Twisted Alexander polynomials and character varieties for 2-bridge knots

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Twisted Alexander invariants and topology of low-dimensional manifolds

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- K : knot in S^3
- $E(K) = S^3 - N(K)$
- $G(K) = \pi_1(E(K))$
- A knot K is fibered if $E(K)$ is fibered over S^1.

Theorem If a knot K is fibered, then $\Delta_K(t)$ is monic and $\deg(\Delta_K(t)) = 2\text{genus}(K)$.

The converse of the above theorem does not hold.

Question Do the twisted Alexander polynomials for K detect if K is fibered?

Theorem (Friedl-Vidussi) The twisted Alexander polynomials for K associated to all finite representations determine if K is fibered.

Question Do the twisted Alexander polynomials for K associated to $SL(2, \mathbb{C})$-representations detect if K is fibered? Do they detect the genus of K?

Twisted Alexander polynomials

- $G(K) = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_{n-1} \rangle$ Wirtinger presentation
- $\alpha : G(K) \to \mathbb{Z} = \langle t \rangle$
- $\rho : G(K) \to SL(2, \mathbb{C})$
- $\tilde{\rho} \otimes \tilde{\alpha} : \mathbb{Z}[G(K)] \to M(2, \mathbb{C}[t^{\pm 1}])$
- $\Phi : \mathbb{Z}[F_n] \to \mathbb{Z}[G(K)] \to M(2, \mathbb{C}[t^{\pm 1}])$

- $M = (m_{ij})$ is an $(n-1) \times n$ matrix where $m_{ij} = \Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M(2, \mathbb{C}[t^{\pm 1}])$
- M_j is the $(n-1) \times (n-1)$ matrix obtained from M by deleting the jth column.
- $M_j \in M(2(n-1) \times 2(n-1), \mathbb{C}[t^{\pm 1}])$

Definition (Wada) The twisted Alexander polynomial of K associated to ρ is

$$\Delta_{K,\rho}(t) = \frac{\det M_j}{\det \Phi(1 - x_j)} \in \mathbb{C}(t)$$

and it is well-defined up to multiplication by t^{2k} $(k \in \mathbb{Z})$.

- If ρ is conjugate to ρ', then $\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$
- For a ring R and a finitely presented group G, the twisted Alexander polynomial of G associated to a $GL(n, R)$-representation is similarly defined.
- There are other versions of the twisted Alexander polynomial: X. S. Lin, Jiang-Wang, Kirk-Livingston, and so on.

Let $\rho : G(K) \to SL(2, \mathbb{C})$ be a nonabelian representation and g the genus of K.

Theorem (Kitano-Morifuji) $\Delta_{K,\rho}(t)$ is a (Laurent) polynomial, i.e., $\Delta_{K,\rho}(t) \in \mathbb{C}[t^{\pm 1}]$.

Definition $\Delta_{K,\rho}(t)$ is monic if the leading coefficient is 1.

Theorem (Friedl-K.) $\deg(\Delta_{K,\rho}(t)) \leq 4g - 2$

Theorem (Goda-Kitano-Morifuji, Kitano-Morifuji) If K is fibered, then $\Delta_{K,\rho}(t)$ is monic and $\deg(\Delta_{K,\rho}(t)) = 4g - 2$.

Character Varieties

- $R(K) = \mathrm{Hom}(G(K), SL(2,\mathbb{C}))$: $SL(2,\mathbb{C})$-representation variety of K
- For $\rho \in R$, the character $\chi_\rho : G(K) \to \mathbb{C}$ is defined by

$$\chi_\rho(x) = \mathrm{tr}\,\rho(x)$$

- $X(K) = \{\chi_\rho \,|\, \rho \in R(K)\}$: $SL(2,\mathbb{C})$-character variety of K
- $R^{nab}(K) = \{\rho \in R(K) \,|\, \rho \text{ is nonabelian}\}$
- $X^{nab}(K) = \{\chi_\rho \,|\, \rho \in R^{nab}(K)\}$

- For $\rho, \rho' \in G(K)$, if $\chi_\rho = \chi_{\rho'}$ and ρ is irreducible, then ρ is conjugate to ρ'.

$$
\begin{array}{ccc}
R(K) & \longrightarrow & X(K) \\
& \searrow \quad \nearrow & \\
& R(K)/\mathrm{conj} &
\end{array}
$$

- In this case, $\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$.

Lemma If $\chi_\rho = \chi_{\rho'}$, then $\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$.

Proof If ρ is reducible, then we may write $\rho(x_i) = \begin{pmatrix} \lambda & \nu_i \\ 0 & \lambda^{-1} \end{pmatrix}$, and hence

$$\Delta_{K,\rho}(t) = \frac{\Delta_K(\lambda t)\Delta_K(\lambda^{-1} t)}{(t-\lambda)(t-\lambda^{-1})}$$

Definition The twisted Alexander polynomial of K associated with $\chi_\rho \in X(K)$ is defined to be $\Delta_{K,\rho}(t)$ and denoted by $\Delta_{K,\chi_\rho}(t)$.

