On the leading coefficients of higher-order Alexander polynomials

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0. Introduction

M : compact orientable 3-manifold w/ empty or troidal ∂
$\psi : \pi_1 M \twoheadrightarrow \langle t \rangle$

$\Delta_{M,\psi}(t) = \text{ord}\, H_1(M; \mathbb{Z}[H_1(M)/\text{ torsion}])$
$\in \mathbb{Z}[H_1(M)/\text{torsion}]/ \pm H_1(M)/\text{torsion}.$

[Cochran '04, Harvey '05]
$\Delta^{(n)}_{M,\psi}(t)$ "$= \text{ord}\, H_1(M; \mathbb{Z}[\pi_1 M/(\pi_1 M)^{(n+1)}])$"
: Higher-order Alexander polynomial of order n
: big indeterminacy

$\deg \Delta^{(n)}_{M,\psi} \in \mathbb{Z}$: Cochran-Harvey invariant of order n

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Aim of the talk

Today we consider non-commutative Reidemeister torsion $\tau_{\rho_n}(M)$ "associated to $\pi_1 M \twoheadrightarrow \pi_1 M/(\pi_1 M)^{(n+1)}$" and introduce "the leading coefficient" $c_n(\psi)$.

[Friedl $'07$]

$$\Delta^{(n)}_{M,\psi} \sim \tau_{\rho_n}(M).$$

$\tau_{\rho_n}(M)$ has <u>smaller indeterminacy</u> than $\Delta^{(n)}_{M,\psi}$.

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· Computations for metabelian cases

· Difference between $c_n(\psi)$ and $c_{n+1}(\psi)$

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Outline

1. Non-commutative Reidemeister torsion
2. The leading coefficient and fiberedness
3. Metabelian examples
4. Monotonicity and realization

1. Non-commutative Reidemeister torsion

$\mathbb{F}$: skew field
$\det\colon GL(n,\mathbb{F}) \to \mathbb{F}^\times_{ab}$ $(:= \mathbb{F}^\times/[\mathbb{F}^\times,\mathbb{F}^\times])$

$\rho\colon \mathbb{Z}[\pi_1 M] \to \mathbb{F}$: homomorphism s. t.
$H^\rho_*(M;\mathbb{F})$ $(:= H_*(C_*(\widetilde{M}) \otimes_{\mathbb{Z}[\pi_1 M]} \mathbb{F})) = 0$
$\rightsquigarrow$ $\tau_\rho(M) \in \mathbb{F}^\times_{ab}/\pm\rho(\pi_1 M)$: Reidemeister torsion

Lemma. (Turaev)

$C_i(\widetilde{M}) \otimes_{\mathbb{Z}[\pi_1 M]} \mathbb{F} = C'_i \oplus C''_i$ s. t.
(i) C'_i, C''_i are spanned by lifts of cells, and
(ii) $pr_{C''_{i-1}} \circ \partial_i\colon C'_i \to C''_{i-1}$ is an isomorphism.

$$\Rightarrow \tau_\rho(X) = \prod_i (\det pr_{C''_{i-1}} \circ \partial_i)^{(-1)^i}.$$

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Rational derived series

π : group

$\rightsquigarrow \pi_r^{(0)} := \pi,$

$\pi_r^{(n+1)} := \{\gamma \in \pi_r^{(n)} \ ; \ \exists k \in \mathbb{Z} \setminus 0 \text{ s. t. } \gamma^k \in [\pi_r^{(n)}, \pi_r^{(n)}]\}.$

: rational derived series

$\pi_r^{(n)}/\pi_r^{(n+1)}$: torsion free abelian

$\rightsquigarrow \pi/\pi_r^{(n+1)}$: poly-torsion-free-abelian

$$1 \triangleleft \pi_r^{(n)}/\pi_r^{(n+1)} \triangleleft \cdots \triangleleft \pi_r^{(1)}/\pi_r^{(n+1)} \triangleleft \pi/\pi_r^{(n+1)}.$$

Theorem. (Passman)

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$\Rightarrow$ the quotient skew field $\mathbb{Q}(\Gamma) := \mathbb{Z}[\Gamma](\mathbb{Z}[\Gamma] \setminus 0)^{-1}$ is defined.

