On the Reidemeister-Turaev torsion of standard Spin^c structures on Seifert fibered 3-manifolds

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Notations

$M :=$ a closed oriented 3-manifold

$S_g :=$ an orientable surface of genus g

$C_* = (0 \xrightarrow{\partial_m} C_m \xrightarrow{\partial_{m-1}} C_{m-1} \to \cdots \xrightarrow{\partial_0} C_0 \xrightarrow{\partial_{-1}} 0)$
: an acyclic, based, finite chain complex / a field F

$\rightsquigarrow \tau(C_*) \in F^\times$: the torsion of C_*

$\hat{M} \to M$: the maximal abelian covering

$\varphi : \mathbb{Z}[H_1(M;\mathbb{Z})] \to F$: a ring homomorphism

$C_*^\varphi(M) := F \otimes_\varphi C_*(\hat{M})$: the twisted chain complex

$$\tau^\varphi(M) := \begin{cases} \tau(C_*^\varphi(M)) \in F^\times / \pm\varphi(H_1(M;\mathbb{Z})) & \text{if } C_*^\varphi(M)\text{: acyclic,} \\ 0 & \text{otherwise.} \end{cases}$$

: the Reidemeister torsion of M

Remark: Remdemeister torsion has an indeterminacy.

The Reidemeister-Turaev torsion

$\mathcal{V}_1, \mathcal{V}_2$: non-singular vector fields on M

$\mathcal{V}_1 \sim \mathcal{V}_2 \Longleftrightarrow \mathcal{V}_1|_{M \setminus B^3} \simeq \mathcal{V}_2|_{M \setminus B^3}$

$\mathrm{Spin}^c(M) := \{\text{non-singular vector fields on } M\}/\sim$

$[\mathcal{V}] \in \mathrm{Spin}^c(M)$: a Spin^c structure represented by $\mathcal{V}$

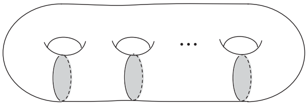
Turaev's idea (1990)

Each Spin^c-structure $[\mathcal{V}]$ determines a basis of $C^{\varphi}(M)$

$\rightsquigarrow \tau^{\varphi}(M, [\mathcal{V}]) \in F/\pm 1$: the Reidemeister-Turaev torsion

$M = V_1 \cup_{S_g} V_2$: a Heegaard splitting of M if

1. V_1, V_2: handlebodies of genus g;
2. $V_1 \cap V_2 = \partial V_1 = \partial V_2 = S_g$.



$M = V_1 \cup_{S_g} V_2$: a Heegaard splitting

$\{D_1^{(i)}, \ldots, D_g^{(i)}\}$: a complete meridian system of V_i

$(S_g; \bigcup_j \partial D_j^{(1)}, \bigcup_k \partial D_k^{(2)})$: a Heegaard diagram

$\rightsquigarrow M = S_g \cup (\bigcup_j D_j^{(1)}) \cup (\bigcup_k D_k^{(2)}) \cup$ (two 3-balls)

Variations of Heegaard diagrams

$(S_g; \alpha, \beta)$: a Heegaard diagram of M, $\alpha := \bigcup_i \alpha_i$, $\beta := \bigcup_j \beta_j$

1. Fix an orientation for each slope of $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$.
2. Choose a point $b_j \in \beta_j \setminus \alpha$ for each β_j.

$\rightsquigarrow (S_g; \vec{\alpha}, \vec{\beta}; \{b_j\})$: a oriented, based Heegaard diagram

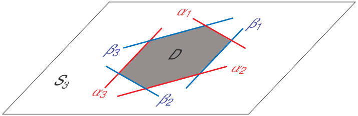
$\vec{\alpha}_i \rightsquigarrow x_i \in \pi_1(M, *)$: a generator, $\vec{\beta}_j \rightsquigarrow r_j \in \pi_1(M, *)$: a relator

$\rightsquigarrow \pi_1(M) \cong \langle x_1, \ldots, x_g \mid r_1, \ldots, r_g \rangle$

