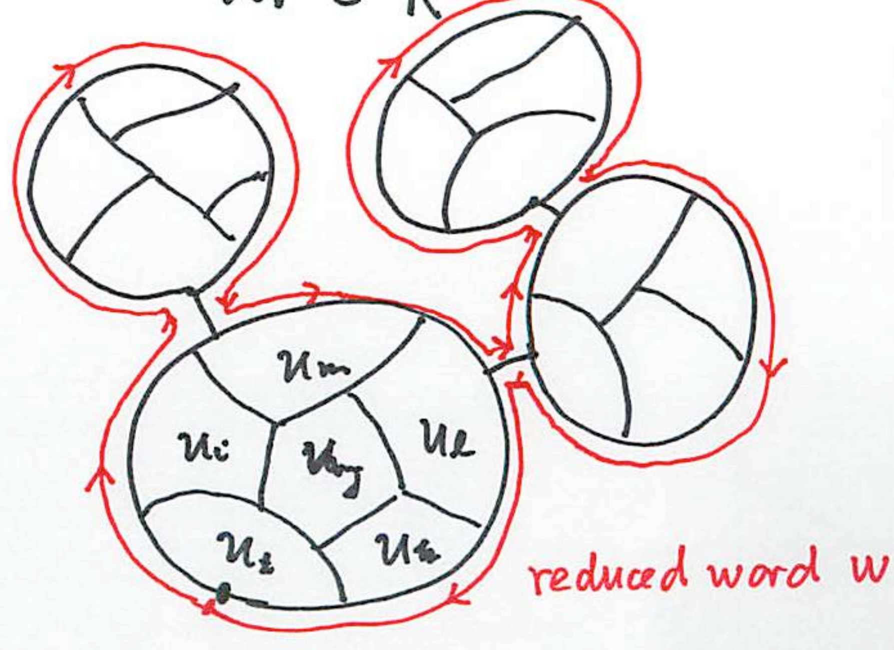
Van-Kampen Diagram

- $G = \langle x, y \mid u \rangle$ $\quad u$: cyclically reduced
- $R := \{$ cyclic permutations of $u, u^{-1} \} \subset \langle x, y \rangle$
- A cyclically reduced word $w \in \langle x, y \rangle$ represents $1 \in G = \langle x, y \mid u \rangle$

$$\Leftrightarrow \quad \underset{\substack{\uparrow \\ \text{reduced}}}{w} = \underset{\substack{\uparrow \\ \text{non-reduced}}}{\prod_{i=1}^{n} p_i u_i p_i^{-1}} \in \langle x, y \rangle \qquad p_i \in \langle x, y \rangle, \quad u_i \in R$$

$\Leftrightarrow$





Def (M, ϕ) is a van-Kampen diagram over $\langle x, y \mid u \rangle$, if

M : a map, ie 2-dim cell complex embedded in $\mathbb{R}^2$,

which is simply connected

$\phi : \{\text{oriented edges of } M\} \to \langle x, y \rangle$

(i) $\phi(e^{-1}) = \phi(e)^{-1}$

(ii) For each 2-cell D of M

$$\phi(\partial D) \in R = \{\text{cyclic permutations of } u^{\pm 1}\}$$

$$\shortparallel$$

$$\phi(e_1) \cdot \phi(e_2) \cdot \cdots \cdot \phi(e_n) \text{ is cyclically reduced}$$

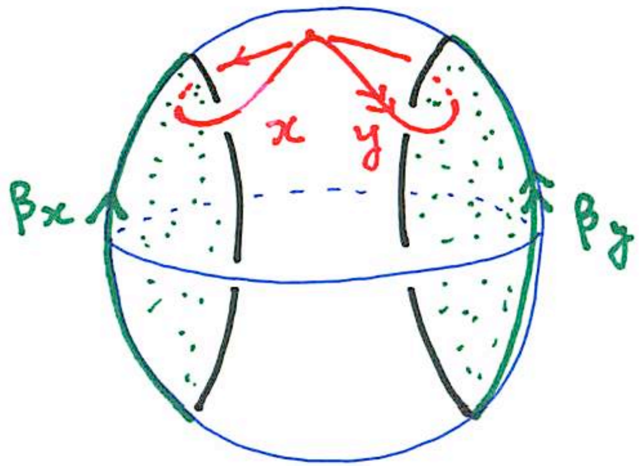
(iii) For the "boundary cycle" $\partial M = e_1 \cdots e_m$

