The Magnus representation and homology cobordism groups of homology cylinders

Takuya SAKASAI (逆井卓也)

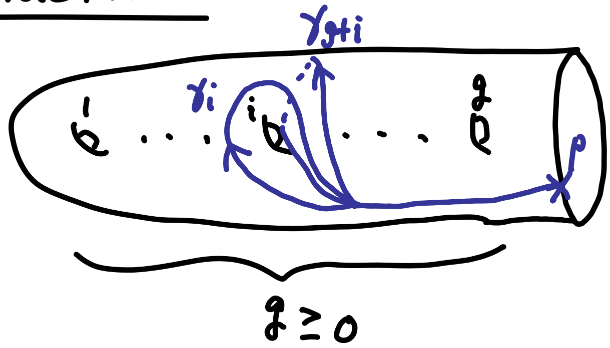
Tokyo Institute of Technology

(Sep. 16, 2010 @ Akita)

- Some results are joint works with Hiroshi Goda.

- Reference (A survey of Magnus representations)
  arXiv. math. GT/1005.5501

§1 Introduction

- $\Sigma_{g,1}$ = 

cpt
oriented
smooth

- $\pi := \pi_1(\Sigma_{g,1}, p) = \langle \gamma_1, \gamma_2, \cdots, \gamma_{2g} \rangle \cong F_{2g}$
- $H := H_1(\Sigma_{g,1}) = \mathrm{Span}_{\mathbb{Z}} \{ \underbrace{\gamma_1, \gamma_2, \cdots, \gamma_{2g}} \}$

symp. basis w.r.t.
$\mu : H \otimes H \longrightarrow \mathbb{Z}$

- $\Gamma_{g,1} = \mathrm{Diff}(\Sigma_{g,1} \text{ rel } \partial\Sigma_{g,1}) / \text{isotopy}$

$= \pi_0 \, \mathrm{Diff}(\Sigma_{g,1} \text{ rel } \partial\Sigma_{g,1})$: the mapping class group of $\Sigma_{g,1}$ (MCG)

- $\Gamma_{g,1} \curvearrowright H$ (preserving μ) yields :

$$\boxed{1 \to \underset{\text{Torelli grp}}{\mathcal{I}_{g,1}} \to \Gamma_{g,1} \xrightarrow{\sigma} \underset{\substack{\rotatebox{90}{$\cong$} \\ Sp(2g, \mathbb{Z})}}{Sp(H, \mu)} \to 1 \quad \text{(exact)}}$$

Th'm (Harer) For $g \geq 3$,

$$H_1(\Gamma_{g,1}) = \Gamma_{g,1}/[\Gamma_{g,1}, \Gamma_{g,1}] = 0$$

as an approximation of $\Gamma_{g,1}$

↪ No non-trivial abelian quotients!!

$$(\Gamma_{g,1} \xrightarrow[\text{hom}]{} A \text{ abelian} \Rightarrow A = \{1\})$$

$\mathbb{Z}/10\mathbb{Z}$

Rem · $\Gamma_{0,1} = \{1\}$, $H_1(\Gamma_{1,1}) = \mathbb{Z}$, $H_1(\Gamma_{2,1}) = \mathbb{Z}_{10}$

· $H_1(\mathcal{J}_{g,1}) = \Lambda^3 H \oplus (2\text{-torsion})$ (Johnson)

§2 Homology cylinders

Def (Goussarov, Habiro, Garoufalidis-Levine)

(M, i_+, i_-) : an homology cylinder (HC) over $\Sigma_{g,1}$

$\overset{\Longleftrightarrow}{\text{def}}$ $\begin{cases} M : \text{a cpt ori. 3-mfd} \\ i_+, i_- : \Sigma_{g,1} \hookrightarrow \partial M \text{ (markings) s.t.} \end{cases}$