$(S_g; \alpha, \beta)$: a Heegaard diagram of M, $\alpha := \bigcup_i \alpha_i$, $\beta := \bigcup_j \beta_j$

$D \subset S_g \setminus (\alpha \cup \beta)$: a disk component

D is said to be joining if $\partial \overline{D} \cap \alpha_i \cong \partial \overline{D} \cap \beta_j \cong I$ for $^\forall i, j$

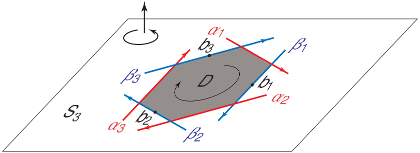


$\rightsquigarrow (S_g; \alpha, \beta; D)$: a punctured Heegaard diagram

From a punctured H-diagram to an oriented, based H-diagram

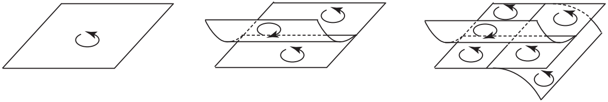
$(S_g; \alpha, \beta; D)$: a punctured Heegaard diagram of M

$$\rightsquigarrow \begin{cases} \text{orientation of each slopes} \\ \text{base points } b_i \text{ on each } \beta_i \end{cases}$$

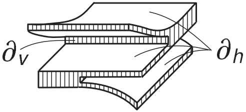


i.e. $(S_g; \alpha, \beta; D)$ determines an oriented, based Heegaard diagram $(S_g; \vec{\alpha}, \vec{\beta}; \{b_k\})$.

$P \subset M$ is an oriented closed branched surface if P is locally modeled one of the following 3 models:



One can regard $N(P)$ as an interval bundle over P.



$\partial N(P) = \partial_h N(P) \cup \partial_v N(P)$

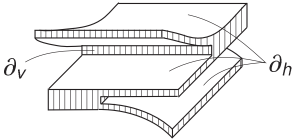
All branched surfaces are assumed to be transversely oriented, i.e. P is equipped with a global orientation on the 1-foliation of $N(P)$.

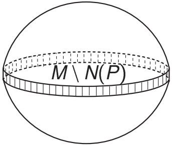
Brached spines of 3-manifolds

A closed branched surface $P \subset M$ is a branched spine if $M \setminus P \cong \mathrm{Int}\, B^3$, and $\partial_v N(P)$ is an annulus.

P: a branched spine of M

$\Longrightarrow$ one can extend the I-fibers of $N(P)$ to the whole of M.





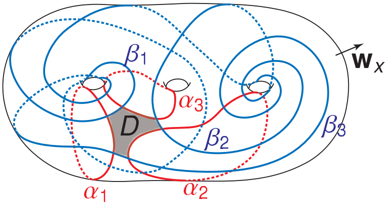
Theorem (Ishii, Benedetti-Petronio)

Every non-singular vector field on a closed 3-manifold M is carried by a branched spine in M.

Punctured H-diagrams and branched spines

Theorem (K)

(i) Every punctured H-diagram of M gives rise to a branched spine of M.
(ii) Every branched spine is constructed from a punctured H-diagram.



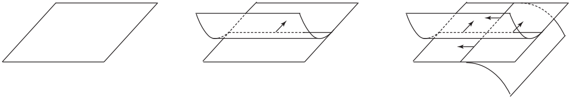
$(S_g; \alpha, \beta; D)$: a punctured H-diagram of M
$\rightsquigarrow$ a branched spine P of M
$\rightsquigarrow$ a non-singular vector field $\mathcal{V}$ on M

Note: Every Spin^c structure is presented by a punctured H-diagram.

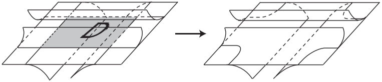
Punctured H-diagrams and branched spines

$(S_g; \alpha, \beta; D)$: a punctured Heegaard diagram

Equip $B := S_g \cup (\bigcup_i D_i^{(1)}) \cup (\bigcup_j D_j^{(2)})$ with a branch structure in such a way that the branch direction points outword from D.