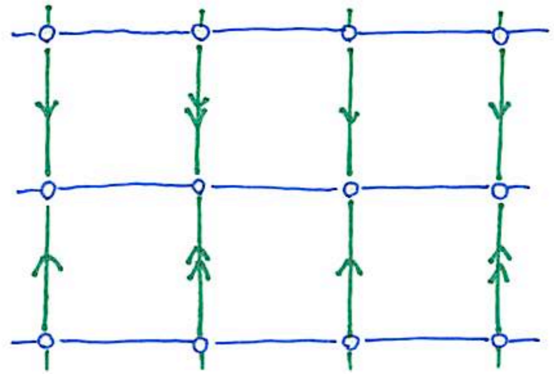
$$\phi(\partial M) := \phi(e_1) \cdots \phi(e_m) \text{ is cyclically reduced.}$$

Prop A cyclically reduced word $w \in \langle x, y \rangle$ represents $1 \in \langle x, y \mid u \rangle$ iff there is a van-Kampen diagram M st $w \equiv \phi(\partial M)$.

Upper presentation of $G(K(r)) = \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle = \pi_1(B^3 - t(\infty)) / \langle\langle \alpha_r \rangle\rangle$

$\pi_1(B^3 - t(\infty)) = \langle x, y \rangle$





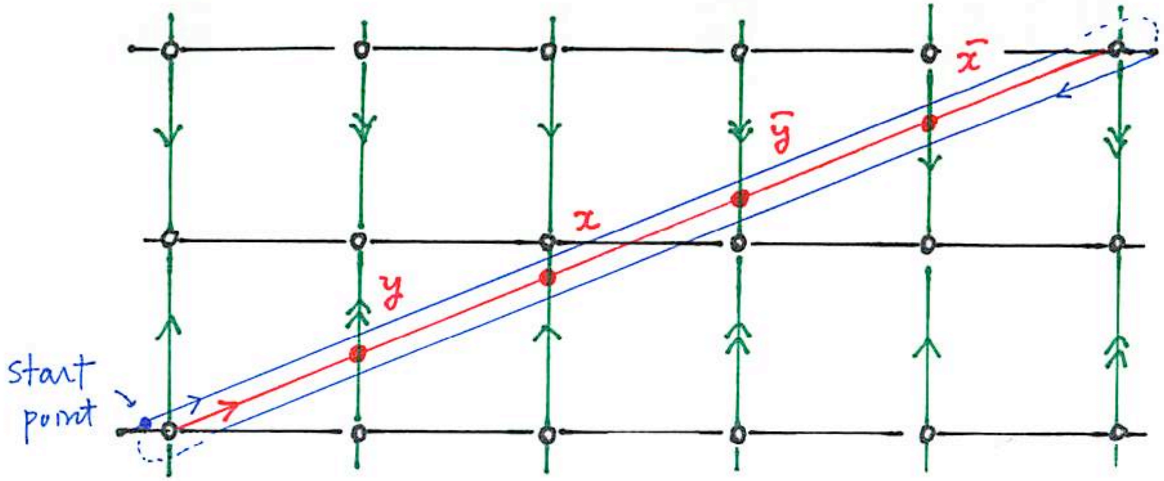
For a loop $\alpha \subset S$,

$[\alpha] \in \pi_1(B^3 - t(\infty)) = \langle x, y \rangle$

is obtained by "reading" the intersections of α

with β_x and β_y.

Example



$$\left(W_{\frac{2}{5}} := y\, x\, \bar{y}\, \bar{x} \right)$$

$$\alpha_{\frac{2}{5}} = x \cdot W_{2/5} \cdot y \cdot \overline{W}_{2/5}$$

$$= x\,(y\,x\,\bar{y}\,\bar{x})\; y\,(x\,y\,\bar{x}\,\bar{y})$$

For $0 < q/p < 1$

$$W_{q/p} = y^{\varepsilon_1}\, x^{\varepsilon_2}\, y^{\varepsilon_3} \cdots\cdots \bigcirc^{\varepsilon_{p-1}} \qquad \varepsilon_i = (-1)^{\left[\frac{q}{p} i\right]}$$

$$\alpha_{q/p} = x\; W_{q/p}\; \square^{(-1)^q}\; \overline{W}_{q/p} \qquad (\text{length} = 2p)$$