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The degree

$\psi\colon \pi_1 M \twoheadrightarrow \langle t \rangle$

$\Gamma_n := \pi_1 M / (\pi_1 M)_r^{(n+1)}$
$\Gamma'_n := \operatorname{Ker}\psi / (\pi_1 M)_r^{(n+1)}$

$\Gamma_n = \Gamma'_n \rtimes_\theta \langle t \rangle, \quad \theta \in \operatorname{Aut}(\Gamma'_n)$
$\rightsquigarrow \ \mathbb{Q}(\Gamma_n) = \mathbb{Q}(\Gamma'_n)(t) \quad (t \cdot x = \theta(x) \cdot t)$

If $H_*^{\rho_n}(M; \mathbb{Q}(\Gamma'_n)(t)) = 0$, then $\tau_{\rho_n}(M) \in \mathbb{Q}(\Gamma'_n)(t)^\times_{ab} / \pm \Gamma'_n \cdot \langle t \rangle$ and the degree are defined.

$$\delta_n(\psi) := \deg \tau_{\rho_n}(M) \ \in \mathbb{Z}$$
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2. The leading coefficient and fiberedness

For a fibered knot K, Δ_K is a monic polynomial with degree $2g(K)$.

Theorem. (Cochran, Friedl, Harvey)

If M is fibered, $M \neq S^1 \times S^2, S^1 \times D^2$, and $\psi\colon \pi_1 M \twoheadrightarrow \langle t \rangle$ ($\in H^1(M;\mathbb{Z})$) is induced by the fibration, then for all n,

$$\delta_n(\psi) = ||\psi||_T.$$

How about a generalization for monicness?
What is the leading coefficient of $\tau_{\rho_n}(M) \in \mathbb{Q}(\Gamma_n')(t)^{\times}_{ab} / \pm \Gamma_n' \cdot \langle t \rangle$?

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How about a generalization for monicness?
What is the leading coefficient of $\tau_{\rho_n}(M) \in \mathbb{Q}(\Gamma'_n)(t)^{\times}_{ab} / \pm \Gamma'_n \cdot \langle t \rangle$?

Key homomorphism

$$c\colon \mathbb{Q}(\Gamma_n')(t)^\times_{ab}/\pm\Gamma_n'\cdot\langle t\rangle \longrightarrow \mathbb{Q}(\Gamma_n')^\times_{ab}/\pm\Gamma_n'\cdot\langle p^{-1}\theta(p)\rangle_{p\in\mathbb{Z}[\Gamma_n']\setminus 0}$$
$$: (a_l t^l + a_{l-1}t^{l-1}+\dots)(b_m t^m + b_{m-1}t^{m-1}+\dots)^{-1} \mapsto a_l b_m^{-1}$$

Lemma.

The map c is a well-defined homomorphism.

Definition.

If $H_*^{\rho_n}(M;\mathbb{Q}(\Gamma_n')(t))=0$, then we set

$$c_n(\psi) := c(\tau_{\rho_n}(M)) \in \mathbb{Q}(\Gamma_n')^\times_{ab}/\pm\Gamma_n'\cdot\langle p^{-1}\theta(p)\rangle_{p\in\mathbb{Z}[\Gamma_n']\setminus 0}.$$

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Theorem.

If M is fibered and $\psi\colon \pi_1 M \twoheadrightarrow \langle t\rangle$ ($\in H^1(M;\mathbb{Z})$) is induced by the fibration, then for all n,

$$c_n(\psi) = 1.$$

Problem.

If $c_n(\psi) = 1$ for all n and $\delta_0(\psi) = \|\psi\|_T$, then is M fibered?
For what class of 3-manifolds is this true?

cf. [Friedl-Vidussi '08]
Twisted Alexander polynomials associated to representations onto finite groups detect fiberedness of M.

