Twisted Alexander polynomials and character varieties for 2-bridge knots

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Twisted Alexander invariants and topology of low-dimensional manifolds

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- $K$: knot in $S^3$
- $E(K) = S^3 - N(K)$
- $G(K) = \pi_1(E(K))$
- A knot $K$ is fibered if $E(K)$ is fibered over $S^1$.

**Theorem** If a knot $K$ is fibered, then $\Delta_K(t)$ is monic and $\deg(\Delta_K(t)) = 2\text{genus}(K)$.

The converse of the above theorem does not hold.
Question Do the twisted Alexander polynomials for $K$ detect if $K$ is fibered?

Theorem (Friedl-Vidussi) The twisted Alexander polynomials for $K$ associated to all finite representations determine if $K$ is fibered.

Question Do the twisted Alexander polynomials for $K$ associated to $SL(2, \mathbb{C})$-representations detect if $K$ is fibered? Do they detect the genus of $K$?
Twisted Alexander polynomials

- $G(K) = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_{n-1} \rangle$ Wirtinger presentation
- $\alpha : G(K) \to \mathbb{Z} = \langle t \rangle$
- $\rho : G(K) \to SL(2, \mathbb{C})$
- $\tilde{\rho} \otimes \tilde{\alpha} : \mathbb{Z}[G(K)] \to M(2, \mathbb{C}[t^{\pm 1}])$
- $\Phi : \mathbb{Z}[F_n] \to \mathbb{Z}[G(K)] \to M(2, \mathbb{C}[t^{\pm 1}])$
\( M = (m_{ij}) \) is an \((n-1) \times n\) matrix where
\[
m_{ij} = \Phi \left( \frac{\partial r_i}{\partial x_j} \right) \in M(2, \mathbb{C}[t^{\pm1}])
\]
\( M_j \) is the \((n-1) \times (n-1)\) matrix obtained from \( M \) by deleting the \( j \)th column.
\( M_j \in M(2(n-1) \times 2(n-1), \mathbb{C}[t^{\pm1}]) \)

**Definition (Wada)** The twisted Alexander polynomial of \( K \) associated to \( \rho \) is
\[
\Delta_{K,\rho}(t) = \frac{\det M_j}{\det \Phi(1-x_j)} \in \mathbb{C}(t)
\]
and it is well-defined up to multiplication by \( t^{2k} \) \((k \in \mathbb{Z})\).
• If $\rho$ is conjugate to $\rho'$, then $\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$

• For a ring $R$ and a finitely presented group $G$, the twisted Alexander polynomial of $G$ associated to a $GL(n, R)$-representation is similarly defined.

• There are other versions of the twisted Alexander polynomial: X. S. Lin, Jiang-Wang, Kirk-Livingston, and so on.
Let $\rho : G(K) \rightarrow SL(2, \mathbb{C})$ be a **nonabelian** representation and $g$ the genus of $K$.

**Theorem** (Kitano-Morifuji) $\Delta_{K,\rho}(t)$ is a (Laurent) polynomial, i.e., $\Delta_{K,\rho}(t) \in \mathbb{C}[t^{\pm 1}]$.

**Definition** $\Delta_{K,\rho}(t)$ is **monic** if the leading coefficient is 1.

**Theorem** (Friedl-K.) $\deg(\Delta_{K,\rho}(t)) \leq 4g - 2$

**Theorem** (Goda-Kitano-Morifuji, Kitano-Morifuji) If $K$ is fibered, then $\Delta_{K,\rho}(t)$ is monic and $\deg(\Delta_{K,\rho}(t)) = 4g - 2$. 

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Character Varieties

- \( R(K) = \text{Hom}(G(K), SL(2, \mathbb{C})) \): \( SL(2, \mathbb{C}) \)-representation variety of \( K \)
- For \( \rho \in R \), the character \( \chi_\rho : G(K) \to \mathbb{C} \) is defined by
  \[
  \chi_\rho(x) = \text{tr} \, \rho(x)
  \]
- \( X(K) = \{ \chi_\rho \mid \rho \in R(K) \} \): \( SL(2, \mathbb{C}) \)-character variety of \( K \)
- \( R^{nab}(K) = \{ \rho \in R(K) \mid \rho \text{ is nonabelian} \} \)
- \( X^{nab}(K) = \{ \chi_\rho \mid \rho \in R^{nab}(K) \} \)
For $\rho, \rho' \in G(K)$, if $\chi_\rho = \chi_{\rho'}$ and $\rho$ is irreducible, then $\rho$ is conjugate to $\rho'$.

In this case, $\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$. 

\[
\begin{array}{ccc}
R(K) & \longrightarrow & X(K) \\
\downarrow & & \downarrow \\
R(K)/\text{conj} & & \\
\end{array}
\]
Lemma If $\chi_\rho = \chi_{\rho'}$, then $\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$.