Note : $x^{\pm 1}$ and $y^{\pm 1}$ appear alternatively in $\alpha_{q/p}$

ie $x^{\pm 2}$ nor $y^{\pm 2}$ doesnot appear in $\alpha_{q/p}$

Fix $0 < r = \frac{q}{p} < 1$ and set $u := \alpha_{q/p} \in \langle x, y \rangle$

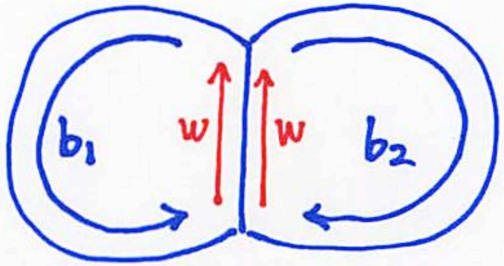
Then $G(K(r)) = \langle x, y \mid u \rangle$

Set $R := \{ \text{cyclic permutations of } u, u^{-1} \}$

<u>Def</u> A reduced word $w \in \langle x, y \rangle$ is a piece of R

$\Leftrightarrow$ $\exists$ distint elements $w_1, w_2 \in R$

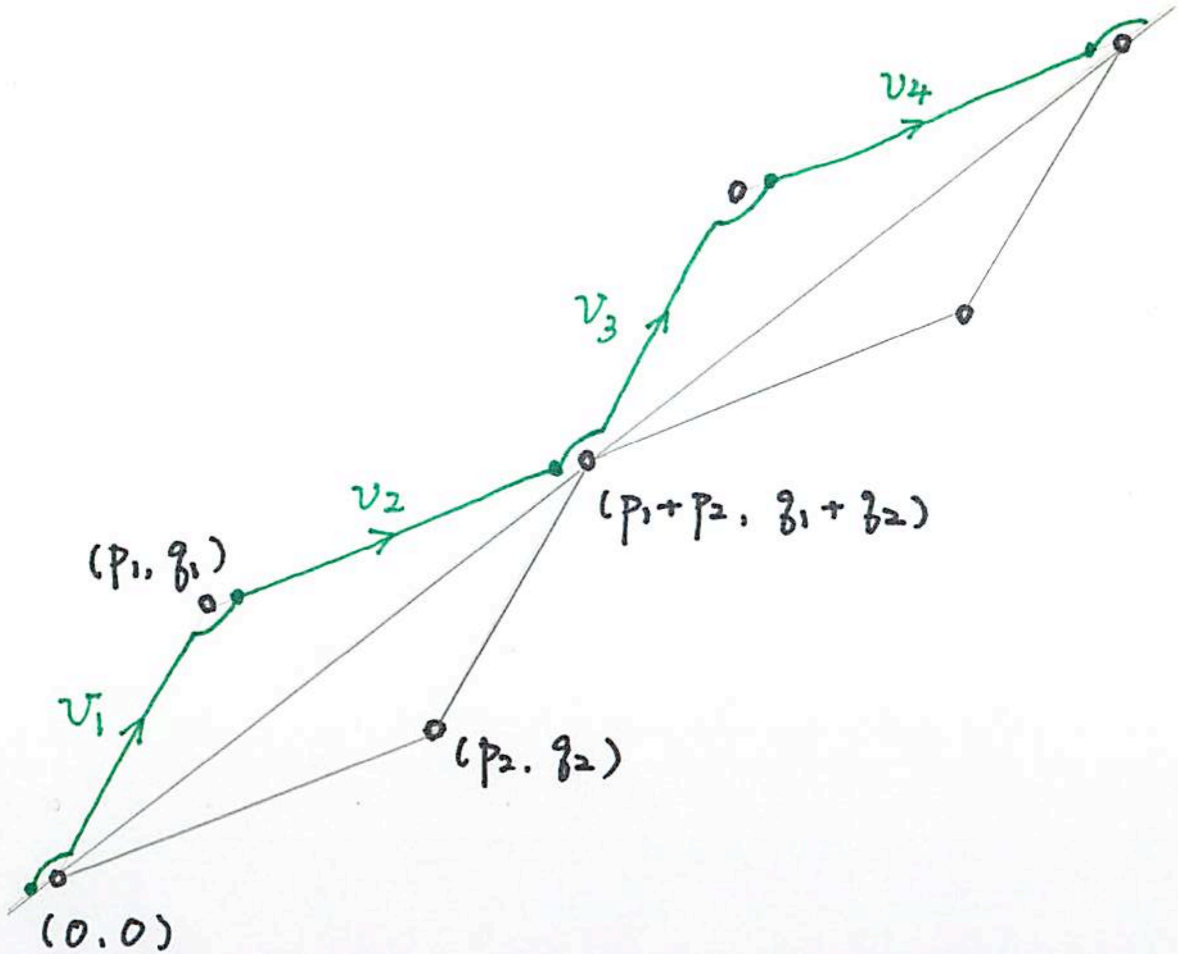
st $w_1 \equiv w b_1$, $w_2 \equiv w b_2$ as reduced words



