L^2-signatures: twisted invariants from duality

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POSTECH

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Twisted topological invariants and topology of low-dimensional manifolds

Outline

- Introduction to L^2-invariants
  - von Neumann group algebra
  - L^2-signatures, Index theorem, L^2-signature defect (= Cheeger-Gromov ρ-invariant)
- Amenable L^2-theoretic methods for homology
  - Invariants under homology cobordism / knot concordance
- Applications to knots and 3-manifolds, including
  - Hidden local torsion of 3-manifold groups
  - Knot concordance: non-concordant (non-slice) knots with vanishing Levine, Casson-Gordon, Cochran-Orr-Teichner obstruction

L^2-homology: fundamentals

Analytic objects as coefficient systems over a group G:

- Classical approach: Use the Hilbert space $\ell^2 G$.

$$\ell^2 G = \left\{ \sum_{g\in G} z_g g \;\middle|\; z_g \in \mathbb{C},\ \sum_{g\in G} |z_g|^2 < \infty \right\}$$

= Hilbert space completion of the group ring $\mathbb{C}G$

$\left\langle \sum_g z_g g, \sum_g w_g g \right\rangle := \sum_g z_g \overline{w_g}$ inner product on $\ell^2 G$

$\mathbb{C}G$ acts on $\ell^2 G$ naturally (on the left) i.e., $\ell^2 G$ is a $\mathbb{C}G$-module.

(classical) L^2-homology for X endowed with $\pi_1(X) \to G$:

$$C_*^{(2)}(X;\ell^2 G) := \ell^2 G \otimes_{\mathbb{Z}G} C_*(X;\mathbb{Z}\pi_1(X)) \cong (\ell^2 G)^{n_*} \quad \text{a Hilbert space}$$

$$H_*^{(2)}(X;\ell^2 G) := \operatorname{Ker}\partial_n / \overline{\operatorname{Im}\partial_{n+1}} \quad \text{where } \overline{(\ )} \text{ denotes the closure.}$$

Atiyah's index theory, L^2-signatures, ...

- Group von Neumann algebra as standard homology coefficients

$\mathcal{B}(\ell^2 G) := \{$bounded linear operators on $\ell^2 G\}$

$\mathbb{C}G \subset \mathcal{B}(\ell^2 G)$ as left multiplication. G also acts on the right of $\ell^2 G$.

$$\mathcal{N}G := \{ a \in \mathcal{B}(\ell^2 G) \mid a \text{ commutes with the right } G\text{-action on } \ell^2 G\}$$

i.e., $a(v\cdot g) = a(v)\cdot g \quad \forall v \in \ell^2 G,\ \forall g \in G$.

$\mathbb{C}G \subset \mathcal{N}G \Rightarrow \mathcal{N}G$ is a $(\mathcal{N}G, \mathbb{C}G)$ bimodule.

The operator adjoint $\langle av, w\rangle = \langle v, a^* w\rangle$ gives an involution on $\mathcal{N}G$.

L^2-homology for X endowed with $\phi\colon \pi_1(X) \to G$:

$$C_*(X;\mathcal{N}G) := \mathcal{N}G \otimes_{\mathbb{C}G} C_*(X;\mathbb{C}G)$$

$$H_*(X;\mathcal{N}G) := H_*(C_*(X;\mathcal{N}G)) = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$$

i.e., standard twisted homology with $\mathcal{N}G$ as coefficients.

Some Advantages of $\mathcal{N}G$

- Spectral theory (from functional analysis)

  Standard inner product on $(\mathcal{N}G)^n$: $\langle x, y\rangle = \sum_{i=1}^{n} y_i x_i^* \in \mathcal{N}G$

  For $a \in \mathcal{N}G$, we write $a \geq 0$ if $a = b^* b$ for some $b \in \mathcal{N}G$.

Theorem: If $A \in M_n(\mathcal{N}G)$ is hermitian ($A^* = A$), then

$\exists\ V_+, V_-, V_0 \subset (\mathcal{N}G)^n$ s.t.