$\rightsquigarrow$ D is removable



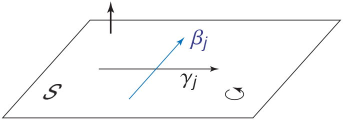
$\rightsquigarrow$ $P := B \setminus D$: a branched spine of M

Dual loops of β_j's

$(S_g; \vec{\alpha}, \vec{\beta}, \{b_j\})$: an oriented, based Heegaard diagram of M

$\gamma_j \subset S_g$: an oriented simple closed curve such that

1. γ_j intersects β_j once and transversely in the direction shown below;
2. $\gamma_j \cap \beta_k = \emptyset$ if $j \neq k$.



$\gamma_j \rightsquigarrow [\gamma_j] \in H_1(M; \mathbb{Z})$

$(S_g; \vec{\alpha}, \vec{\beta}; \{b_j\})$, $\varphi : \mathbb{Z}[H_1(M;\mathbb{Z})] \to F$: a ring homomorphism
Let the twisted chain complex $C_*^{\varphi}(M)$ be acyclic.
$1 \leq^{\exists} l_1, l_2 \leq g$ such that

$$\tau^{\varphi}(S_g; \vec{\alpha}, \vec{\beta}; \{b_k\}) := \frac{\det\left(\varphi\left(\left[\frac{\partial r_j}{\partial x_j}\right]\right)\right)_{l_1, l_2}}{(\varphi([x_{l_1}]) - 1)(\varphi([\gamma_{l_2}]) - 1)} \in F^{\times} / \pm 1,$$

where $\left(\varphi\left(\left[\frac{\partial r_i}{\partial x_j}\right]\right)\right)_{l_1, l_2}$ is the matrix obtained by removing l_1-th row and l_2-th column from the matrix $\left(\varphi\left(\left[\frac{\partial r_i}{\partial x_j}\right]\right)\right)_{1 \leq i,j \leq g}$.

Fox derivation:

$$\begin{cases} \frac{\partial}{\partial x_j}(x_i) = \delta_{i,j} & (\delta_{i,j}\text{: Kronecker delta}) \\ \frac{\partial}{\partial x_n}(uv) = \frac{\partial}{\partial x_n}(u) + u\frac{\partial}{\partial x_n}(v) & \end{cases}$$

$\rightsquigarrow$ the Reidemeister-Turaev torsion of $(S_g; \vec{\alpha}, \vec{\beta}; \{b_k\})$

Formula

$(S_g; \alpha, \beta, D)$: a punctured Heegaard diagram

1. $\rightsquigarrow (S_g; \vec{\alpha}, \vec{\beta}, \{b_j\})$
2. $\rightsquigarrow P$: a branched spine $\rightsquigarrow [\mathcal{V}]$: Spin^c structure

Theorem (K)

Under the above setting, we have

$$\tau^{\varphi}(M, [\mathcal{V}]) = \tau^{\varphi}(S_g; \vec{\alpha}, \vec{\beta}; \{b_j\}) \in F/\pm 1.$$

Key: The following diagram is commutative.

$$\begin{array}{ccc} \mathrm{PH}(M) & \xrightarrow{\Phi_{\mathrm{PO}}} & \mathrm{OBH}(M) \\ \Phi_{\mathrm{PB}} \downarrow & & \downarrow \Phi_{\mathrm{OS}} \\ \mathrm{BrSp}(M) & \xrightarrow[\Phi_{\mathrm{BS}}]{} & \mathrm{Spin}^c(M) \end{array}$$

Seifert fibered 3-manifolds

A Seifert fibered manifold is a 3-manifold which admits a foliation by circles.
A non-singular vector field $\mathcal{V}$ is said to be standard if it is everywhere tangent to a leaf.
$[\mathcal{V}]$ a standard Spin^c structure