in van-Kampen diagram

Canonical decomposition of the relator $u = \alpha_{q/p}$

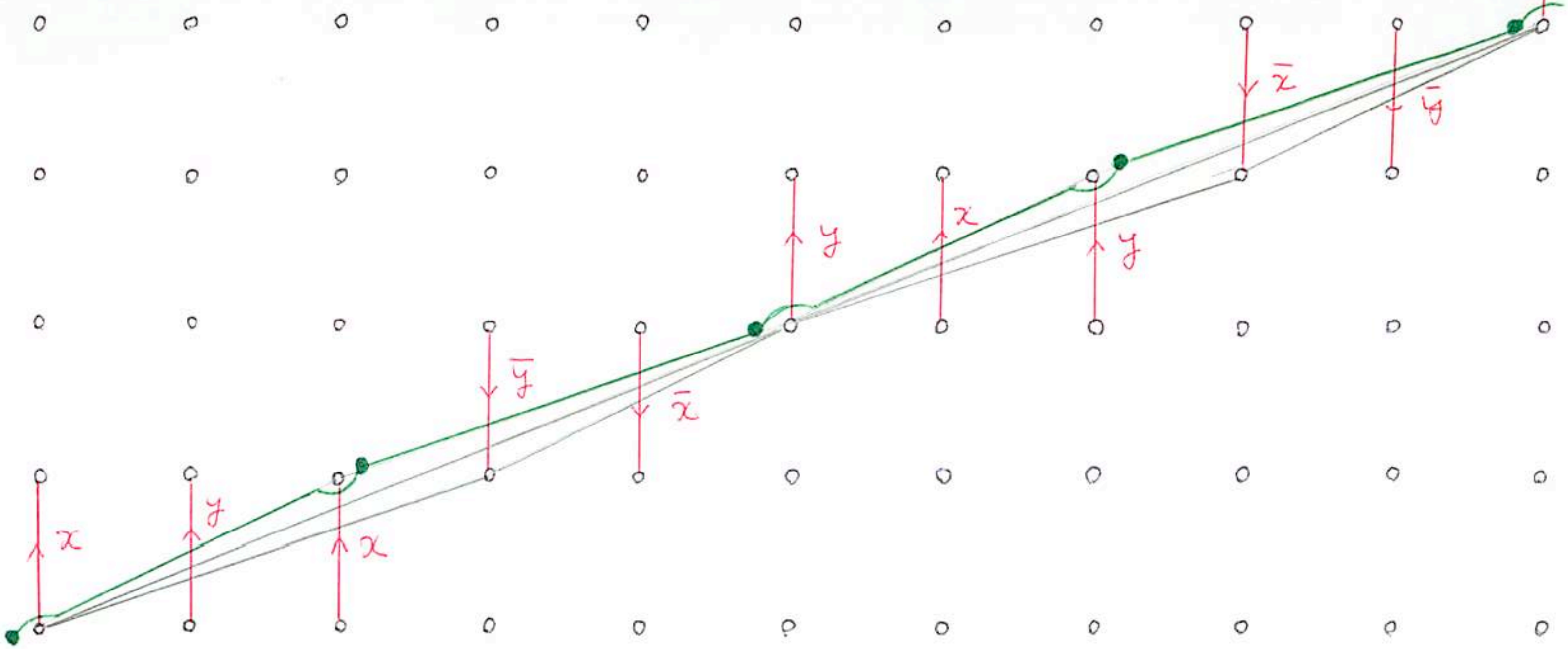
q_i/p_i $(i=1,2)$: Farey neighbors of q/p, st $\frac{q_2}{p_2} < \frac{q}{p} < \frac{q_1}{p_1}$



$u = v_1 v_2 v_3 v_4$, $|v_1| = |v_3| = p_1 + 1$

$|v_2| = |v_4| = p_2 - 1$

Example $\frac{2}{5} = \frac{1}{2} \oplus \frac{1}{3}$



$$u = \alpha_{2/5} = \underbrace{x\ y\ x}_{v_1}\ \underbrace{\bar{y}\ \bar{x}}_{v_2}\ \underbrace{y\ x\ y}_{v_3}\ \underbrace{\bar{x}\ \bar{y}}_{v_4}$$