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3. Metabelian examples

$K \subset S^3$: tame knot with monic Δ_K
$M = E$: exterior
$\psi : \pi_1 E \twoheadrightarrow \langle t \rangle$: abelianization

$\Gamma_1 = \pi_1 E / (\pi_1 E)^{(2)}$
$\Gamma_1' = (\pi_1 E)^{(1)} / (\pi_1 E)^{(2)} \cong \mathbb{Z}^d, \quad d := \deg \Delta_K$

$\mathbb{Z}[\Gamma_1'] = \mathbb{Z}[s_1^{\pm 1}, \ldots, s_d^{\pm 1}]$: UFD

$$\mathbb{Q}(\Gamma_1')^\times / \pm\Gamma_1' \cdot \langle p^{-1}\theta(p)\rangle_{p \in \mathbb{Z}[\Gamma_1'] \setminus 0} = \langle p \rangle_{p \in \mathbb{Z}[s_1^{\pm 1}, \ldots, s_d^{\pm 1}] : \text{prime}} / \pm\langle s_1, \ldots s_d \rangle \cdot \langle \tfrac{\theta(p)}{p} \rangle$$
$$\cong \mathbb{Z}^\infty \text{ (if } d > 0)$$

Using prime decomposition, we can determine an element in $\mathbb{Q}(\Gamma_1')^\times / \pm \Gamma_1' \cdot \langle p^{-1}\theta(p)\rangle_{p \in \mathbb{Z}[\Gamma_1'] \setminus 0}$ is 1 or not.

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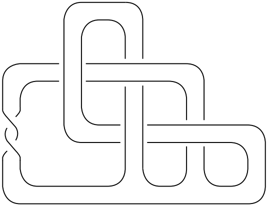
Computations

EX. 1 K = the following, g(K) = 1, $\Delta_K = t^2 - t - 1$



K has the same Alexander module as that of 3_1.

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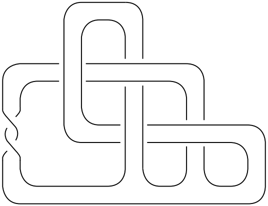

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$$\tau_{\rho_1}(E) = \det \begin{pmatrix} \rho_1(\overline{\frac{\partial r_1}{\partial a}}) & \rho_1(\overline{\frac{\partial r_2}{\partial a}}) \\ \rho_1(\overline{\frac{\partial r_1}{\partial b}}) & \rho_1(\overline{\frac{\partial r_2}{\partial b}}) \end{pmatrix} (t-1)^{-1}.$$

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$$c_1(\psi) = a^2 - ab + b^2 \neq 1, \quad \delta_1(\psi) = 2g(K) - 1 = 1.$$

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$$c_1(\psi) = a^2 - ab + b^2 \neq 1, \quad \delta_1(\psi) = 2g(K) - 1 = 1.$$

EX. 2 $\quad K = 12_{n57}$, g(K) = 2, $\quad \Delta_K = t^4 - 2t^3 + 3t^2 - 2t + 1$

$\pi_1 E = \langle x, a, b \mid r_1, r_2 \rangle$,
$r_1 =$
$b^{-1}axa^{-1}bx^2ba^{-1}xab^{-1}x^{-2}b^{-1}ax^{-1}a^{-1}bx^2bx^{-1}a^{-1}x^{-1}bx^2ba^{-1}x^{-1}ab^{-1}x^{-2}$,
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$\Gamma'_1 = \langle a_n, b_n \mid a_n a_{n+1}^{-3} a_{n+2} b_{n+1}^2, a_{n+1}^{-2} b_n b_{n+1} b_{n+2} \rangle_{ab} = \langle a_0, a_1, b_0, b_1 \rangle_{ab}$,