Fact (Seifert) Closed orientable Seifert fibered 3-manifolds are parametrized as $(g; b; (p_1, q_1), (p_2, q_2), \ldots, (p_r, q_r))$

g: genus of the base surface , b: obstruction class

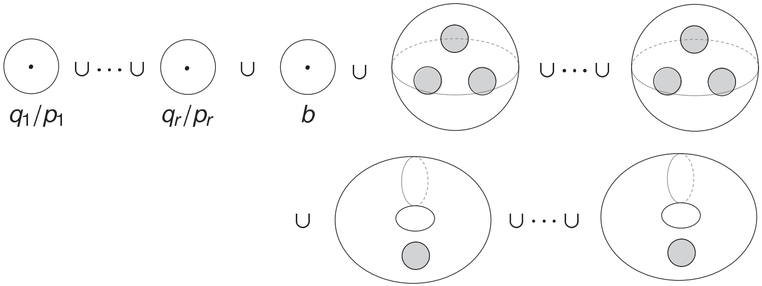
(p_i, q_i): type of the singular fiber such that $p_i > 1$, $\gcd(p_i, q_i) = 1$

M: a closed 3-manifold, $\mathcal{V}$: a non-singular vector field on M
$\mathrm{HG}(\mathcal{V}) :=$ min{genera of punctured H-diagrams carrying $\mathcal{V}$}
$\rightsquigarrow \mathrm{HG}(\mathcal{V}) \geqslant \mathrm{HG}(M)$ (By definition)

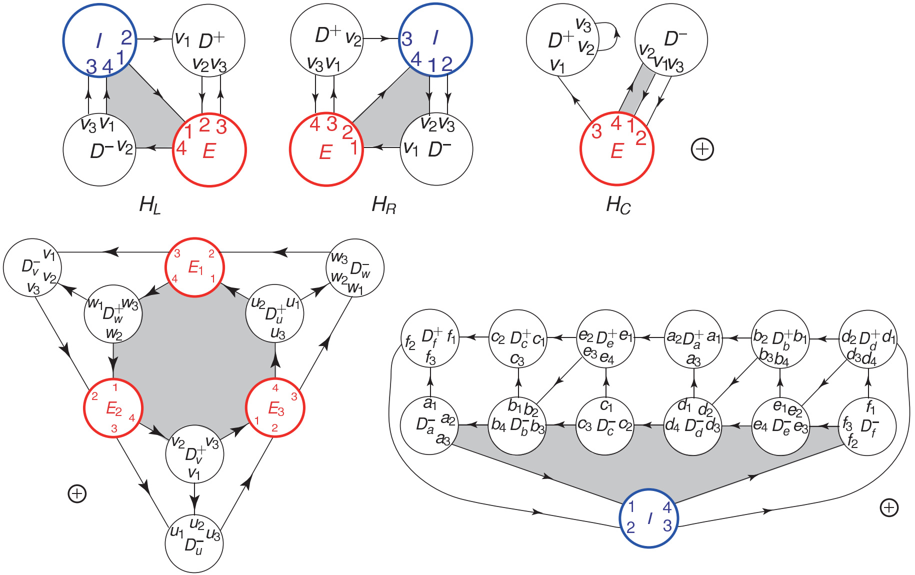
Theorem

M a Seifert fibered 3-manifold, $[\mathcal{V}]$: a standard Spin^c structure on M
$\Longrightarrow \mathrm{HG}(\mathcal{V}) = \mathrm{HG}(M)$

Decomposition of Seifert manifolds



Decomposition of Seifert manifolds



Lemma

The above pieces give rise to a punctured Heegaard diagram of any Seifert fibered 3-manifolds equipped with standard Spin^c structure.

$p, q \in \mathbb{N}$: mutually coprime numbers with $p > q$

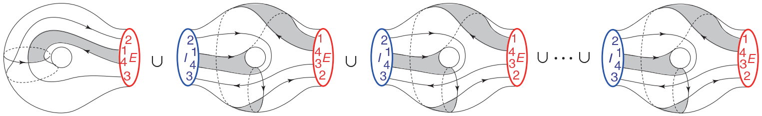
Define a word $w(p, q)$ of letters L and R as:

$$w(p,q) := \begin{cases} L^{a_1} R^{a_2} L^{a_3} \cdots L^{a_{n-2}} R^{a_{n-1}} L^{a_n} & \text{(if } n \text{ is odd)} \\ L^{a_1} R^{a_2} L^{a_3} \cdots R^{a_{n-2}} L^{a_{n-1}} R^{a_n} & \text{(if } n \text{ is even)}, \end{cases}$$

where $q/p = [a_1, a_2, \ldots, a_n, 1] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + 1}}}}$.