$v_1,\ v_3 \leftrightarrow 1/2$, $|v_1| = |v_3| = 2+1 = 3$

$v_2,\ v_4 \leftrightarrow 1/3$, $|v_2| = |v_3| = 3-1 = 2$

Example $\quad r = \frac{2}{5} \qquad u = x \; yx\bar{y}\bar{x} \; y \; xy\bar{x}\bar{y}$

$$
R: \quad
\begin{array}{ll}
\boxed{x\,y}\; x\,\bar{y}\,\bar{x}\,y\,x\,y\,\bar{x}\,\bar{y} & \boxed{y\,x\,\bar{y}\,\bar{x}}\;\bar{y}\,x\,y\,\bar{x}\,\bar{y}\,\bar{x} \\
\boxed{y\,x\,\bar{y}\,\bar{x}}\; y\,x\,y\,\bar{x}\,\bar{y}\,x & x\,\bar{y}\,\bar{x}\,\bar{y}\,x\,y\,\bar{x}\,\bar{y}\,\bar{x}\,y \\
x\,\bar{y}\,\bar{x}\,y\,x\,y\,\bar{x}\,\bar{y}\,x\,y & \bar{y}\,\bar{x}\,\bar{y}\,x\,y\,\bar{x}\,\bar{y}\,\bar{x}\,y\,x \\
\bar{y}\,\bar{x}\,y\,x\,y\,\bar{x}\,\bar{y}\,x\,y\,x & \bar{x}\,\bar{y}\,x\,y\,\bar{x}\,\bar{y}\,\bar{x}\,y\,x\,\bar{y} \\
\bar{x}\,y\,x\,y\,\bar{x}\,\bar{y}\,x\,y\,x\,\bar{y} & \bar{y}\,x\,y\,\bar{x}\,\bar{y}\,\bar{x}\,y\,x\,\bar{y}\,\bar{x} \\
y\,x\,y\,\bar{x}\,\bar{y}\,x\,y\,x\,\bar{y}\,\bar{x} & \boxed{x\,y}\;\bar{x}\,\bar{y}\,\bar{x}\,y\,x\,\bar{y}\,\bar{x}\,\bar{y} \\
x\,y\,\bar{x}\,\bar{y}\,x\,y\,x\,\bar{y}\,\bar{x}\,y & y\,\bar{x}\,\bar{y}\,\bar{x}\,y\,x\,\bar{y}\,\bar{x}\,\bar{y}\,x \\
y\,\bar{x}\,\bar{y}\,x\,y\,x\,\bar{y}\,\bar{x}\,y\,x & \bar{x}\,\bar{y}\,\bar{x}\,y\,x\,\bar{y}\,\bar{x}\,\bar{y}\,x\,y \\
\bar{x}\,\bar{y}\,x\,y\,x\,\bar{y}\,\bar{x}\,y\,x\,y & \bar{y}\,\bar{x}\,y\,x\,\bar{y}\,\bar{x}\,\bar{y}\,x\,y\,\bar{x} \\
\bar{y}\,x\,y\,x\,\bar{y}\,\bar{x}\,y\,x\,y\,\bar{x} & \bar{x}\,y\,x\,\bar{y}\,\bar{x}\,\bar{y}\,x\,y\,\bar{x}\,\bar{y}
\end{array}
$$

- $w = yx\bar{y}\bar{x}$, xy, $\cdots$ are pieces

Observe: $u = (xyx)(\bar{y}\bar{x})(yxy)(\bar{x}\bar{y}) =: v_1 v_2 v_3 v_4$

Then $v_1 := xyx$ is not a piece.

Key Lemma (Complete characterization of the pieces)

Suppose $d(\infty, r) \geqq 3$, and recall the decomposition

$u = v_1 v_2 v_3 v_4$ arising from $\frac{q}{p} = \frac{q_1}{p_1} \oplus \frac{q_2}{p_2}$. Then:

(a) No piece can contain v_1 or v_3

(b) No piece can contain v_2 or v_4 in its "interior",

i.e. $v_{1e} v_2 v_{3b}$ is not a piece,

where v_{1e} is a non-empty terminal subword of v_1

v_{3b} " " initial " " v_3.

(c) v_2 and v_4 are pieces.