Proof If $\rho$ is reducible, then we may write $\rho(x_i) = \begin{pmatrix} \lambda & \nu_i \\ 0 & \lambda^{-1} \end{pmatrix}$, and hence

$$\Delta_{K,\rho}(t) = \frac{\Delta_K(\lambda t)\Delta_K(\lambda^{-1} t)}{(t - \lambda)(t - \lambda^{-1})}$$

Definition The twisted Alexander polynomial of $K$ associated with $\chi_\rho \in X(K)$ is defined to be $\Delta_{K,\rho}(t)$ and denoted by $\Delta_{K,\chi_\rho}(t)$. 
2-bridge knots

- 2-bridge knots are parameterized by a pair of relatively prime integers $\alpha$ and $\beta$ such that $\alpha$ is odd and $-\alpha < \beta < \alpha$.
- 2-bridge knots $K(\alpha, \beta)$ and $K(\alpha', \beta')$ have the same type iff $\alpha = \alpha'$ and $\beta = \beta'$ or $\beta \beta' \equiv 1 \mod \alpha$.
- A 2-bridge knot is fibered if and only if $\Delta_K(t)$ is monic.
- For a 2-bridge knot $K$ of genus $g$, $\deg(\Delta_K(t)) = 2g$. 
\begin{itemize}
  \item $G(K(\alpha, \beta)) = \langle a, b \mid wa = bw \rangle$, \quad $w = a^{\epsilon_1} b^{\epsilon_2} \cdots a^{\epsilon_{\alpha-2}} b^{\epsilon_{\alpha-1}}$
  
  \item Up to conjugacy, for $\rho \in R^{nab}(K)$, we can take

  $$
  \rho(a) = \begin{pmatrix}
  s & 1 \\
  0 & s^{-1}
  \end{pmatrix}
  \quad \text{and} \quad
  \rho(b) = \begin{pmatrix}
  s & 0 \\
  2 - y & s^{-1}
  \end{pmatrix}
  $$

  where $(s, y) \in \mathbb{C}^2$ is a solution of the Riley polynomial $\phi(s, y) \in \mathbb{Z}[s^{\pm 1}, y]$.

  \item $\chi_\rho(a) = s + s^{-1}$ and $\chi_\rho(ab^{-1}) = y$
\end{itemize}

\textbf{Lemma} (Culler-Shalen) $\chi_\rho \in X^{nab}(K)$ is uniquely determined by $\chi_\rho(a)$ and $\chi_\rho(ab^{-1})$. 
\( \phi(s, y) = \phi(s^{-1}, y) \Rightarrow \) we use the Riley polynomial \( \phi(x, y) \in \mathbb{Z}[x, y] \) where \( x = s + s^{-1} \).

We identify

\[
X^{nab}(K) = \{(x, y) \in \mathbb{C}^2 \mid \phi(x, y) = 0\}
\]

and write

\[
\Delta_{K, \chi_\rho}(t) = \sum_{i=n}^{m} \psi_i(x, y) t^i
\]

for some \( \psi_i(x, y) \in \mathbb{C}[x, y] \).
**Main Results**

**Theorem** (K.-Morifuji) A 2-bridge knot $K$ is fibered if and only if $\Delta_{K,\rho}(t)$ is monic for any nonabelian representation $\rho : G(K) \to SL(2, \mathbb{C})$. 
Lemma (Burde, De Rham) Let \( \eta_0 : G(K) \rightarrow SL(2, \mathbb{C}) \) be an abelian representation of a knot \( K \) given by \( \eta_0(\mu) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \), where \( \mu \) is the meridian of \( K \) and \( \lambda \neq 0 \in \mathbb{C} \). Then there is a reducible nonabelian representation \( \rho : G(K) \rightarrow SL(2, \mathbb{C}) \) such that \( \chi_\rho = \chi_{\eta_0} \) if and only if \( \Delta_K(\lambda^2) = 0 \).

Corollary For a 2-bridge knot \( K \), \( R^{nab}(K) \) has a reducible representation.

Proof of Theorem For a reducible representation \( \rho \in R^{nab}(K) \),

\[
\Delta_{K,\rho}(t) = \frac{\Delta_K(\lambda t)\Delta_K(\lambda^{-1} t)}{(t - \lambda)(t - \lambda^{-1})}
\]

which is nonmonic for a nonfibered \( K \).
**Theorem** (K.-Morifuji) For a nonfibered 2-bridge knot $K$, there exists an irreducible curve component in $X^{nab}(K)$ which contains only a finite number of monic characters.

**Proof**

- $\Delta_{K,\chi}(t) = 
  \psi_{4g-2}(x, y)t^{4g-2} + \psi_{4g-3}(x, y)t^{4g-3} + \cdots + \psi_1(x, y)t + \psi_0(x, y)$

- $\phi_1(x, y) = 0$: irreducible component containing a reducible character

- $\{(x, y) \in \mathbb{C}^2 \mid \phi_1(x, y) = 0 \text{ and } \psi_{4g-2}(x, y) = 1\} \cup \{(x, y) \in \mathbb{C}^2 \mid \phi_1(x, y) = 0 \text{ and } \psi_{4g-2}(x, y) = 0\}$ is finite.
**Theorem** (K.-Morifuji) For a 2-bridge knot $K$ of genus $g$, there exists an irreducible curve component $X_1$ in $X^{nab}(K)$ such that $\deg(\Delta_{K,\chi}(t)) = 4g - 2$ for all but finitely many $\chi \in X_1$.