$\rightsquigarrow$ the part corresponding to a singular torus of type (p, q) can be constructed by attaching H_L's and H_R's following the word $w(p, q)$:

$H_C \cup (H_L \cup \cdots \cup H_L) \cup (H_R \cup \cdots \cup H_R) \cup \cdots \cup (H_L \cup \cdot \cup H_L)$



H_C $\quad$ H_L $\quad$ H_L

Algorithm

Step 1 Orient α and β, and take base points of β.

Step 2 Get a rigid presentation $\langle x_1, \ldots, x_g \mid r_1, \ldots, r_g \rangle$ of $\pi_1((S(g; b; (p_1, q_1), (p_2, q_2), \ldots, (p_r, q_r)), *)$.

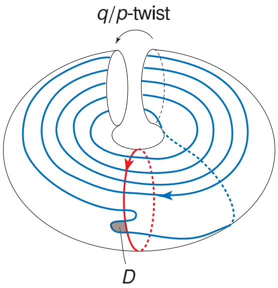
Step 3 Find an arbitrary dual system γ of β and relate a word $y_i(x_1, \ldots, x_g)$ to each loop γ_i in γ.

Step 4 If there exist two integers $k, l \in \{1, \ldots, g\}$ such that all of $\det B_{k,l}$, $\varphi([y_l]) - 1$ and $\varphi([y_l]) - 1$ are non-zero, then we have

$$\begin{aligned} \tau^{\varphi} & \left(S(g; b; (p_1, q_1), (p_2, q_2), \ldots, (p_r, q_r)), \mathcal{V}_{st}\right) \\ = & \pm \frac{\det B_{k,l}}{(\varphi([x_k]) - 1)(\varphi([y_l]) - 1)} \in F^{\times} / \pm 1, \end{aligned}$$

If not, then $C^{\varphi}((S(g; b; (p_1, q_1), (p_2, q_2), \ldots, (p_r, q_r)))$ is not acyclic, hence we have $\tau^{\varphi} = 0$ by definition.

Example (Lens spaces)



$\pi_1(L(p,q)) = \langle x \mid x^p \rangle$
$[\gamma] = x^r$, where $qr = 1 \pmod p$
$\zeta := \varphi([x])$

Example (Lens spaces)

$$\tau^\varphi(L(p,q),[\mathcal{V}_{st}]) = \pm \frac{1}{(\zeta - 1)(\zeta^r - 1)}$$

$L(11,1) = L(11,10)$

- $1/(\zeta - 1)^2$
- $\zeta/(\zeta - 1)^2$
- $\zeta^2/(\zeta - 1)^2$
- $\zeta^3/(\zeta - 1)^2$
- $\zeta^4/(\zeta - 1)^2$
- $\zeta^5/(\zeta - 1)^2$
- $\zeta^6/(\zeta - 1)^2$
- $\zeta^7/(\zeta - 1)^2$
- $\zeta^8/(\zeta - 1)^2$
- $\zeta^9/(\zeta - 1)^2$
- $\zeta^{10}/(\zeta - 1)^2$

Spin^c structures of $L(11,1)$

$L(11,1) = L(11,10)$

- $1/(\zeta - 1)^2$
- $\zeta/(\zeta - 1)^2$
- $\zeta^2/(\zeta - 1)^2 = 1/(\zeta^{10} - 1)^2$
- $\zeta^3/(\zeta - 1)^2$
- $\zeta^4/(\zeta - 1)^2$
- $\zeta^5/(\zeta - 1)^2$
- $\zeta^6/(\zeta - 1)^2$
- $\zeta^7/(\zeta - 1)^2$
- $\zeta^8/(\zeta - 1)^2$
- $\zeta^9/(\zeta - 1)^2$
- $\zeta^{10}/(\zeta - 1)^2$