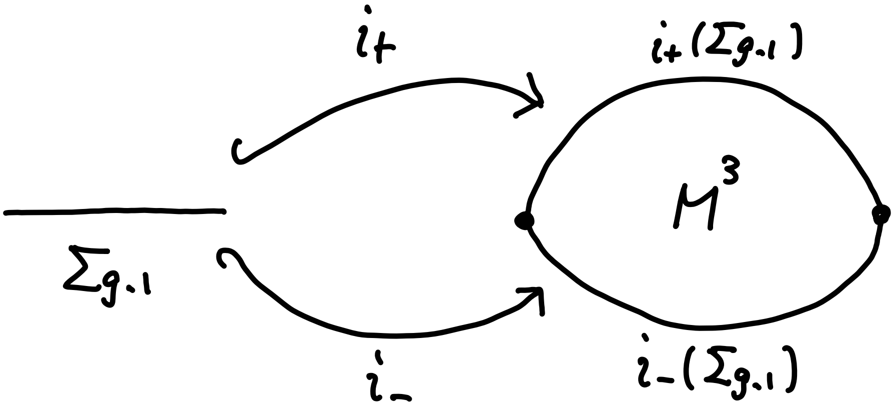
① i_+ : ori. pres. i_- : ori. rev.

② $i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1}) = \partial M$, $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_\pm(\partial\Sigma_{g,1})$

③ $i_+|_{\partial\Sigma_{g,1}} = i_-|_{\partial\Sigma_{g,1}}$

④ $H_*(M, i_+(\Sigma_{g,1})) = H_*(M, i_-(\Sigma_{g,1})) = 0$

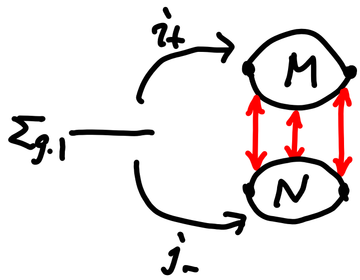
Def $\mathcal{C}_{g,1} :=$ HC's over $\Sigma_{g,1}$ / marking pres. diffeo.



Note
$(M, i_\pm(\partial\Sigma_{g,1}))$ is $\underbrace{\text{a sutured}}_{\text{balanced}}$ mfd.

Stacking

$$(M, i_+, i_-)\cdot(N, j_+, j_-) := (M \cup_{i_-=j_+} N,\ i_+, j_-)$$



↳ $\mathcal{C}_{g,1}$ becomes a monoid

w/ unit

$$1_{\mathcal{C}_{g,1}} = (\Sigma_{g,1}\times[0,1],\ id\times 1,\ id\times 0).$$

$\Gamma_{g,1}$-action

$$\begin{array}{ccc} \mathcal{C}_{g,1} & \curvearrowleft & \Gamma_{g,1} \\ \Psi & & \Psi \\ (M, i_+, i_-) & & [\varphi] \end{array}$$

$$(M, i_+, i_-)\cdot[\varphi] = (M, i_+, i_- \circ \varphi)$$

$\rightsquigarrow$ can check

$$\begin{array}{ccc} \Gamma_{g,1} & \hookrightarrow & \mathcal{C}_{g,1} \\ \Psi & & \Psi \\ [\varphi] & \longmapsto & 1_{\mathcal{C}_{g,1}}\cdot[\varphi] \end{array}$$

<u>inj. monoid hom</u>

"mapping cylinder"

From monoid to group

Def $(M, i_+, i_-) \sim (N, j_+, j_-)$ <u>homology cobordant</u>

$\overset{\text{def}}{\iff}$ $\exists W$: cpt oriented smooth 4-mfd

s.t.
$$\partial W = M \cup (-N)$$
$$H_*(W, M) = H_*(W, N) = 0$$

Def (Garoufalidis-Levine)

$$\mathcal{H}_{g,1} := \mathcal{C}_{g,1} / \text{H-cob} \qquad \text{group} \qquad (M, i_+, i_-)^{-1} = (-M, i_-, i_+)$$

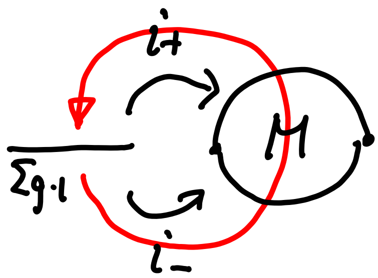
Can check

$$\begin{tikzcd}
\underset{\text{grp}}{\Gamma_{g,1}} \arrow[r, hook] \arrow[dr, hook, bend right] & \overset{\text{monoid}}{\mathcal{C}_{g,1}} \arrow[d] \\
 & \underset{\text{grp}}{\mathcal{H}_{g,1}}
\end{tikzcd}$$