$(\mathcal{N}G)^n = V_+ \oplus V_- \oplus V_0$ orthogonal sum

where $V_0 = \ker\{A : (\mathcal{N}G)^n \to (\mathcal{N}G)^n,\ x \mapsto xA^*\}$,

A is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ definite on $\left\{\begin{matrix}V_+\\ V_-\end{matrix}\right\}$, i.e.

for $\underset{\neq 0}{x} \in \left\{\begin{matrix}V_+\\ V_-\end{matrix}\right\}$, $\langle xA^*, x\rangle$ is $\left\{\begin{matrix}>0\\ <0\end{matrix}\right\}$.

- Dimension theory over $\mathcal{N}G$ [W. Lück]

  von Neumann trace $\mathrm{tr}^{(2)} : \mathcal{N}G \to \mathbb{C}$ is defined by $\mathrm{tr}^{(2)} a = \langle a(e), e\rangle_{\ell^2 G}$.

F.g. projective case : $P \oplus Q = (\mathcal{N}G)^n$

Choose a projection $p : (\mathcal{N}G)^n \to (\mathcal{N}G)^n$ onto P.

Regard p as a matrix $p = (p_{ij}) \in M_n(\mathcal{N}G) \cong \mathrm{End}((\mathcal{N}G)^n)$.

Define $\dim^{(2)} P := \mathrm{tr}^{(2)} p = \sum_i \mathrm{tr}^{(2)} p_{ii}$ (always real, nonnegative!)

General case : $\dim^{(2)} M := \sup\{\dim^{(2)} P \mid P \subset M,\ P \text{ f.g. projective}\}$

Theorem [Lück] $\dim^{(2)} : \{\mathcal{N}G\text{-modules}\} \longrightarrow \mathbb{R}_{\geq 0}$ satisfies:

1. $\dim^{(2)} 0 = 0$, $\dim^{(2)} \mathcal{N}G = 1$
2. $0 \to A \to B \to C \to 0 \Rightarrow \dim^{(2)} B = \dim^{(2)} A + \dim^{(2)} C$.

Underlying reason : There are "sufficiently many" projective modules $/\mathcal{N}G$.

since for any subsp $V \subset \ell^2 G$, $\exists$ proj. $p : \ell^2 G \to \overline{V}$ = closure of V.

Twisted intersection and L^2-signature

W: compact topological 4-manifold (with ∂), $\phi : \pi_1 W \to G$

Duality : $H_2(W, \partial W; \mathcal{N}G) \underset{\text{P.D.}}{\cong} H^2(W; \mathcal{N}G) \underset{\text{Kronecker}}{\longrightarrow} \mathrm{Hom}(H_2(W; \mathcal{N}G), \mathcal{N}G)$

$\uparrow$

$\rightsquigarrow$ $\lambda : H_2(W; \mathcal{N}G) \times H_2(W; \mathcal{N}G) \to \mathcal{N}G$ hermitian over $\mathcal{N}G$

Another advantage of $\mathcal{N}G$: $H_2(W; \mathcal{N}G) \cong P \oplus T$ $\quad$ P: f.g. proj. $\dim^{(2)} T = 0$

λ gives rise to $\lambda : P \times P \to \mathcal{N}G$.

Writing $P \oplus Q = (\mathcal{N}G)^n$, $\lambda \oplus 0 : (\mathcal{N}G)^n \times (\mathcal{N}G)^n \to \mathcal{N}G$

$\qquad (\mathcal{N}G)^n = V_+ \oplus V_- \oplus V_0$ spectral decomposition

Definition: $\mathrm{sign}^{(2)}_G(W) := \dim^{(2)} V_+ - \dim^{(2)} V_- \in \mathbb{R}$.

L^2-induction : if $G \hookrightarrow \Gamma$, then $\mathrm{sign}^{(2)}_G(W) = \mathrm{sign}^{(2)}_\Gamma(W)$.

L^2-index Theorem and signature defect

Topological index theorem : [Lück-Schick, Weinberger]

If W^4 is closed, then $\mathrm{sign}^{(2)}(W)$ = ordinary signature $\sigma(W)$.