2-bridge knots

- 2-bridge knots are parameterized by a pair of relatively prime integers α and β such that α is odd and $-\alpha < \beta < \alpha$.
- 2-bridge knots $K(\alpha, \beta)$ and $K(\alpha', \beta')$ have the same type iff $\alpha = \alpha'$ and $\beta = \beta'$ or $\beta\beta' \equiv 1 \bmod \alpha$.
- A 2-bridge knot is fibered if and only if $\Delta_K(t)$ is monic.
- For a 2-bridge knot K of genus g, $\deg(\Delta_K(t)) = 2g$.

- $G(K(\alpha,\beta)) = \langle a, b \mid wa = bw \rangle, \quad w = a^{\epsilon_1} b^{\epsilon_2} \cdots a^{\epsilon_{\alpha-2}} b^{\epsilon_{\alpha-1}}$
- Up to conjugacy, for $\rho \in R^{nab}(K)$, we can take

$$\rho(a) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \text{ and } \rho(b) = \begin{pmatrix} s & 0 \\ 2-y & s^{-1} \end{pmatrix}$$

  where $(s, y) \in \mathbb{C}^2$ is a solution of the Riley polynomial $\phi(s, y) \in \mathbb{Z}[s^{\pm 1}, y]$.
- $\chi_\rho(a) = s + s^{-1}$ and $\chi_\rho(ab^{-1}) = y$

Lemma (Culler-Shalen) $\chi_\rho \in X^{nab}(K)$ is uniquely determined by $\chi_\rho(a)$ and $\chi_\rho(ab^{-1})$.

- $\phi(s, y) = \phi(s^{-1}, y) \Rightarrow$ we use the Riley polynomial $\phi(x, y) \in \mathbb{Z}[x, y]$ where $x = s + s^{-1}$.
- We identify

$$X^{nab}(K) = \{(x, y) \in \mathbb{C}^2 \mid \phi(x, y) = 0\}$$

and write

$$\Delta_{K,\chi_\rho}(t) = \sum_{i=n}^{m} \psi_i(x, y) t^i$$

for some $\psi_i(x, y) \in \mathbb{C}[x, y]$.

Main Results

Theorem (K.-Morifuji) A 2-bridge knot K is fibered if and only if $\Delta_{K,\rho}(t)$ is monic for any nonabelian representation $\rho : G(K) \to SL(2, \mathbb{C})$.

Lemma (Burde, De Rham) Let $\eta_0 : G(K) \to SL(2,\mathbb{C})$ be an abelian representation of a knot K given by $\eta_0(\mu) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where μ is the meridian of K and $\lambda \neq 0 \in \mathbb{C}$. Then there is a reducible nonabelian representation $\rho : G(K) \to SL(2,\mathbb{C})$ such that $\chi_\rho = \chi_{\eta_0}$ if and only if $\Delta_K(\lambda^2) = 0$.

Corollary For a 2-bridge knot K, $R^{nab}(K)$ has a reducible representation.

Proof of Theorem For a reducible representation $\rho \in R^{nab}(K)$,

$$\Delta_{K,\rho}(t) = \frac{\Delta_K(\lambda t)\Delta_K(\lambda^{-1} t)}{(t-\lambda)(t-\lambda^{-1})}$$

which is nonmonic for a nonfibered K.

Theorem (K.-Morifuji) For a nonfibered 2-bridge knot K, there exists an irreducible curve component in $X^{\mathrm{nab}}(K)$ which contains only a finite number of monic characters.

Proof

- $\Delta_{K,\chi}(t) =$
  $\psi_{4g-2}(x, y)t^{4g-2} + \psi_{4g-3}(x, y)t^{4g-3} + \cdots + \psi_1(x, y)t + \psi_0(x, y)$
- $\phi_1(x, y) = 0$: irreducible component containing a reducible character
- $\{(x, y) \in \mathbb{C}^2 \mid \phi_1(x, y) = 0 \text{ and } \psi_{4g-2}(x, y) = 1\} \cup$
  $\{(x, y) \in \mathbb{C}^2 \mid \phi_1(x, y) = 0 \text{ and } \psi_{4g-2}(x, y) = 0\}$ is finite.

Theorem (K.-Morifuji) For a 2-bridge knot K of genus g, there exists an irreducible curve component X_1 in $X^{\mathrm{nab}}(K)$ such that $\deg(\Delta_{K,\chi}(t)) = 4g - 2$ for all but finitely many $\chi \in X_1$.

Proof $\{(x, y) \in \mathbb{C}^2 \mid \phi_1(x, y) = 0\} \cap \{(x, y) \in \mathbb{C}^2 \mid \psi_{4g-2}(x, y) = 0\}$ is finite.

