Cobordism on homology cylinders and combinatorial torsions

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Twisted Alexander invariants and topology of low-dimensional manifolds

September 16, 2010

Homology cylinder: definitions

- For $g, k \geq 0$, let $\Sigma_{g,k}$:= oriented compact surface of genus g with k boundary components.
- A homology cylinder (M, i_+, i_-) over $\Sigma_{g,k}$ is a triple (M, i_+, i_-) where M is a compact oriented 3-manifold and $i_+, i_- : \Sigma_{g,k} \to \partial M$ are embeddings such that
  (1) i_+ is orientation preserving and i_- is orientation reversing,
  (2) $\partial M = i_+(\Sigma_{g,k}) \cup i_-(\Sigma_{g,k})$ and $i_+(\Sigma_{g,k}) \cap i_-(\Sigma_{g,k}) = i_+(\partial\Sigma_{g,k}) = i_-(\partial\Sigma_{g,k})$,
  (3) $i_+|_{\partial\Sigma_{g,k}} = i_-|_{\partial\Sigma_{g,k}}$,
  (4) $i_+, i_- : H_*(\Sigma_{g,k}; \mathbb{Z}) \to H_*(M; \mathbb{Z})$ are isomorphisms.

Homology cylinder: examples

- $(\Sigma_{g,k} \times [0,1],\ i_+ = \mathrm{id} \times 0,\ i_- = \mathrm{id} \times 1)$
  where collars of $i_+(\Sigma_{g,k})$ and $i_-(\Sigma_{g,k})$ are stretched half-way along $(\partial\Sigma_{g,k}) \times [0,1)$.
- $\mathcal{M}_{g,k}$:= the mapping class group.
  For a diffeomorphism $\varphi \in \mathcal{M}_{g,k}$,

$$M(\varphi) = (\Sigma_{g,k} \times [0,1],\ i_+ = \mathrm{id} \times 0,\ i_- = \varphi \times 1).$$

Homology cylinder: examples

- Let K be a knot of genus g such that
  (1) $\Delta_K(t)$ is monic,
  (2) $\deg(\Delta_K(t)) = 2g$.

  Then for a minimal genus Seifert surface $\Sigma \subset S^3 - N(K)$, $S^3 - N(\Sigma \times [0,1])$ is a homology cylinder over Σ.
- For a knot K, $\partial(N(K)) = A_1 \cup A_2$ where A_i are annuli and $A_1 \cap A_2 = \mu_1 \cup \mu_2$. Then $S^3 - N(K)$ is a homology cylinder over an annulus.

Homology cylinder: monoid structure

- $\mathcal{C}_{g,k}$:= the monoid of all isomorphism classes of homology cylinders over $\Sigma_{g,k}$.
- The product operation on $\mathcal{C}_{g,k}$:

$$(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-).$$

Homology cobordism of homology cylinders

- (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma_{g,k}$ are homology cobordant if there exists a compact oriented smooth 4-manifold W such that
  (1) $\partial W = M \cup (-N) \,/\, (i_+(x) = j_+(x), i_-(x) = j_-(x), x \in \Sigma_{g,k})$,
  (2) $H_*(M;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ and $H_*(N;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ are isomorphisms.
- $\mathcal{H}_{g,k}$:= the group of homology cobordism classes of elements in $\mathcal{C}_{g,k}$ with $(M, i_+, i_-)^{-1} = (-M, i_-, i_+)$.

Theorem (Garoufalidis-Levine) $\mathcal{M}_{g,k}$ embeds into $\mathcal{C}_{g,k}$ and $\mathcal{H}_{g,k}$ via $\varphi \mapsto M(\varphi)$.

Theorem (Powell) If $g \geq 3$, then the abelianization of $\mathcal{M}_{g,k}$ is trivial.

Theorem (Goda-Sakasai, '2009) For $g \geq 1$, $\mathcal{C}_{g,1}$ surjects to $\mathbb{Z}^\infty$.