L^2-signature defect (= von Neumann-Cheeger-Gromov ρ-invariant)

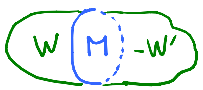
M: closed 3-manifold, $\phi : \pi_1 M \to G$.

$\Rightarrow \exists\, W^4$ s.t. $\partial W = rM$ and $\exists$ diagram

$$\begin{tikzcd} \pi_1(rM) \arrow[r, "\phi"] \arrow[d, "i_*"] & G \arrow[d, hook, "\exists"] \\ \pi_1(W) \arrow[r, "\exists"] & \Gamma \end{tikzcd}$$

Define $\rho^{(2)}(M, \phi) = \frac{1}{r}\left(\mathrm{sign}^{(2)}_\Gamma W - \sigma(W)\right)$.

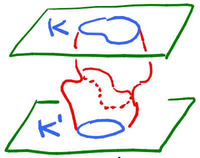
Index theorem + L^2-induction $\Rightarrow \rho^{(2)}$ is well-defined.

$V =$ 

$\Rightarrow \mathrm{sign}^{(2)} V = \mathrm{sign}^{(2)} W - \mathrm{sign}^{(2)} W'$

$\qquad\qquad \sigma(V) = \sigma(W) - \sigma(W')$

Recall:



concordance

$\xrightarrow{\text{0-surgery}}$

W: $\mathbb{Z}$-homology cobordism

$$H_*(M_K;\mathbb{Z}) \cong H_*(W;\mathbb{Z}) \cong H_*(M_{K'};\mathbb{Z})$$

Main question:

Given an R-homology cobordism W and

comm. ring

$$\begin{tikzcd} \pi_1(M) \arrow[d] \arrow[dr, "\phi"] & \\ \pi_1(W) \arrow[r] & G \\ \pi_1(M') \arrow[u] \arrow[ur, "\phi'"'] & \end{tikzcd},$$

for which groups G is $\rho^{(2)}(M,\phi)$ equal to $\rho^{(2)}(M',\phi')$?

Known cases: p-groups, poly-torsion-free-abelian (PTFA)

G is PTFA $\Leftrightarrow$ $\exists\ G=G_0 \triangleright G_1 \triangleright \cdots \triangleright G_{n+1}=\{e\}$ s.t. each G_i/G_{i+1} is torsion-free abelian.

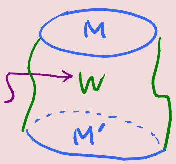
Amenable L^2-theoretic method [C.-Orr]

G is amenable if $\exists$ finitely additive invariant measure on G

G is in Strebel's class $D(R)$ if for $\alpha: P\to Q$ with P, Q RG-proj.,

$1_R\otimes\alpha: R\otimes_{RG} P \to R\otimes_{RG} Q$ is 1-1 $\Rightarrow$ α is 1-1.

Theorem [C.-Orr] Suppose



R-homology cobordism,

$$\begin{tikzcd} \pi_1 M \arrow[d] \arrow[dr, "\phi"] & \\ \pi_1 W \arrow[r, dashed] & \Gamma \\ \pi_1 M' \arrow[u] \arrow[ur, "\phi'"'] & \end{tikzcd}$$

Γ: amenable, in $D(R)$

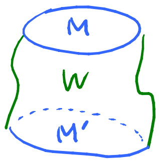
Then $\rho^{(2)}(M,\phi) = \rho^{(2)}(M',\phi')$.