$\mathcal{C}_{g,1}$ & $\mathcal{H}_{g,1}$ are "enlargement" of $\Gamma_{g,1}$

$\Gamma_{g,1}$ and $\mathcal{H}_{g,1}$ share many property:

$$\sigma : \begin{array}{ccc} \mathcal{H}_{g,1} & \longrightarrow & Sp(H, \mu) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (M, i_+, i_-) & \longmapsto & i_+^{-1} \circ i_- \end{array}$$



$$\begin{tikzcd}
1 \arrow[r] & \mathcal{I}\mathcal{H}_{g,1} \arrow[r] & \mathcal{H}_{g,1} \arrow[r, "\sigma"] & Sp(H, \mu) \arrow[r] & 1 \\
1 \arrow[r] & \mathcal{I}_{g,1} \arrow[r] \arrow[u, phantom, "\cup" rotate=90] & \Gamma_{g,1} \arrow[r] \arrow[u, phantom, "\cup" rotate=90] & Sp(H, \mu) \arrow[r] \arrow[u, phantom, "\parallel"] & 1
\end{tikzcd}$$

More generally

$$\begin{tikzcd} \Gamma_{g,1} \arrow[r] & \mathrm{Aut}\, N_k \\ \mathcal{H}_{g,1} \arrow[u, phantom, "\cap"] \arrow[ur, "\exists"] & \end{tikzcd}$$

Stallings' th'm

$$N_k = \pi \big/ \underbrace{[\pi,[\pi,[\pi, \cdots [\pi,\pi]\cdots]]}_{k}$$

$\Downarrow$ connection to

- Johnson homomorphism
- MMM-classes

Dehn-Nielsen

$$\begin{tikzcd} \Gamma_{g,1} \arrow[r, hook] & \mathrm{Aut}\, \pi \\ \mathcal{H}_{g,1} \arrow[u, phantom, "\cap"] \arrow[r] & \mathrm{Aut}\, {}^{\exists}\pi^{acy} \arrow[u, phantom, "\cap"] \end{tikzcd}$$

Levine's acyclic closure of π

§3 Invariants of HCs

Q. What is $H_1(\mathcal{H}_{g,1})$? (Recall $H_1(\Gamma_{g,1}) = 0$ for $g \geq 3$)

Main theme of this talk

Notation

$$\begin{tikzcd} & \mathbb{Z}H \arrow[d, hook] \\ H \arrow[ur, hook, "\text{monomial}"] \arrow[r, hook] & \mathcal{K}_H \end{tikzcd}$$

$$\mathbb{Z}H \cong \mathbb{Z}[\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \cdots, \gamma_{2g}^{\pm 1}]$$

$$\mathcal{K}_H = \mathrm{Frac}(\mathbb{Z}H) \cong \mathbb{Q}(\gamma_1, \gamma_2, \cdots, \gamma_{2g})$$

- $(M, i_+, i_-) \in \mathcal{C}_{g,1}$

$$\pi_1 M \longrightarrow H_1(M) \xrightarrow[\cong]{i_+^{-1}} H$$

$\rightsquigarrow$ $\mathbb{Z}H$, $\mathcal{K}_H$ are $\pi_1 M$-module

$$\boxed{\underline{\text{Lemma}} \quad H_*(M, i_\pm(\Sigma_{g,1}); \mathcal{K}_H) = 0}$$

i.e (M, i_+, i_-) is a "$\mathcal{K}_H$-homology cobordism".

We use this lemma to define:

① <u>torsion</u> (Reidemeister torsion)

$$\tau(M) := \tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_H)) \in GL(\mathcal{K}_H)/\sim$$

$$\cong \downarrow$$

$$\mathcal{K}_H^\times/\pm H$$

② Magnus matrix

$$
\begin{tikzcd}
\mathcal{K}_H^{2g} \cong H_1(\Sigma_{g,1}, p ; i_-^*\mathcal{K}_H) \arrow[d, "r(M)\cdot"] \arrow[dr, "i_-", "\cong"'] & \\
 & H_1(M, p ; \mathcal{K}_H) \arrow[dl, "i_+^{-1}", "\cong"'] \\
\mathcal{K}_H^{2g} \cong H_1(\Sigma_{g,1}, p ; i_+^*\mathcal{K}_H) &
\end{tikzcd}
$$

$$
\Longrightarrow \quad r : \mathcal{C}_{g,1} \longrightarrow GL(2g, \mathcal{K}_H)
$$