Questions (Garoufalidis-Levine, Goda-Sakasai) Is $\mathcal{H}_{g,k}$ infinitely generated? Is the abelianization of $\mathcal{H}_{g,k}$ infinitely generated?

- $\Sigma_{0,0} = S^2$, $\Sigma_{0,1} = D^2$, $\Sigma_{0.2} =$ annulus
- $\mathcal{H}_{0,0}^{\text{smooth}} \cong \mathcal{H}_{0,1}^{\text{smooth}} \cong \Theta_3^{\text{smooth}}$ and it has infinite rank.
- $\mathcal{H}_{0,0}^{\text{top}} \cong \mathcal{H}_{0,1}^{\text{top}} \cong \Theta_3^{\text{top}} = 0$
- $\mathcal{H}_{0,2} \cong \mathbb{Z} \oplus$ (the concordance group of framed knots in $\mathbb{Z}$HS) $\twoheadrightarrow$ $(\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty \oplus \mathbb{Z}^\infty$

Theorem (Cha-Friedl-K.) Let $g, n \geq 0$. Then the kernel of the epimorphism $\mathcal{H}_{g,n}^{\text{smooth}} \to \mathcal{H}_{g,n}^{\text{top}}$ contains an abelian group of infinite rank. If $g = 0$, then there exists in fact a homomorphism $\mathcal{F}\colon \mathcal{H}_{0,n}^{\text{smooth}} \to A$ onto an abelian group of infinite rank such that the restriction of $\mathcal{F}$ to the kernel of the projection map $\mathcal{H}_{0,n}^{\text{smooth}} \to \mathcal{H}_{0,n}^{\text{top}}$ is also surjective.

Main Theorem (Cha-Friedl-K.)

(1) If $b_1(\Sigma_{g,k}) > 0$, then there exists an epimorphism

$$\mathcal{H}_{g,k} \to (\mathbb{Z}/2)^\infty$$

which splits. In particular, the abelianization of $\mathcal{H}_{g,k}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$.

(2) If $k > 1$, then there exists an epimorphism

$$\mathcal{H}_{g,k} \to \mathbb{Z}^\infty.$$

Furthermore, the abelianization of $\mathcal{H}_{g,k}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty \bigoplus \mathbb{Z}^\infty$.

The Torelli group of $\mathcal{H}_{g,k}$

- The Torelli group $\mathcal{I}_{g,k}$ of $\mathcal{M}_{g,k}$ is

$$\mathcal{I}_{g,k} := \{g \in \mathcal{M}_{g,k} \mid g \text{ acts trivially on } H_1(\Sigma_{g,k})\}.$$

- For a homology cylinder (M, i_+, i_-) over Σ, we can define the automorphism

$$\varphi(M) := (i_+)_*^{-1}(i_-)_* \colon H_1(\Sigma) \xrightarrow[(i_-)_*]{\cong} H_1(M) \xrightarrow[(i_+)_*^{-1}]{\cong} H_1(\Sigma).$$

- The Torelli group of $\mathcal{H}_{g,k}$ is

$$\mathcal{IH}_{g,k} := \{(M, i_+, i_-) \in \mathcal{H}_{g,k} \mid \varphi(M) = \mathrm{id}\}.$$

Theorem (Morita) The abelianization of $\mathcal{IH}_{g,1}$ has infinite rank.

Theorem (Cha-Friedl-K.)

(1) If $b_1(\Sigma_{g,k}) > 0$, then there exists an epimorphism

$$\mathcal{IH}_{g,k} \to (\mathbb{Z}/2)^\infty$$

which splits. In particular, the abelianization of $\mathcal{IH}_{g,k}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$.

(2) If $g > 1$ or $k > 1$, then there exists an epimorphism

$$\mathcal{IH}_{g,k} \to \mathbb{Z}^\infty.$$

Furthermore, the abelianization of $\mathcal{IH}_{g,k}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty \bigoplus \mathbb{Z}^\infty$.