$AD(R) = \{$amenable groups in $D(R)\}$ is large:

$\underbrace{p\text{-groups } (R=\mathbb{Z}_p), \text{ PTFA}}_{}$ groups, and many infinite groups with torsion

only previously known case from which homology cob. invariants are

Key idea:



Compare $\rho^{(2)}(M,\phi)$ and $\rho^{(2)}(M',\phi')$ using W

$$\rho^{(2)}(M,\phi) - \rho^{(2)}(M',\phi') = \operatorname{sign}^{(2)}_\Gamma W - \sigma(W)$$

$H_*(W,M;R)=0 \Rightarrow$ ordinary intersection form of $W \equiv 0$
$\Rightarrow \sigma(W)=0$

$H_*(W,M;R\Gamma)=0$? In general, No!

or other coeff. system over Γ

(c.f. Γ = p-group, PTFA: algebraic results)

Theorem [C.-Orr] Suppose $\Gamma \in AD(R)$, C_* = f.g. free (proj.) chain cpx $/\mathbb{Z}\Gamma$

Then $H_*(R\otimes_{\mathbb{Z}G} C_*)=0 \Rightarrow \dim^{(2)}_\Gamma H_*(\mathcal{N}G\otimes_{\mathbb{Z}G} C_*)=0$

Theorem [C.] Γ, C_*: as above $\Rightarrow \dim^{(2)}_\Gamma H_*(\mathcal{N}G\otimes_{\mathbb{Z}G} C_*) \le \dim_R H_*(R\otimes_{\mathbb{Z}G} C_*)$.
(and $R=\mathbb{Z}_p$ or $\mathbb{Q}$)

Applications due to [C.-Orr] include:

- Homology Chang-Weinberger theorem (high dim'l)
  For any $(4k+3)$-dim'l manifold M having torsion in $\pi_1(M)$ with certain property, $\exists$ infinitely many $M_0=M, M_1, M_2, \cdots$ that are simple homotopy equivalent but not mutually homology cobordant.
- Homology cobordism of homology spherical space forms (dim=3)
- Knots in general 3-manifolds

In what follows we discuss further applications.

Hidden local torsion in 3-manifold groups

An advantage of new L^2-techniques for homology: groups with torsion

3-dim'l perspective: Torsion in "generic" 3-manifold groups is rare.
(e.g. M^3: closed, irreducible, nonspherical $\Rightarrow$ $\pi_1(M)$: torsion-free)

But, we will illustrate that the study of the interplay between dimension 3 and 4 has significantly different flavor:

> Torsion may appear in a natural homological context, even for "generic" (e.g. hyperbolic) 3-manifolds.

Homology localization (algebraic closure)

$$\Omega^R := \left\{ \alpha : \pi \to G \;\middle|\; \begin{array}{l} \pi, G \text{ f.p., } \alpha \text{ is 2-connected on } H_*(-;R) \\ \text{i.e., } \cong \text{ on } H_1(-;R), \text{ onto on } H_2(-;R) \end{array} \right\}$$

e.g. If W is an R-homology cobordism of M, then $\pi_1(M) \xrightarrow[i_*]{} \pi_1(W) \in \Omega^R$

> $\exists$ functorial assignment $\pi \mapsto \hat{\pi}$ = homology localization of π (endowed with $\pi \to \hat{\pi}$)
> s.t. ① $\pi \to G$ is in Ω^R $\Rightarrow$ $\hat{\pi} \xrightarrow{\cong} \hat{G}$
> ② $\hat{\pi}$ is universal among such ones in an "appropriate" sense.

e.g. If W is an R-homology cobordism of M, then $\forall \phi : \pi_1(M) \to \Gamma$ factoring through $\widehat{\pi_1(M)}$, we have

$$\begin{tikzcd} \pi_1(M) \arrow[r, "\phi"] \arrow[d, "i_*"'] & \Gamma \\ \pi_1(W) \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Def $g \in \pi_1(M^3)$ is hidden local torsion if $\pi_1(M)$ is torsion-free but $\pi_1(M) \longrightarrow \widehat{\pi_1(M)}$, $g \longmapsto$ torsion element (i.e. order $< \infty$)

Denote the lower central series by

$$G_1 = G, \quad G_{g+1} = [G, G_g], \quad G_\omega = \bigcap_{g<\infty} G_g \quad (\omega = 1^{st} \text{ infinite ordinal})$$

> **Theorem** [C.-Orr] $\exists$ closed hyperbolic 3-manifolds M which have hidden local torsion lying in $\pi_1(M)_\omega$.