**Proof** $\{(x, y) \in \mathbb{C}^2 | \phi_1(x, y) = 0\} \cap \{(x, y) \in \mathbb{C}^2 | \psi_{4g-2}(x, y) = 0\}$ is finite.
**Theorem** (K.-Morifuji) Let $K = K(\alpha, \beta)$ be a nonfibered 2-bridge knot and $c \in \mathbb{Z}$ the leading coefficient of $\Delta_K(t)$. Suppose that for any odd prime divisor $p$ of $\alpha$, $c \not\equiv 0$ and $c^2 \not\equiv \pm 1 \mod p$. Then the number of monic characters in $X^{nab}(K)$ is finite.

**Theorem** (K.-Morifuji) Let $K = K(\alpha, \beta)$ be a 2-bridge knot of genus $g$ and $c \in \mathbb{Z}$ the leading coefficient of $\Delta_K(t)$. Suppose that for any odd prime divisor $p$ of $\alpha$, $c \not\equiv 0 \mod p$. Then $\deg(\Delta_K, \chi(t)) = 4g - 2$ for all but finitely many $\chi \in X^{nab}(K)$. 
Example $K = K(7, 3) = 5_2$. $\Delta_K(t) = 2t^2 - 3t + 2$ and genus($K$) = 1.
Take $p = 7, c = 2$. Then $c \not\equiv 0$ and $c^2 \not\equiv \pm 1 \mod p$. Therefore

- The number of monic characters is finite.
- For all but finitely many characters, $\deg(\Delta_K, \chi(t)) = 4 \cdot 1 - 2 = 2$.

Example $K = (15, 11) = 7_4$. $\Delta_K(t) = 4t^2 - 7t + 4$ and genus($K$) = 1.

- $X^{nab}(K)$ has two components and one of them has no reducible representations.
- Take $p = 3, 5, c = 4$. Then $c \not\equiv 0 \mod p$. So for all but finitely many characters, $\deg(\Delta_K, \chi(t)) = 4 \cdot 1 - 2 = 2$. 
Example: $J(k, \ell)$

- $J(k, \ell)$ is a knot $\iff k\ell$ is even
- $J(k, \ell) = K(\alpha, \beta)$ where $\frac{\beta}{\alpha} = \frac{\ell}{1-k\ell}$
- $J(k, \ell)$ is the mirror image of $J(-k, -\ell)$
- $J(K, \ell) = J(\ell, k)$
- $J(2, 2q)$ is a twist knot
From now on, we consider $J(k, 2q)$ with $k > 0$.

**Theorem** (Morifuji, ’2008) Let $K = J(2, 2q)$ and $|q| > 1$. Then $X^{nab}(K)$ has at most $2|q| - 4$ monic characters if $q$ is odd and $2|q| - 2$ monic characters if $q$ is even.

**Theorem** (Macasieb-Petersen-Van Luijk) For $K = J(k, 2q)$, if $k \neq 2q$ and $k \geq 2$, then $X^{nab}(K)$ is irreducible. If $k = 2q$, $X^{nab}(K)$ has two components.
Theorem (K.-Morifuji) Let $K = J(k, 2q)$ be a nonfibered knot where $k \neq 2q$. Then the number of monic characters is bounded above by

$$2(k + 1)^2q^2 - (k + 1)(k + 4)|q|$$

if $k$ is even or

$$(k + 1)(k - 1)|q|$$

if $k$ is odd.
Future work

- A 2-bridge knot is a torus knot or a hyperbolic knot $\Rightarrow$ A nonfibered 2-bridge knot is a hyperbolic knot

- For a hyperbolic knot $K$, $X^{nab}(K)$ has the canonical component $X_0^{nab}(K)$ containing the character of $\rho_0 : G(K) \to \text{SL}(2, \mathbb{C})$ which is a lift of the discrete faithful representation to $\text{PSL}(2, \mathbb{C})$.

Conjecture (Dunfield-Friedl-Jackson) For a hyperbolic knot $K$, $\Delta_{K, \chi_{\rho_0}}(t)$ detects if $K$ is fibered.

- Dunfield-Friedl-Jackson proved the conjecture for all knots with at most 15 crossings.
They also showed that for a knot $K$ of genus $g$ the set of monic characters in $X(K)$ is Zariski closed and 
\[ \{ \chi \in X(K) \mid \deg(\Delta_{K,\chi}(t)) = 4g - 2 \} \] is Zariski open.

For a hyperbolic knot $K$, it is known that the canonical component $X_0^{nab}(K)$ is a curve.

**Conjecture** (K.-Morifuji) For a nonfibered knot $K$, there exists a curve component $X_1(K)$ in $X^{nab}(K)$ so that 
\[ \{ \chi \in X_1(K) \mid \Delta_{K,\chi}(t) \text{ is monic} \} \] is a finite set.