Strategy for proving Main Theorem

(1) Let $H := H_1(\Sigma_{g,k})$ and $Q(H)$ the quotient field of $\mathbb{Z}[H]$. Construct a homomorphism

$$\mathcal{H}_{g,k} \to Q(H)^\times / \sim$$

using the Reidemeister torsion.

(2) Show that $\mathcal{H}_{g,k}$ has (the desired) nontrivial image in the abelian group $Q(H)^\times / \sim$ under the homomorphism.

Reidemeister torsion

- Considering $\pi_1(M) \to H_1(M) \overset{\cong}{\longleftarrow} H_1(\Sigma_+) \overset{i_+}{\longleftarrow} H$, the torsion of (M, i_+, i_-) is defined by

$$\tau(M) := \tau(M, \Sigma_+; Q(H)) := \tau(C_*(\widetilde{M}, \widetilde{i_+(\Sigma_{g,k})}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} Q(H)).$$

- For any homology cylinder M, we have $\tau(M) = \text{ord}\, H_1(M, \Sigma_+; \mathbb{Z}[H])$.

Theorem For $(M, i_+, i_-), (N, j_+, j_-) \in \mathcal{C}_{g,k}$,

$$\tau(M \cdot N) = \tau(M) \cdot \varphi(M)(\tau(N)).$$

- In fact, there is a homomorphism

$$\begin{array}{rcl} \mathcal{H}_{g,k} & \to & \mathrm{Aut}^*(H) \ltimes (Q(H)^\times / N(H)) \\ M & \mapsto & (\varphi(M), \tau(M)). \end{array}$$

Theorem For $(M, i_+, i_-), (N, j_+, j_-) \in \mathcal{C}_{g,k}$, if (M, i_+, i_-) and (N, j_+, j_-) are homology cobordant, then

$$\tau(M) = \tau(N) \cdot q \cdot \bar{q}$$

for some $q \in Q(H)^\times$.

- $A(H) := \{\pm h \cdot p^{-1} \cdot \varphi(p) \mid h \in H,\ p \in Q(H)^{\times},\ \text{and } \varphi \in \mathrm{Aut}^*(H)\}$ where $\mathrm{Aut}^*(H)$ is the set of $\varphi \in \mathrm{Aut}(H)$ which preserves the intersection form of Σ and $\varphi|_{H_1(\partial\Sigma)} = \mathrm{id}$.
- $N(H) := \{\pm h \cdot q \cdot \bar{q} \mid q \in Q(H)^{\times},\ h \in H\}$.
- Denote $A(H)N(H)$ by AN.

Theorem The torsion invariant gives rise to a group homomorphism

$$\tau \colon \mathcal{H}_{g,k} \to Q(H)^{\times}/AN.$$

Structure of $Q(H)^\times/AN$

- $Q(H)^{sym} = \{p \in Q(H)^\times \mid p = \bar{p} \text{ in } Q(H)^\times/A\}$.
- There is an exact sequence:

$$1 \to \frac{Q(H)^{sym}}{AN} \to \frac{Q(H)^\times}{AN} \to \frac{Q(H)^\times}{Q(H)^{sym}} \to 1$$

Theorem

$$\begin{aligned} (1) Q(H)^{sym}/AN &\xrightarrow{\cong} \bigoplus_{[\lambda]} \mathbb{Z}/2, \\ (2) Q(H)^\times/Q(H)^{sym} &\xrightarrow{\cong} \bigoplus_{\{[\mu],[\bar{\mu}]\}} \mathbb{Z}. \end{aligned}$$

We revisit:

Main Theorem (Cha-Friedl-K.)

(1) If $b_1(\Sigma_{g,k}) > 0$, then there exists an epimorphism

$$\mathcal{H}_{g,k} \to (\mathbb{Z}/2)^\infty$$

which splits. In particular, the abelianization of $\mathcal{H}_{g,k}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$.