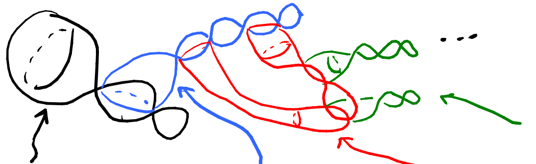
hidden local torsion $\in \pi_1(M)_\omega$ $\Rightarrow$ invisible in any residually nilpotent coeff. systems (e.g. p-groups)

The examples are constructed by "enlarging" the fundamental group of the twisted torus bundle $N = S^1 \times S^1 \times [0,1] \big/ (z,w,0) \sim (-z,-w,1)$.

> **Theorem** [C.-Orr] $\exists$ closed hyperbolic 3-manifolds M which have hidden local torsion lying in $\pi_1(M)_\omega$.

> **Theorem** [C.-Orr] $\exists$ closed hyperbolic 3-mfds $M = M_0, M_1, \cdots$ s.t.
> ① $\exists$ homology equiv. $f_i : M_i \to M$ $\forall i$.
> ② $\forall \phi : \pi_1(M) \to G$ = torsion free, $\rho^{(2)}(M, \phi) = \rho^{(2)}(M, \phi \circ f_{i*})$
> ③ Similary, all prior homology cob. invariants related to p-groups
> $\left\{ \begin{array}{l} \text{Wall's multisignature [Gilmer, Livingston, Ruberman]} \\ \text{Atiyah-Patodi-Singer } \tilde{\eta}\text{-invariants [Levine, Friedl]} \\ \text{L-group invariants from iterated } p\text{-covers [C.]} \\ \text{Twisted torsion invariants [C.-Friedl]} \end{array} \right\}$ have the same value for M_i.
> ④ M_i is not homology cob. to M_j $\forall i \neq j$

Whitney towers in 4-manifolds : approximation to an honest disk



$\mathcal{C}_3 \cdots \mathcal{C}_n$ (n = height)

$\mathcal{C}_0 = \{$immersed 2-spheres$\}$ $\quad$ $\mathcal{C}_1 = \{$immersed Whitney disks$\}$ $\quad$ $\mathcal{C}_2$

$\mathcal{C}_k \cap \left(\bigcup_{i=1}^{k} \mathcal{C}_i\right) = \phi$

K is called (n)-solvable if M_K bounds W^4 for which $\exists$ Whitney tower $\{\mathcal{C}_k\}$ of height n s.t. $|\mathcal{C}_0| = \frac{1}{2} b_2(W)$ and the intersection form $\lambda : H_2(W) \times H_2(W) \to \mathbb{Z}$ vanishes on $\mathcal{C}_0$.

K is called (n.5)-solvable if $\exists$ W^4 and Whitney tower $\{\mathcal{C}_k\}$ of height n+1, allowing $\mathcal{C}_{n+1}$ may intersect $\mathcal{C}_n$.

W is called an (n) - solution.
(n.5)

Brief History (topological category)

1950's →	1960's →	1970's →	1980's →	1990's (2003)
Fox-Milnor	Levine (n=0.5)	Casson-Gordon (n=1.5)	Freedman (top. slicing)	Cochran-Orr-Teichner (n>1.5)

(n)-solvable filtration :

$$0 \subset \cdots \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_{1.5} \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C} = \{\text{knots}\}/\text{conc.}$$

slice knots $\quad$ n-solvable knots

... Casson-Gordon $\equiv 0$ $\quad$ ... Levine $\equiv 0$

[C-O-T '03] $\exists$ non-slice knots ($\notin \mathcal{F}_{2.5}$) with vanishing $\left\{ \begin{array}{l} \text{Levine} \\ \text{Casson-Gordon} \end{array} \right\}$

[Cochran-Teichner '07] nonslice examples in level n
($\mathcal{F}_n / \mathcal{F}_{n.5} \neq 0$)

[Cochran-Harvey-Leidy '09] More about the structure of $\mathcal{F}_n / \mathcal{F}_{n.5}$

