Why is the twisted invariant related to fiberability?

Jae Choon Cha
POSTECH

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RIMS Seminar on
Twisted topological invariants and topology of low-dimensional manifolds

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Main topic: twisted invariants and fibered knots

Particularly I want to discuss:

1. Original motivation and ideas that led me to the first result on fibered knots [TAMS'03, arXiv '01]
2. Viewpoints from general twisted coefficients
3. Some subsequent works

The treatments will be:

1. As elementary as possible
2. Mainly for the simplest but most essential case

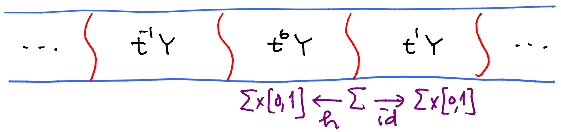
Alexander polynomial and fibered knots

$K \subset S^3$ is fibered $\Rightarrow$ $\Delta_K(t)$ is monic (and $\deg \Delta_K(t) = 2g(K)$)

Let $X = S^3 -$ (open tub. nbhd. of K), $\Sigma \subset S^3$ be a fiber surface for K.

$Y := \left(X \text{ cut along } \Sigma\right) \cong \Sigma \times [0,1]$ i.e., $X = \Sigma\times[0,1] / (x,0) \sim (h(x),1)$
where $h : \Sigma \xrightarrow[\cong]{} \Sigma$ is the monodromy.

Compute $H_1(\bar{X})$, $\bar{X}$ = infinite cyclic cover of X:

$\bar{X}$ = 

Therefore $H_1(X) = H_1(\Sigma)[t^{\pm 1}] / tI - h_* \;=\; \mathbb{Z}[t^{\pm1}]^{2g} / tI - h_*$ $\quad (g = g(\Sigma))$

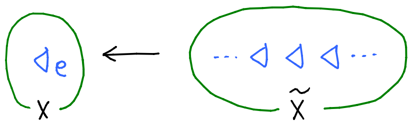
$\Delta_K(t) = \det(tI - h_*)$ is monic and $\deg = 2g$.

Twisted coefficients and coverings : the ultimate source of "twisting".

X : finite CW complex with $\pi = \pi_1(X)$.

$\mathbb{Z}\pi := \left\{ \sum_{g\in\pi} n_g g \;\middle|\; n_g \in \mathbb{Z} \right\}$ group ring of π
(= free abelian group generated by π)

$C_*(\tilde{X}; \mathbb{Z})$:= cellular chain complex of the universal cover $\tilde{X}$
= free abelian group generated by cells of $\tilde{X}$
= (left) free $\mathbb{Z}\pi$-module generated by cells of X



$\{\text{cells in } \tilde{X} \text{ over } e\} \approx \pi$
(indexed by $g \in \pi$)

In facts, π acts on cells in $\tilde{X}$ over e as deck transformation

Often we write $C_*(X;\mathbb{Z}\pi) := C_*(\tilde{X};\mathbb{Z})$ viewing it as a $\mathbb{Z}\pi$-module.

For any right $\mathbb{Z}\pi$-module M, define

$C_*(X;M) := M \otimes_{\mathbb{Z}\pi} C_*(X;\mathbb{Z}\pi)$ twisted chain complex

$H_*(X;M) := H_*(C_*(X;M))$ twisted homology

(Note any left $\mathbb{Z}\pi$-module becomes a right $\mathbb{Z}\pi$-module via $x \cdot g := g^{-1}x$)

Invariants from $C_*(X;-)$ or $H_*(X;-)$ are often called "twisted invariants".

If our twisted coefficient system is "good enough", we may consider

- Invariants of modules $H_*(X;-)$: rank (= Betti number)
  "torsion part" of a module
- Invariants of the based chain complex $C_*(X;-)$: combinatorial torsion
- ⋮

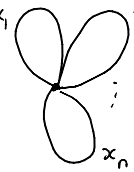
Fox calculus fits into this viewpoint (c.f. [Wada])