$\rho_1(\overline{\frac{\partial r_1}{\partial a}}) = a_0^2 a_1^{-1} b_0^{-2} b_1 t^4 - (a_0^2 a_1^{-1} b_0^{-2} b_1 + a_0 a_1^{-1} b_0^{-1} b_1) t^3 + (a_0^2 a_1^{-4} b_0^{-1} b_1^3 - a_1^{-1} + a_0 a_1^{-1} b_0^{-1} b_1) t + b_0^{-1}$
$\rho_1(\overline{\frac{\partial r_1}{\partial b}}) = -a_0^2 a_1^{-2} b_0^{-2} b_1^2 t^4 + (a_0 a_1^{-1} b_0^{-1} b_1 + a_0 a_1^{-2} b_0^{-1} b_1^2 + a_1^{-1} b_1) t^3 - (a_0^2 a_1^{-4} b_0^{-1} b_1^3 + 1) t^2 + (a_0 a_1^{-1} b_0^{-1} + a_0 a_1^{-2} b_0^{-1} b_1 + a_1^{-1}) t - b_0^{-1}$,
$\rho_1(\overline{\frac{\partial r_2}{\partial a}}) = a_0 a_1^{-1} b_0^{-1} t^2 - a_0 a_1^{-1} b_0^{-1} t + b_0^{-1}$,
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$c_1(\psi) = a_0 b_1 + a_1 b_0 - b_0 b_1 \neq 1, \quad \delta_1(\psi) = 2g(K) - 1 = 3.$

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$\rho_1(\overline{\frac{\partial r_2}{\partial a}}) = a_0 a_1^{-1} b_0^{-1} t^2 - a_0 a_1^{-1} b_0^{-1} t + b_0^{-1},$
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$c_1(\psi) = a_0 b_1 + a_1 b_0 - b_0 b_1 \neq 1, \quad \delta_1(\psi) = 2g(K) - 1 = 3.$

EX. 2 $K = 12_{n57}$, g(K) = 2, $\Delta_K = t^4 - 2t^3 + 3t^2 - 2t + 1$

$\pi_1 E = \langle x, a, b \mid r_1, r_2 \rangle$,
$r_1 =$
$b^{-1}axa^{-1}bx^2ba^{-1}xab^{-1}x^{-2}b^{-1}ax^{-1}a^{-1}bx^2bx^{-1}a^{-1}x^{-1}bx^2ba^{-1}x^{-1}ab^{-1}x^{-2}$,
$r_2 = b^{-1}axa^{-1}xab^{-1}x^{-1}bx^{-1}$.

$\Gamma_1' = \langle a_n, b_n \mid a_n a_{n+1}^{-3} a_{n+2} b_{n+1}^2, a_{n+1}^{-2} b_n b_{n+1} b_{n+2} \rangle_{ab} = \langle a_0, a_1, b_0, b_1 \rangle_{ab}$,

$\rho_1(\overline{\frac{\partial r_1}{\partial a}}) = a_0^2 a_1^{-1} b_0^{-2} b_1 t^4 - (a_0^2 a_1^{-1} b_0^{-2} b_1 + a_0 a_1^{-1} b_0^{-1} b_1) t^3 + (a_0^2 a_1^{-4} b_0^{-1} b_1^3 - a_1^{-1} + a_0 a_1^{-1} b_0^{-1} b_1) t + b_0^{-1}$
$\rho_1(\overline{\frac{\partial r_1}{\partial b}}) = -a_0^2 a_1^{-2} b_0^{-2} b_1^2 t^4 + (a_0 a_1^{-1} b_0^{-1} b_1 + a_0 a_1^{-2} b_0^{-1} b_1^2 + a_1^{-1} b_1) t^3 - (a_0^2 a_1^{-4} b_0^{-1} b_1^3 + 1) t^2 + (a_0 a_1^{-1} b_0^{-1} + a_0 a_1^{-2} b_0^{-1} b_1 + a_1^{-1}) t - b_0^{-1}$,
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(Mutants of $K = 12_{n57}$)