Knot concordance and amenable L^2-method

Theorem [C.] K is slice, $\Gamma \in AD(R)$ for some R,
$\phi : \pi_1 M_K \to \Gamma$ factors through $D^4 -$ (slice disk)
$\Rightarrow \rho^{(2)}(M_K, \phi) = 0$

Theorem [C.] K is (n.5)-solvable, $\Gamma \in AD(R)$, $R = \mathbb{Q}$ or $\mathbb{Z}/p$, $\Gamma^{(n+1)} = \{e\}$,
$\phi : \pi_1 M_K \to \Gamma$ factors through an (n.5)-solution
$\mu \mapsto \infty$-order
$\Rightarrow \rho^{(2)}(M_K, \phi) = 0$.

Remark : The slice obstruction is not a consequence of the solvability obstruction.

As a special case, our result specializes to :

[Cochran-Orr-Teichner '03 Ann. Math.] Γ : PTFA, $\Gamma^{(n+1)} = \{e\}$,
$\phi : \pi_1 M_K \to \Gamma$ factors through an (n.5)-solution W
$\Rightarrow \rho^{(2)}(M_K, \phi) = 0$.

If $\exists$ (n)-solution W for K satisfying the above, we say
"K is an (n)-solvable knot with (C-O-T) PTFA L^2-signature = 0."

Let $\mathcal{V}_n = \{$(n)-solvable knots with PTFA L^2-sign. = 0$\}$: $\mathcal{F}_{n.5} \subset \mathcal{V}_n \subset \mathcal{F}_n$

Theorem [C.] $\forall n$, $\exists$ (n)-solvable knots $K_1, K_2, \cdots$ satisfying
① Any linear combination $\#_i a_i K_i$ lies in $\mathcal{V}_n$.
② If $a_i \neq 0$ for some i, then $\#_i a_i K_i \notin \mathcal{F}_{n.5}$.
i.e., $\mathcal{V}_n / \mathcal{F}_{n.5}$ has infinite rank.

Construction of twisted coefficient systems

For a sequence $\mathcal{P} = (R_0, R_1, \cdots)$ of comm. rings and a group G, define mixed-coefficient commutator series as follows:

$$\mathcal{P}^0 G = G, \quad \mathcal{P}^{n+1} G = \operatorname{Ker}\left\{ G^n \to G^n/[G^n, G^n] \otimes_{\mathbb{Z}} R_n \right\}$$

If $R_i = \mathbb{Q}$ or $\mathbb{Z}_{(p)}$ for $\forall i$, $G/\mathcal{P}^n G$ is in $AD(\mathbb{Z}/p)$.

We apply this to groups of 4-manifolds we construct from a solution. Our L^2-signatures are obtained from an associated tower of covers:

$$M_K = M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_{n-1} \leftarrow M_n$$

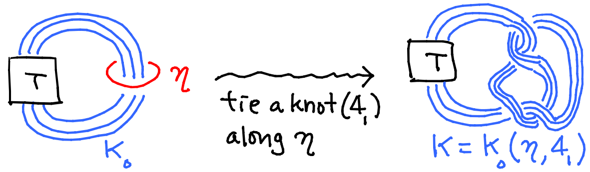
infinite cyclic

torsion-free or p-torsion cover

c.f. Casson-Gordon : $n=2$, C-O-T : n high but always torsion-free

A problem

Satellite construction : (infection)



Does there exist a slice knot K_0 and $[\eta] \in \pi_1(S^3 - K_0)^{(n)}$, n high (>2) for which $K_0(\eta, 4_1)$ is not (topologically) slice ?

c.f. When K_0 is a link, there are several recent methods :

[C. '09, 10] L-group valued invariants from iterated p-covers

[C.-Livingston-Ruberman '08, C.-Kim '08, Van Cott '09, A Levine '09]

Covering link calculus for iterated Bing / Whitehead doubles

[C.-Friedl] Twisted torsion invariants

For knots (and n high), any new methods ?

Thank You