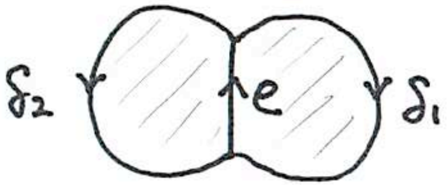
Moreover, every subword of the form

$$v_{1e} v_2, \quad v_2 v_{3b}, \quad v_{3e} v_4, \quad v_4 v_{1b}$$

are pieces.

Convention for a van-Kampen diagram M

(0) M is reduced, ie for every inner edge e of M,



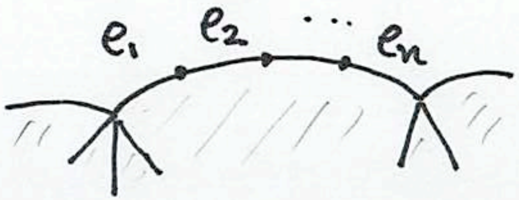
$\phi(\delta_1) \neq \phi(\delta_2)$. (Because, otherwise we can simplify M.)

(1) Every inner vertex has degree ≥ 3.

($\because$)

(2) For every boundary edge e, $\phi(e)$ is a piece.

(3)



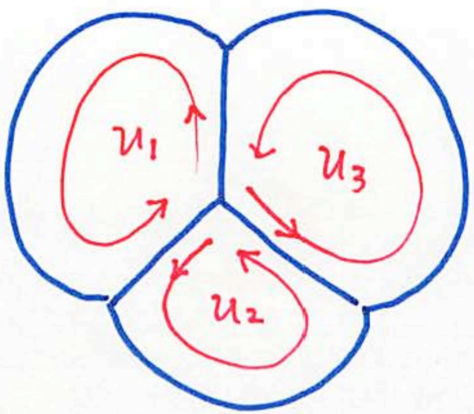
$\phi(e_1)\phi(e_2)\cdots\phi(e_n)$ cannot be expressed as a product of less than n pieces.

Key Lemma 1 R satisfies the conditions $C(4)$ and $T(4)$.

$C(4)$: Any $u \in R$ is not a product of $3\ (= 4-1)$ pieces.

$T(4)$: If $u_1, u_2, u_3 \in R$ and if $u_{i+1} \neq u_i^{-1}$ $(1 \leq i \leq 3)$, then at least one of $u_1 u_2$, $u_2 u_3$, $u_3 u_1$ is reduced.

ie the following situation does not occur in van Kampen diagram.



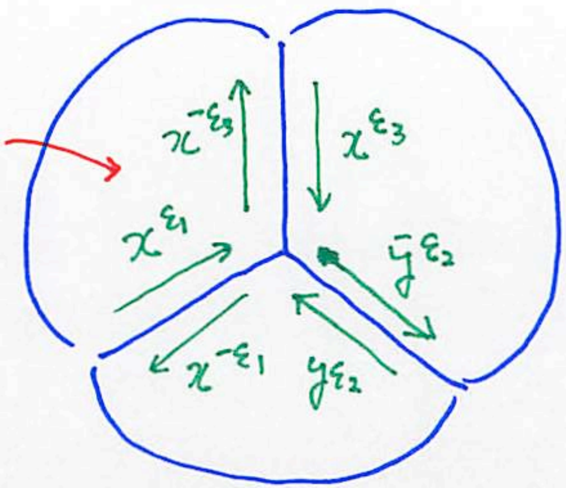
(Proof of Key Lemma)

- $C(4)$ follows from the characterization of the pieces.

  (A shortest decomposition of $u = v_1 v_2 v_3 v_4$ into pieces is $v_{1b} \cdot v_{1e} v_2 \cdot v_{3b} \cdot v_{3e} v_4$, which has length $4 > 3$.)

- $T(4)$ follows from the fact that u is cyclically alternating, i.e. $u = x^{\varepsilon_1} y^{\varepsilon_2} x^{\varepsilon_3} \cdots x^{\varepsilon_{2p-1}} y^{\varepsilon_{2p}}$ $(\varepsilon_i = \pm 1)$

non-alternating



Def M is a $[p, q]$-map if

(1) $\deg(v) \geq p$ for every interior vertex v.

(2) Every face D has at leat q edges.

Cor to Key Lemma 1

Any reduced van Kampen diagram over $G(K(r)) = \langle x, y \mid u \rangle$ is a $[4, 4]$-map.