<u>Magnus representation</u>

$$
(M, i_+, i_-) \longmapsto r(M)
$$

← Magnus matrix

<u>Prop</u>

ⓐ
$$
\begin{tikzcd}
\mathcal{C}_{g,1} \arrow[r, "r"] \arrow[d] & GL(2g, \mathcal{K}_H) \\
\mathcal{H}_{g,1} \arrow[ur, "r"'] \arrow[u, phantom, shift left=6ex, "\circlearrowleft"] &
\end{tikzcd}
$$
, τ does NOT factor through $\mathcal{H}_{g,1}$

ⓑ For $f = r, \tau$

$$
f(M_1 \cdot M_2) = f(M_1) \cdot {}^{\sigma(M_1)} f(M_2)
$$

i.e. crossed hom.

ⓒ For $[\varphi] \in \Gamma_{g,1}$, $\tau([\varphi]) = 1$ and

$$
r([\varphi]) = \left(\overline{\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i}} \right)_{i,j} \in GL(2g, \mathbb{Z}H)
$$

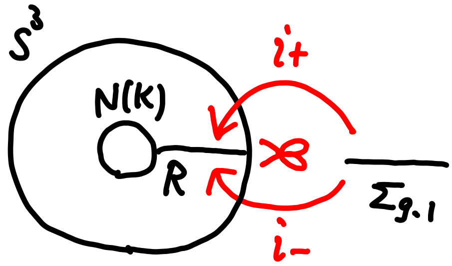
Example ("Homologically fibered knots")

$K \subset S^3$: a knot w/ $\begin{cases} \deg \Delta_K(t) = 2g & g = \text{genus}(K) \\ \Delta_K(t) \text{ is monic.} \end{cases}$

R : a min. genus Seifert surface of K

$\cong \uparrow i$ fix

$\Sigma_{g,1}$



$\Longrightarrow$ $M_R = (S^3 - N(R), i_+, i_-)$

is an HC.

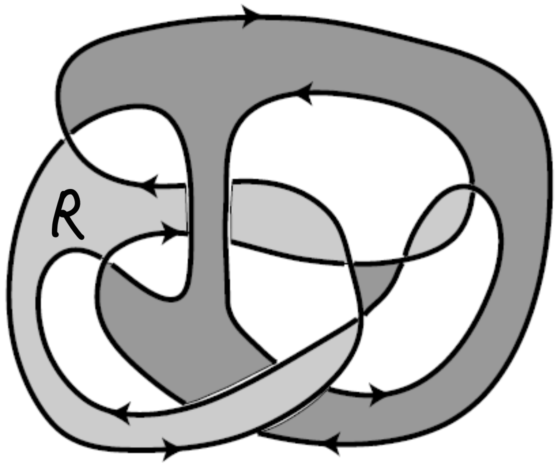
ambiguity : ① $i : \Sigma_{g,1} \xrightarrow{\cong} R$

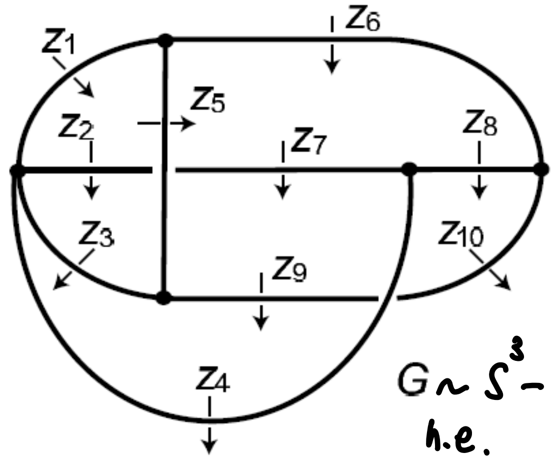
② Uniqueness of R.

But, $\boxed{\begin{array}{l} \underline{\text{Prop}} \quad K : \text{as above} \\ \qquad\quad R_1, R_2 : \text{min genus Seifert surfaces} \\ \qquad\qquad\quad \text{w/ fixed id. with } \Sigma_{g,1} \\ \Longrightarrow M_{R_1} \text{ and } M_{R_2} \text{ are conjugate in } \mathcal{H}_{g,1} \end{array}}$

$\hookrightarrow$ $H_1(\mathcal{H}_{g,1})$ gives invariants of HF knots.