Theorem (K.-Morifuji) Let $K = K(\alpha, \beta)$ be a nonfibered 2-bridge knot and $c \in \mathbb{Z}$ the leading coefficient of $\Delta_K(t)$. Suppose that for any odd prime divisor p of α, $c \not\equiv 0$ and $c^2 \not\equiv \pm 1 \mod p$. Then the number of monic characters in $X^{\mathrm{nab}}(K)$ is finite.

Theorem (K.-Morifuji) Let $K = K(\alpha, \beta)$ be a 2-bridge knot of genus g and $c \in \mathbb{Z}$ the leading coefficient of $\Delta_K(t)$. Suppose that for any odd prime divisor p of α, $c \not\equiv 0 \mod p$. Then $\deg(\Delta_{K,\chi}(t)) = 4g - 2$ for all but finitely many $\chi \in X^{\mathrm{nab}}(K)$.

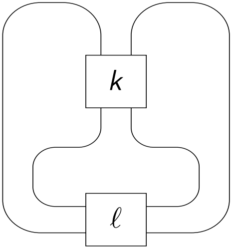
Example $K = K(7,3) = 5_2$. $\Delta_K(t) = 2t^2 - 3t + 2$ and $\mathrm{genus}(K) = 1$. Take $p = 7, c = 2$. Then $c \not\equiv 0$ and $c^2 \not\equiv \pm 1 \mod p$. Therefore

- The number of monic characters is finite.
- For all but finitely many characters, $\deg(\Delta_{K,\chi}(t)) = 4 \cdot 1 - 2 = 2$.

Example $K = (15,11) = 7_4$. $\Delta_K(t) = 4t^2 - 7t + 4$ and $\mathrm{genus}(K) = 1$.

- $X^{nab}(K)$ has two components and one of them has no reducible representations.
- Take $p = 3, 5$, $c = 4$. Then $c \not\equiv 0 \mod p$. So for all but finitely many characters, $\deg(\Delta_{K,\chi}(t)) = 4 \cdot 1 - 2 = 2$.

Example: $J(k, \ell)$



- $J(k, \ell)$ is a knot $\Leftrightarrow$ $k\ell$ is even
- $J(k, \ell) = K(\alpha, \beta)$ where $\frac{\beta}{\alpha} = \frac{\ell}{1 - k\ell}$
- $J(k, \ell)$ is the mirror image of $J(-k, -\ell)$
- $J(K, \ell) = J(\ell, k)$
- $J(2, 2q)$ is a twist knot

From now on, we consider $J(k, 2q)$ with $k > 0$.

Theorem (Morifuji, '2008) Let $K = J(2, 2q)$ and $|q| > 1$. Then $X^{nab}(K)$ has at most $2|q| - 4$ monic characters if q is odd and $2|q| - 2$ monic characters if q is even.

Theorem (Macasieb-Petersen-Van Luijk) For $K = J(k, 2q)$, if $k \neq 2q$ and $k \geq 2$, then $X^{nab}(K)$ is irreducible. If $k = 2q$, $X^{nab}(K)$ has two components.

Theorem (K.-Morifuji) Let $K = J(k, 2q)$ be a nonfibered knot where $k \neq 2q$. Then the number of monic characters is bounded above by

$$2(k+1)^2q^2 - (k+1)(k+4)|q|$$

if k is even or

$$(k+1)(k-1)|q|$$

if k is odd.

Future work

- A 2-bridge knot is a torus knot or a hyperbolic knot $\Rightarrow$ A nonfibered 2-bridge knot is a hyperbolic knot
- For a hyperbolic knot K, $X^{nab}(K)$ has the canonical component $X_0^{nab}(K)$ containing the character of $\rho_0 : G(K) \to SL(2, \mathbb{C})$ which is a lift of the discrete faithful representation to $PSL(2, \mathbb{C})$.

Conjecture (Dunfield-Friedl-Jackson) For a hyperbolic knot K, $\Delta_{K,\chi_{\rho_0}}(t)$ detects if K is fibered.

- Dunfield-Friedl-Jackson proved the conjecture for all knots with at most 15 crossings.

- They also showed that for a knot K of genus g the set of monic characters in $X(K)$ is Zariski closed and $\{\chi \in X(K) \mid \deg(\Delta_{K,\chi}(t)) = 4g-2\}$ is Zariski open.
- For a hyperbolic knot K, it is known that the canonical component $X_0^{nab}(K)$ is a curve.

Conjecture (K.-Morifuji) For a nonfibered knot K, there exists a curve component $X_1(K)$ in $X^{nab}(K)$ so that $\{\chi \in X_1(K) \,|\, \Delta_{K,\chi}(t) \text{ is monic}\}$ is a finite set.