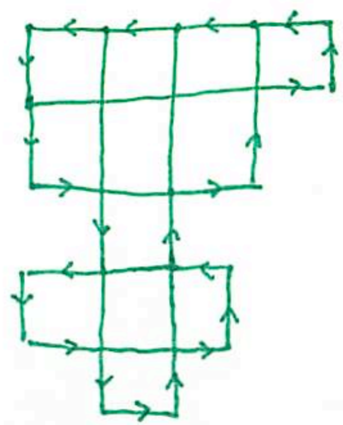
Curvature Formula

For every van-Kampen diagram M which is a $[4, 4]$-map,

$$\sum_{v \in \partial M} (3 - d(v)) \geqq 4$$

(Proof of the curvature theorem)

- Suppose M is a subdiagram of the tessellation of $\mathbb{R}^2$ by squares.



$$\begin{cases} \text{The inner angle of } M \text{ at } v \in \partial M = \frac{\pi}{2}(d(v)-1) \\ \text{the exterior} \quad : \quad : \quad : \quad = \pi - \frac{\pi}{2}(d(v)-1) \\ \qquad\qquad\qquad\qquad\qquad = \frac{\pi}{2}(3-d(v)) \end{cases}$$

$$2\pi = \text{total curvature of } \partial M = \sum_{v \in \partial M} \frac{\pi}{2}(3-d(v))$$

Hence $\displaystyle\sum_{v \in \partial M}(3-d(v)) = 4$

- If M has an interior point v with $\deg(v) > 4$, then M has negative curvature at v, and hence

$$\sum_{v \in \partial M} \frac{\pi}{2}(3-d(v)) = \text{curvature of } \partial M \underset{\uparrow}{=} 2\pi - \int_M K\, dA \geq 2\pi$$

(Gauss-Bonnet)

Hence $\displaystyle\sum_{v \in \partial M}(3-d(v)) \geq 4$

$\square$

Key Lemma 2

If w is a reduced cyclically alternating word in $\langle x, y \rangle$

st $[W] = 1$ in $G(K(r)) = \langle x, y \mid u \rangle$,

then w contains some "special subword" w_0,

st w_0 is a subword of some $u_i \in R$

with $|w_0| > \frac{1}{2}|u_i| = \frac{1}{2}|u| = p$

In fact, w_0 properly contains

$v_1 v_2$, $v_3 v_4$, $v_2^{-1} v_1^{-1}$ or $v_4^{-1} v_3^{-1}$,

where $u = v_1 v_2 v_3 v_4$ is the canonical decomposition.

(Proof of Key Lemma 2) Assume $M \cong D^2$.

$$A := \{ v \in \partial M \mid \deg(v) = 2 \}$$

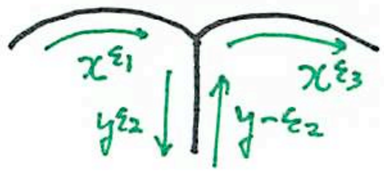
$$B := \{ v \in \partial M \mid \deg(v) \geq 4 \}$$

Then $\partial M^{(0)} = A \sqcup B$

($\because$) Otherwise

$M \not\cong D^2$

or

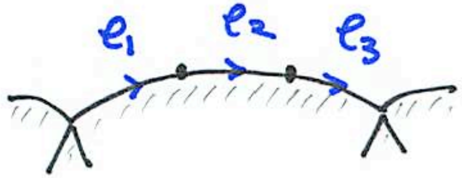


w is not alternating

On the other hand,

$$A - B \geq \sum_{v \in \partial M} (3 - d(v)) \geq 4$$ by the curvature formula

So ∂M contains



more than two successive vertices of degree 2.

Consider the subword $\phi(e_1)\phi(e_2)\phi(e_3)$ of w.

By Convention, this is a product of 3 pieces and cannot be expressed as a product of 2 pieces.