(2) If $k > 1$, then there exists an epimorphism

$$\mathcal{H}_{g,k} \to \mathbb{Z}^\infty.$$

Furthermore, the abelianization of $\mathcal{H}_{g,k}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty \bigoplus \mathbb{Z}^\infty$.

Symmetry of τ

- $\Delta_K(t)$ is of order 2 in $\mathbb{Z}[t^{\pm 1}]^\times / \{f(t)\overline{f(t)} \mid f(t) \in \mathbb{Z}[t^{\pm 1}]\}$.
- If Σ has at most one boundary component, then

$$\tau(M) := \tau(M, \Sigma_+) = \overline{\tau(M, \Sigma_+)} = \overline{\tau(M)} \in Q(H)^\times / AN$$

  since $h \mapsto h^{-1}$ $(h \in H)$ is in $\mathrm{Aut}^*(H)$.
  $\Rightarrow \tau(M)$ is of order 2 in $Q(H)^\times / AN$.
- If Σ has more than one boundary component, then in general

$$\tau(M) := \tau(M, \Sigma_+) = \overline{\tau(M, \Sigma_-)} \neq \overline{\tau(M, \Sigma_+)} = \overline{\tau(M)} \in Q(H)^\times / AN.$$

  $\Rightarrow \tau(M)$ may not be of order 2.

Constructing elements of order 2

- $U :=$ unknot. $E(K) := S^3 - N(K)$.
- Choose an embedding $f : S^1 \times D^2 \cong E(U) \to M := \Sigma \times [0,1]$ representing $h \neq 0 \in H$. For a knot K, define

$$M(K) := \big(M - f(\operatorname{int} E(U))\big) \underset{f(\partial E(U)) = \partial E(K)}{\cup} E(K).$$

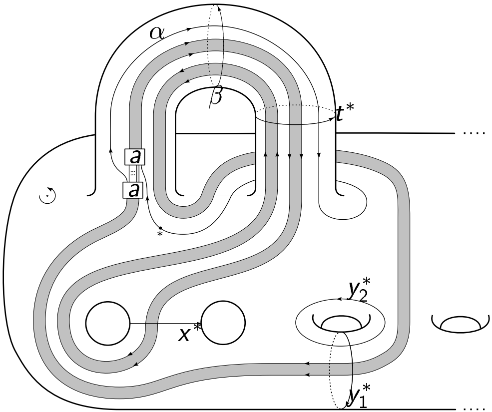
Lemma $\tau(M(K)) = \Delta_K(h) \in \mathbb{Z}[H]$.

Constructing examples of infinite order

- The examples are obtained using handle decomposition:

$$M(a) := \Sigma \times [0,1] \cup \text{1-handle} \cup \text{2-handle}$$

- The case $\Sigma_{g,2}$ with $g > 0$:



Examples of infinite order

- $H := H_1(\Sigma) = \langle x, y_1, y_2, \ldots, y_{2g} \rangle$,
- $\partial \colon C_2(M(a), \Sigma_+; \mathbb{Z}[H]) = \mathbb{Z}[H] \to C_1(M(a), \Sigma_+; \mathbb{Z}[H]) = \mathbb{Z}[H]$
- $\tau(M(a)) = \det(\partial) = \mathbb{Z}[H]$-valued intersection number $\lambda(\alpha, \beta) = 1 + (y_1 - 1)x + y_1(y_1 - 1)x^2 + \cdots + y_1^{a-1}(y_1 - 1)x^a$.
- $\tau(M(a)) \neq \overline{\tau(M(a))}$ in $Q(H)^\times/AN$.
- $\mathrm{ord}(\tau(M(a)) = \infty$ in $Q(H)^\times/AN$.
- If $a \neq b$, $\tau(M(a)) \neq \tau(M(b)) \neq \overline{\tau(M(a))}$ in $Q(H)^\times/AN$.