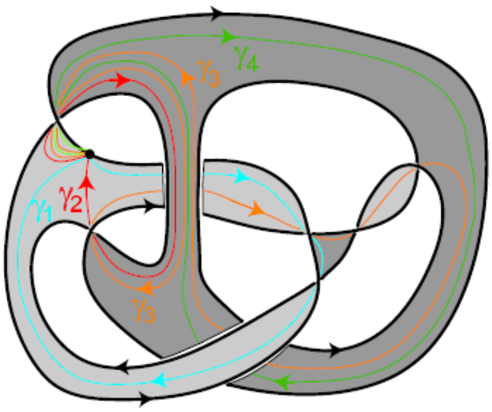
Example of calculation of "admissible presentation"





$G \sim S^3 - N(R)$
h.e.

12n0057



12n0057

Generators	$i_-(\gamma_1), \ldots, i_-(\gamma_4),\ z_1, \ldots, z_{10},\ i_+(\gamma_1), \ldots, i_+(\gamma_4)$
Relations	$z_1 z_5 z_6^{-1},\ z_2 z_3 z_4 z_1,\ z_3 z_9^{-1} z_5^{-1},\ z_7 z_4 z_8^{-1},\ z_8 z_{10} z_6,$ $z_2 z_5 z_7^{-1} z_5^{-1},\ z_9 z_4 z_{10}^{-1} z_4^{-1},\ i_-(\gamma_1) z_1^{-1} z_5^{-1},\ i_-(\gamma_2) z_2,$ $i_-(\gamma_3) z_4 z_8 z_7 z_5^{-1},\ i_-(\gamma_4) z_4,\ i_+(\gamma_1) z_5^{-1},\ i_+(\gamma_2) z_9^{-1} z_6^{-1},$ $i_+(\gamma_3) z_6 z_4 z_7 z_5^{-1} z_3^{-1} z_5 z_6^{-1},\ i_+(\gamma_4) z_6 z_7^{-1} z_6^{-1}$

$$r(M_R) = \begin{pmatrix} \frac{x_3+x_1 x_2^2(-1+x_2(-1+x_4))-x_2 x_3 x_4}{x_1 x_2^2(-1+x_2(-1+x_4))} & -\frac{(-1+x_4)(-1+x_2 x_4)}{-1+x_2(-1+x_4)} & \frac{x_4}{1+x_2-x_2 x_4} & 0 \\ -\frac{(1+x_1 x_2)x_3}{x_1^2 x_2(-1+x_2(-1+x_4))} & -\frac{x_2(1+x_1 x_2)(-1+x_4)}{x_1(-1+x_2(-1+x_4))} & -\frac{(1+x_2)(1+x_1 x_2^2(-1+x_4))}{x_1 x_2(-1+x_2(-1+x_4))} & \frac{1}{x_4} \\ \frac{x_3}{x_1(-1+x_2(-1+x_4))} & \frac{x_2^2(-1+x_4)}{-1+x_2(-1+x_4)} & \frac{x_2(1+x_2)(-1+x_4)}{-1+x_2(-1+x_4)} & 0 \\ \frac{(x_1 x_2^2-x_3)x_4}{x_1^2 x_2(-1+x_2(-1+x_4))} & \frac{x_2 x_4(x_1 x_2+x_3-x_3 x_4)}{x_1 x_3(-1+x_2(-1+x_4))} & \frac{(1+x_2)(x_1 x_2^2-x_3)x_4}{x_1 x_2 x_3(-1+x_2(-1+x_4))} & 1 \end{pmatrix},$$

where $x_i = i_+(\gamma_i)$

and

$$\det(\tau(M_R)) = x_1 x_2^4 + x_1 x_2^5 - x_1 x_2^5 x_4 .$$

(These results say that 12n57 is NOT fibered.)

§4 Abelian quotients of $\mathcal{H}_{g,1}$

@ Previously known :

- $\mathcal{JH}_{g,1} \overset{\exists}{\twoheadrightarrow} \mathbb{Z}^\infty$ (Morita)
- $\mathcal{C}_{g,1} \overset{\exists}{\twoheadrightarrow} \mathbb{Z}_{\geq 0}^\infty$ (Goda - S.)

@ Recall : we have crossed homs

r (Magnus rep.) : H-cob. inv

τ (torsion) : NOT H-cob. inv.