So, it properly contains a "maximal" "2-piece", w^*,

ie (i) w^* is a product of 2 pieces

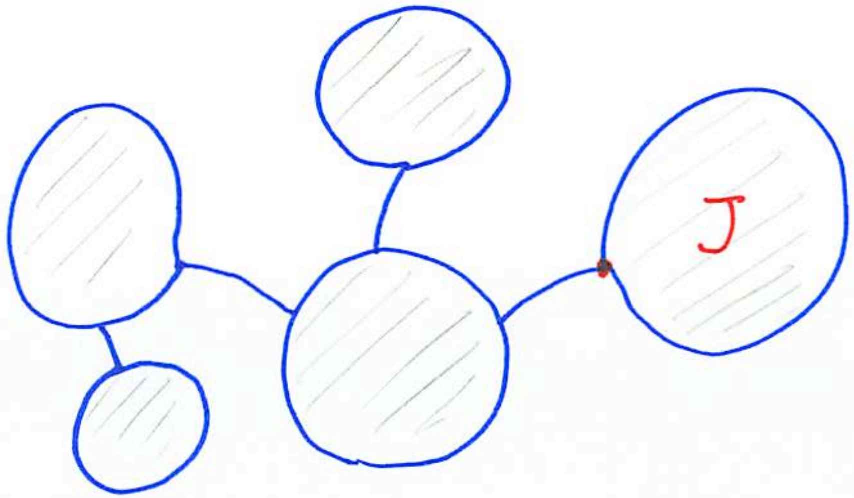
(ii) Any word w' containg w^* as an intial subword cannot expressed as a product of 2 pieces.

By the classification of the pieces, we can list all maximal 2-pieces.

This implies that $\phi(e_1)\phi(e_2)\phi(e_3)$ properly contains

$v_1 v_2$, $v_3 v_4$, $v_2^{-1} v_1^{-1}$ or $v_4^{-1} v_3^{-1}$.

If $M \ncong D^2$, we apply the above arguments to an extremal disk, J, of D^2.



$$|J| \cong D^2 \quad , \quad |J| \cap \overline{|M - J|} = \{ \text{a vertex of } M \}$$

(Proof of Theorem 1′)

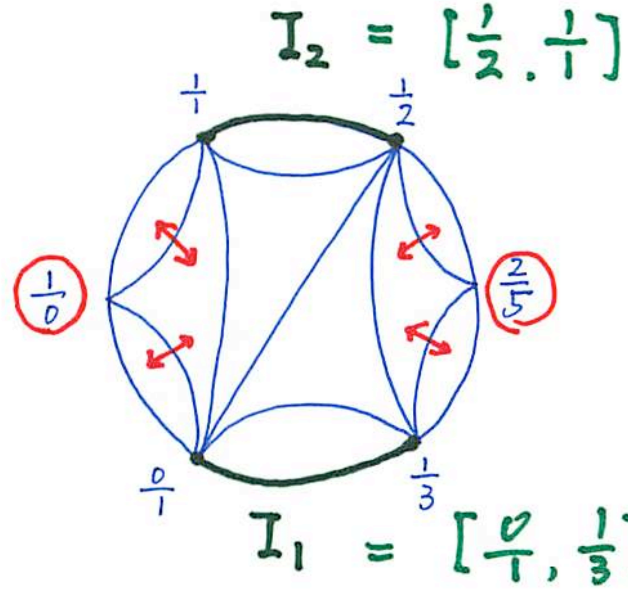
Pick $s \in I_1 \cup I_2$. Then we can show that $\alpha_s \in \langle x, y \rangle$ does not satisfy the condition in Key lemma 2.

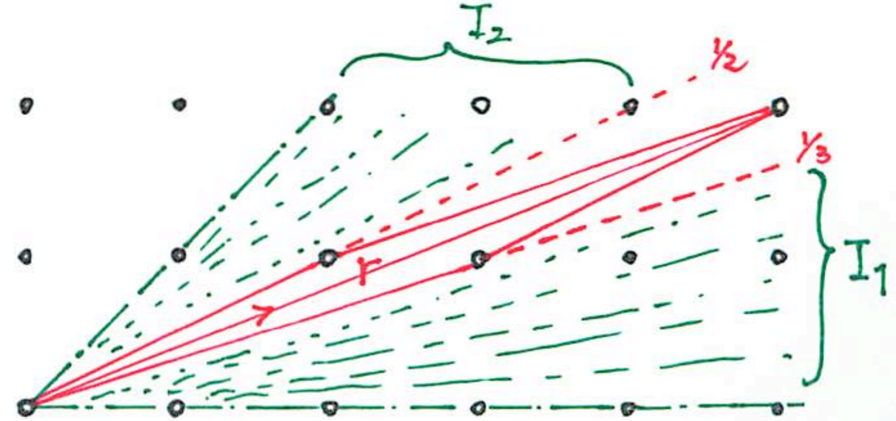
(Intuitive explanation)

$s \in I_1 \cup I_2$

$\Leftrightarrow$ The slope s is far from r

$\Leftrightarrow$ α_s and $\alpha_r = u$ cannot share a long subword.





$\square$