Spin^c structures of $L(11,2)$

$L(11,2) = L(11,5) = L(11,6) = L(11,9)$

- $1/(\zeta-1)(\zeta^6-1)$
- $\zeta/(\zeta-1)(\zeta^6-1)$
- $\zeta^2/(\zeta-1)(\zeta^6-1)$
- $\zeta^3/(\zeta-1)(\zeta^6-1)$
- $\zeta^4/(\zeta-1)(\zeta^6-1)$
- $\zeta^5/(\zeta-1)(\zeta^6-1)$
- $\zeta^6/(\zeta-1)(\zeta^6-1)$
- $\zeta^7/(\zeta-1)(\zeta^6-1)$
- $\zeta^8/(\zeta-1)(\zeta^6-1)$
- $\zeta^9/(\zeta-1)(\zeta^6-1)$
- $\zeta^{10}/(\zeta-1)(\zeta^6-1)$

Spin^c structures of $L(11,2)$

$L(11,2) = L(11,5) = L(11,6) = L(11,9)$

- $1/(\zeta - 1)(\zeta^6 - 1)$
- $\zeta/(\zeta - 1)(\zeta^6 - 1) = 1/(\zeta^{10} - 1)(\zeta^6 - 1)$
- $\zeta^2/(\zeta - 1)(\zeta^6 - 1)$
- $\zeta^3/(\zeta - 1)(\zeta^6 - 1)$
- $\zeta^4/(\zeta - 1)(\zeta^6 - 1)$
- $\zeta^5/(\zeta - 1)(\zeta^6 - 1)$
- $\zeta^6/(\zeta - 1)(\zeta^6 - 1) = 1/(\zeta - 1)(\zeta^5 - 1)$
- $\zeta^7/(\zeta - 1)(\zeta^6 - 1) = 1/(\zeta^{10} - 1)(\zeta^5 - 1)$
- $\zeta^8/(\zeta - 1)(\zeta^6 - 1)$
- $\zeta^9/(\zeta - 1)(\zeta^6 - 1)$
- $\zeta^{10}/(\zeta - 1)(\zeta^6 - 1)$

Example ($S_g \times S^1$)

S_g: a closed surface of genus g

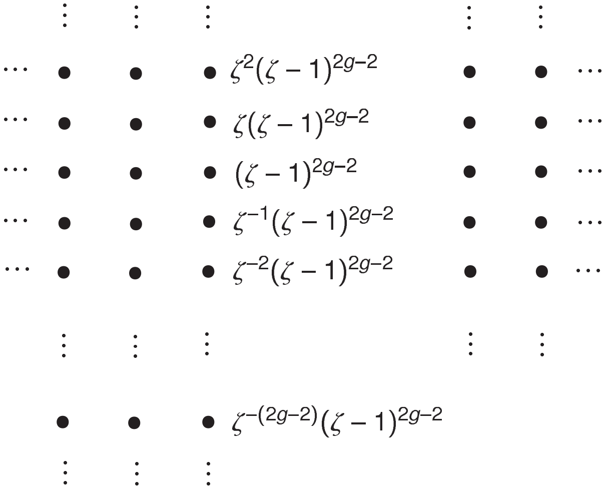
$\pi_1(S_g) = \langle x_1, x_2, \ldots, x_{2g}, y \mid [x_i x_{2g+1}]\ (i = 1, 2, \ldots, 2g),\ \prod_{i=1^g}[x_{2i-1}, x_{2i}] \rangle$

$\zeta_i := \varphi([x_i])$, $i = 1, 2, \ldots, 2g$, $\zeta := \varphi([y])$

Example ($S_g \times S^1$)