<u>Th'm</u> (Cha - Friedl - Kim) $g \geq 1$

The composition

$$\tilde{\tau} : \mathcal{C}_{g,1} \xrightarrow{\det \circ \tau} \mathcal{K}_H / \pm H \longrightarrow\!\!\!\!\!\rightarrow \mathcal{K}_H / \pm HAN \cong \mathbb{Z}_2^{\infty}$$

gives a <u>homomorphism</u> and <u>factors through $\mathcal{H}_{g,1}$</u>.

Moreover, $\operatorname{Im} \tilde{\tau} \cong \mathbb{Z}_2^{\infty}$.

↳ $\mathcal{H}_{g,1}$ & $H_1(\mathcal{H}_{g,1})$ are NOT fin. generated

Here

- $A := \{ \varphi(f) \cdot f^{-1} \mid f \in \mathcal{K}_H, \varphi \in Sp(H) \}$

  kills the action of $Sp(H)$

- $N := \{ f \cdot \bar{f} \mid f \in \mathcal{K}_H \}$

  yields H-cov. invariance

- $\mathcal{K}_H^\times / \pm H \cong \mathbb{Z}^\infty$ (Count exponents of irred. polyn.)
  $\mathbb{Z}H$: UFD

($\mathcal{K}_H^\times / \pm HA \cong \mathbb{Z}^\infty$)

- $\varphi = -I_{2g} \Rightarrow \varphi(f) = \bar{f}$

  so $\mathcal{K}_H^\times / \pm HAN \cong \mathbb{Z}_2^\infty$

- They show (Alex. polyn. of knots) $\subset \mathcal{K}_H^\times$ //

§5 Magnus representation and $\mathcal{H}_{g,1}$

Using Cha-Friedl-Kim's idea, we have

$$\hat{r} : \mathcal{I}\mathcal{H}_{g,1} \xrightarrow{r} GL(2g, \mathcal{K}_H) \longrightarrow \mathcal{K}_H^\times \longrightarrow \mathcal{K}_H^\times / \pm H \cong \mathbb{Z}^\infty$$

$$\tilde{r} : \mathcal{H}_{g,1} \longrightarrow GL(2g, \mathcal{K}_H) \longrightarrow \mathcal{K}_H^\times \longrightarrow \mathcal{K}_H^\times / \pm HA \cong \mathbb{Z}^\infty$$

$\hat{r}, \tilde{r}$ are homomorphisms.

Thm ① $\mathrm{Im}(\hat{r} : \mathcal{J}\mathcal{H}_{g,1} \longrightarrow \mathbb{Z}^\infty) \cong \mathbb{Z}^\infty$

② $\tilde{r} : \mathcal{H}_{g,1} \longrightarrow \mathbb{Z}^\infty$ is trivial.

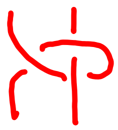
cf. Morita showed

$\exists\, t : \mathcal{J}\mathcal{H}_{g,1} \twoheadrightarrow \mathbb{Z}^\infty$.

(Sketch of Proof)

① Levine's construction:

pure string link $\rightsquigarrow$ HC



Using this, we show $\mathrm{Im}\,\hat{r} \supset \left\{ \dfrac{\gamma_2^{m}\gamma_3 + \gamma_4 - 1}{\gamma_2^{-m}\gamma_3^{-1} + \gamma_4^{-1} - 1} \right\}_{m=0}^{\infty}$

② Twisted symplecticity of r: $\exists\, \tilde{J} \in GL(2g, \mathbb{Z}H)$

$\overline{r(M)} \cdot \tilde{J} \cdot r(M) = {}^{\sigma(M)}\tilde{J} \Rightarrow \tilde{r}(M)^2 = 1 \in K_H^\times / \pm HA$ //

The proof of Th'm ② implies

$$\det(r(M)) = h\,\frac{\bar{p}}{p} \qquad h \in H,\ p \in \mathbb{Z}H$$

$\rightsquigarrow\ p \in \mathcal{K}_H^\times / \pm H\,\mathcal{S}$ is uniquely defined.

$$\mathcal{S} = \{ f \in \mathcal{K}_H^\times \mid f = \bar{f} \}$$

$M \mapsto p$ defines a hom. $\sqrt{\tilde{r}} : \mathcal{H}_{g,1} \to \mathcal{K}_H^\times / \pm H A N \mathcal{S} \cong \mathbb{Z}_2^\infty$