<u>EX. 3</u> $K = 12_{n56}$, $g(K) = 3$, $\Delta_K = t^4 - 2t^3 + 3t^2 - 2t + 1$

$\Gamma'_1 = \langle a_0, a_1, b_0, b_1 \rangle_{ab}$

$c_1(\psi) = a_0 - 1 \neq 1$, $\delta_1(\psi) = 2g(K) - 1 = 5.$

<u>EX. 4</u> $K = 12_{n221}$, $g(K) = 3$, $\Delta_K = t^4 - 2t^3 + 3t^2 - 2t + 1$

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τ_{ρ_1} detects the difference of some mutants.
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4. Monotonicity and realization

For any $f(t) \in \mathbb{Z}[t, t^{-1}]/\langle \pm t \rangle$ w/ $f(t^{-1}) = f(t)$ and $f(1) = 1$, there exists a knot K s. t. $\Delta_K(t) = f(t)$.

Theorem. (Cochran, Friedl-Kim, Harvey)

If $H_*^{\rho_n}(M; \mathbb{Q}(\Gamma_n')(t)) = 0$, then

$$\delta_n(\psi) \leq \delta_{n+1}(\psi) \leq \cdots \leq \|\psi\|_T.$$

and all the differences are even.

Theorem. (Cochran, Friedl-Kim)

For any never decreasing sequence $(m_i)_{i \in \mathbb{Z}_{\geq 0}}$ of positive odd integers which is bounded, there exists a knot K s. t. $\delta_i(\psi) = m_i$ for all i.

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Monotonicity

Problem.

If $c_{n+1}(\psi) = 1$, then $c_n(\psi) = 1$?

There is a natural surjection $\mathbb{Z}[\Gamma'_{n+1}] \twoheadrightarrow \mathbb{Z}[\Gamma'_n]$, but there is no extension $\mathbb{Q}(\Gamma'_{n+1}) \to \mathbb{Q}(\Gamma'_n)$ in general.

If $b_1(M) = 1$, then $\Gamma'_1 = (\pi_1 M)_r^{(1)}/(\pi_1 M)_r^{(2)}$ is abelian.
$R := \mathbb{Z}[\Gamma'_1](\mathbb{Z}[\Gamma'_1] \setminus \operatorname{Ker} \epsilon)^{-1}, \quad \epsilon \colon \mathbb{Z}[\Gamma'_1] \twoheadrightarrow \mathbb{Z}[\Gamma'_0] = \mathbb{Z}$

Proposition.

If $b_1(M) = 1$, then $c_1(\psi) \in R^\times / \pm \Gamma'_1 \cdot \langle \frac{\theta(p)}{p} \rangle_{p \in R^\times}$.

Corollary.

If $b_1(M) = 1$ and $c_1(\psi) = 1$, then $c_0(\psi) = 1$ and $\delta_1(\psi) = \delta_0(\psi)$.

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Theorem.

For any n, there are infinitely many knots s. t. $c_i(\psi) = 1$ for $i \leq n$ and $c_{n+1}(\psi) \neq 1$.

$K, J \subset S^3$: tame knots with nontrivial Δ_K, Δ_J
$\eta \subset S^3$: unknot representing an element of $(\pi_1 E_K)^{(n)} \setminus (\pi_1 E_K)^{(n+1)}$
$\rightsquigarrow$ K_0 : result of the infection of K by J along η, i.e.,
$E_{K_0} := E_{K \sqcup \eta} \cup_{\mu_\eta = \lambda_J, \lambda_\eta = \mu_J} E_J$.

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(i) $\tau_{\rho_i}(E_{K_0}) = \Delta_J(\eta)\tau_{\rho_i}(E_K)$ for $i \leq n$
(ii) $\Delta_J(\eta) \neq 1 \in \mathbb{Q}(\Gamma'_n)^\times_{ab} / \pm \Gamma'_n \cdot \langle p^{-1}\theta(p) \rangle_{p \in \mathbb{Z}[\Gamma'_n] \setminus 0}$.

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