$\tau^\varphi(S_g \times S^1, \mathcal{V}_{st}) = \pm(\zeta - 1)^{2g-2}$

Spin^c structures of $S_g \times S^1$



Spin^c structures of $S_g \times S^1$

$$\zeta^2(\zeta - 1)^{2g-2}$$

$$\zeta(\zeta - 1)^{2g-2}$$

$$(\zeta - 1)^{2g-2}$$

$$\zeta^{-1}(\zeta - 1)^{2g-2}$$

$$\zeta^{-2}(\zeta - 1)^{2g-2}$$

$$\vdots$$

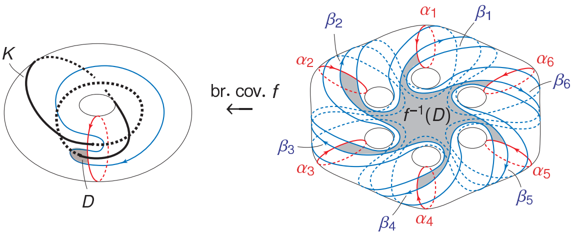
$$\zeta^{-(2g-2)}(\zeta - 1)^{2g-2} = (\zeta^{-1} - 1)^{2g-2}$$

Example (Brieskorn 3-manifolds)

$p, q, r \in \mathbb{N}_{\geqslant 2}$
$\Sigma(p, q, r) := \{(x, y, z) \in \mathbb{C}^3 \mid |x|^2 + |y|^2 + |z|^2 = 1, x^p + y^q + z^r = 0\}.$
: the Brieskorn 3-manifold of type (p, q, r)

$\Sigma(p, q, r)$ is the r-fold branched covering of S^3 branched along a torus knot or link of type (p, q)

$$H_1(\Sigma(p, q, r); \mathbb{Z}) = \begin{cases} 1 & n = \pm 1 \pmod 6 \\ \mathbb{Z}/3\mathbb{Z} & n = \pm 2 \pmod 6 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = \pm 3 \pmod 6 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 0 \pmod 6 \end{cases}.$$



$\pi_1(\Sigma(2,3,6n)) = \langle x_1, \ldots, x_{6n} \mid x_i x_{i+6n-1}^{-1} x_{i+1}^{-1},\ 1 \leqslant i \leqslant 6n \rangle$,
$H_1(M;\mathbb{Z}) = \mathbb{Z}\langle [x_1] \rangle \oplus \mathbb{Z}\langle [x_2] \rangle$
$\zeta_1 = \varphi([x_1])$, $\zeta_2 = \varphi([x_2])$

Example ($\Sigma(2,3,6n)$)

$$\tau^{\varphi}(\Sigma(2,3,6n), [\mathcal{V}_{st}]) = \pm \frac{\det\left(\varphi\left(\left[\frac{\partial x_i x_{i+6i-1}^{-1} x_{i+1}^{-1}}{\partial x_j}\right]\right)\right)_{1,1}}{(\zeta_1^{-1} - 1)(\zeta_1 - 1)} = \pm n$$

Spin^c structures of $\Sigma(2,3,6n)$

$$
\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & \zeta_1^{-2}\zeta_2^{2}\cdot n & \zeta_1^{-1}\zeta_2^{2}\cdot n & \zeta_2^{2}\cdot n & \zeta_1\zeta_2^{2}\cdot n & \zeta_1^{2}\zeta_2^{2}\cdot n & \cdots \\
\cdots & \zeta_1^{-2}\zeta_2\cdot n & \zeta_1^{-1}\zeta_2\cdot n & \zeta_2\cdot n & \zeta_1\zeta_2\cdot n & \zeta_1^{2}\zeta_2\cdot n & \cdots \\
\cdots & \zeta_1^{-2}\cdot n & \zeta_1^{-1}\cdot n & \color{red}{n} & \zeta_1\cdot n & \zeta_1^{2}\cdot n & \cdots \\
\cdots & \zeta_1^{-2}\zeta_2^{-1}\cdot n & \zeta_1^{-1}\zeta_2^{-1}\cdot n & \zeta_2^{-1}\cdot n & \zeta_1\zeta_2^{-1}\cdot n & \zeta_1^{2}\zeta_2^{-1}\cdot n & \cdots \\
\cdots & \zeta_1^{-2}\zeta_2^{-2}\cdot n & \zeta_1^{-1}\zeta_2^{-2}\cdot n & \zeta_2^{-2}\cdot n & \zeta_1\zeta_2^{-2}\cdot n & \zeta_1^{2}\zeta_2^{-2}\cdot n & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$