$\boxed{\begin{array}{l} \underline{\text{Th'm}} \\ \qquad \det(r(M)) = \dfrac{\det(\overline{\tau(M)})}{\det(\tau(M))} \in \mathcal{K}_H^\times / \pm H \\ \qquad \text{In particular, } \sqrt{\tilde{r}} = \tilde{\tau} : \mathcal{H}_{g,1} \longrightarrow \mathcal{K}_H^\times / \pm H A N \underset{\sim}{\mathcal{S}} \end{array}}$

<u>Rem</u> $\{\text{Alexander polyn.}\} \subset \mathcal{S}$

§6 Higher dimensional cases

$k \geq 2$, $n \geq 1$

$$X_n^k := \#_n (S^1 \times S^{k-1}) - \mathring{D}^k \qquad X_n^2 = \Sigma_{n,1}$$

When $k \geq 3$, $\pi_1 X_n^k \cong F_n = \langle x_1, x_2, \cdots, x_n \rangle$

$$H_1 := H_1(X_n^k) \cong \mathbb{Z}^n$$

We can define homology cylinders over X_n^k.

$$\begin{tikzcd} \Gamma(X_n^k) \arrow[r] \arrow[dr] & \mathcal{C}(X_n^k) \arrow[d] \\ & \mathcal{H}(X_n^k) \end{tikzcd}$$

Also the Magnus rep

$$r : \mathcal{H}(X_n^k) \longrightarrow GL(n, \mathcal{K}_{H_1})$$

is defined.

Consider the homomorphism

$$\tilde{r} : \mathcal{H}(X_n^k) \xrightarrow{r} GL(n, \mathcal{K}_{H_1}) \xrightarrow{\det} \mathcal{K}_H^\times \longrightarrow \mathcal{K}_H^\times / \pm HA \cong \mathbb{Z}^\infty$$

$$\boxed{\begin{array}{l} \underline{\text{Thm}} \quad \text{For } k \geq 3 \text{ and } n \geq 2, \\[1em] \qquad\qquad \operatorname{Im} \tilde{r} \cong \mathbb{Z}^\infty \end{array}}$$

(Sketch of Proof)

<u>Step 1</u> Show

$$\begin{tikzcd} \mathcal{H}(X_n^k) \arrow[r, two heads, "\text{Onto}"] & \operatorname{Aut} F_n^{acy} \\ \Gamma(X_n^k) \arrow[u] \arrow[r, two heads] & \operatorname{Aut} F_n \arrow[u, hook] \end{tikzcd}$$

$\tilde{r}$ factors through $\operatorname{Aut} F_n^{acy}$

<u>Step 2</u> $\operatorname{Aut} F_n^{acy}$ includes

H_1-isom endomorphisms of F_n

Consider

$$\begin{array}{rccc} f_m : & F_n & \longrightarrow & F_n \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & x_1 & \longmapsto & (x_1 x_2^{-1} x_1^{-1} x_2^{-1})^m x_1 x_2^{2m} \\ & x_i & \longmapsto & x_i \quad (2 \leq i \leq n) \end{array}$$

Then $f_m \in \mathrm{Aut}\, F_n^{acy}$ and

$$\tilde{r}(f_m) = 1 - x_2 + x_2^2 - x_2^3 + \cdots + x_2^{2m}.$$

Well-known fact on cyclotomic polynomials:

$1 - x_2 + x_2^2 - x_2^3 + \cdots + x_2^{2m}$ is irreducible if $2m+1$ is prime.

$\rightsquigarrow$ conclusion //

§7 Questions

① $H^1(\mathcal{H}_{g,1}) \overset{?}{=} 0$

② $H^2(\mathcal{H}_{g,1}) \overset{?}{\cong} \mathbb{Z}$ ($\supset$ is o.k.)

③ (From A. Campo)

$$\begin{tikzcd} \Gamma_{g,1} \arrow[r] & \mathcal{C}_{g,1} \arrow[l, bend left, "\exists ?\ \text{retraction}"] \end{tikzcd}$$

④ Is any knot "twisted" homologically fibered ?

⑤ $H_3(F_n^{acy}) \overset{?}{=} 0$

⑥ Is F_n^{acy} finitely generated ?
(Unlikely)

FIN