2-skeleton $X^{(2)}$ of X ⇝ group presentation of $\pi = \pi_1(X)$

a description of $\partial_2 : C_2(X;-) \to C_1(X;-)$ ← Fox calculus

(often presentation of $H(X;-)$) ⇝ invariants of π and M

$X^{(1)} =$



x_1, x_2, x_n

a 2-cell $e_j^2 =$

attaching map $= $ a word $r_j(x_1, \dots, x_n)$ $=$ a relator in $\pi_1(X)$

x_n ‖ a generator of $\pi_1(X)$

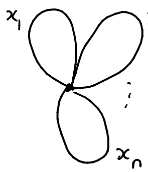
Kernel = normal subgp gen. by r_j

$$\downarrow$$

free group F on $x_1, \dots, x_n$

$$\downarrow p$$

$\pi_1(X) = \pi_1(X^{(2)})$



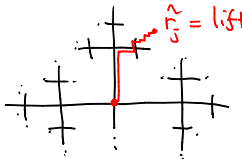
x_1, x_2, x_n

a 2-cell $e_j^2 =$

attaching map $=$ a word r_j

Fox calculus computes $\tilde{r}_j$

universal cover of $X^{(1)}$



$\tilde{r}_j =$ lift of r_j

$\{$1-cells in $X^{(1)}$ over $x_i\}$ $\approx$ free group $F = \langle x_1, \dots, x_n \rangle$

So $\tilde{r}_j \in (\mathbb{Z}F)^{\#\text{ of 1-cells}}$

$\tilde{r}_j = \left(\frac{\partial r_j}{\partial x_i}\right)_{i=1,\dots,n}$

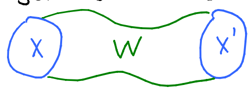
universal cover of $X^{(2)}$

$\partial(\text{lift of } e_j) = \phi_*(\tilde{r}_j)$

$= \phi_*\left(\frac{\partial r_j}{\partial x_i}\right)$

If X is a manifold, we have a lot more:

- Poincaré duality (chain level and homology level)
  $\cap S_X$: $C^i(X;\mathbb{Z}\pi) \xrightarrow{\simeq} C_{n-i}(X;\mathbb{Z}\pi)$
  $H^i(X;\mathbb{Z}\pi) \xrightarrow{\simeq} H_{n-i}(X;\mathbb{Z}\pi)$
- L-theory: Algebraic cobordism of chain complexes



⇝ chain complex analogue

- Twisted intersection form $H_{2k}(X;\mathbb{Z}\pi) \times H_{2k}(X;\mathbb{Z}\pi) \to \mathbb{Z}\pi$
- Signature invariants ... if our twisted coefficients are good enough

Coming soon: More about "twisted duality"

Special cases:

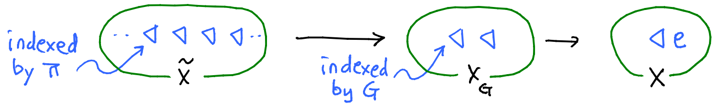
① Group ring coefficients

Given a homomorphism $\phi: \pi = \pi_1(X) \to G$ and a commutative ring R,

$M := RG$ is an $(RG, \mathbb{Z}\pi)$-bimodule.

Now $C_*(X; RG) = RG \underset{\mathbb{Z}\pi}{\otimes} C_*(\tilde{X}; \mathbb{Z}) \cong C_*(X_G; R)$ left RG-module

G-cover of X



i.e., $H_*(X; RG)$ = ordinary homology $H_*(X_G; R)$

e.g. Alexander module $= H_1(S^3 - K; \mathbb{Z}[t^{\pm 1}])$ where $\phi: \pi_1(S^3 - K) \to \mathbb{Z} = \langle t | \cdot \rangle$, $\mu \mapsto t$

② Representations as modules over group rings

Given a representation

$\rho: \pi = \pi_1(X) \to GL(n, \mathcal{R})$ where $\mathcal{R}$ = a (possibly noncomm.) ring,

$GL(n,\mathcal{R}) = \{$endomorphisms on the right of $\mathcal{R}^n\}$

we regard $\mathcal{R}^n$ as an $(\mathcal{R}, \mathbb{Z}\pi)$-bimodule.

So $H_*(X; \mathcal{R}^n) = H_*(\mathcal{R}^n \underset{\mathbb{Z}\pi}{\otimes} C_*(X; \mathbb{Z}\pi))$ is a left $\mathcal{R}$-module.

Why finite dimensional representations?

X is finite (as a CW-complex)

$\Rightarrow C_i(X; \mathcal{R}^n) = \mathcal{R}^n \underset{\mathbb{Z}\pi}{\otimes} C_i(X; \mathbb{Z}\pi) \cong \mathcal{R}^{n \cdot (\#\text{of } i\text{-cells})}$

is f.g. over $\mathcal{R}$

$\Rightarrow H_i(X; \mathcal{R}^n)$ is f.g. over $\mathcal{R}$ if $\mathcal{R}$ is Noetherian.

Twisted Alexander module of knots

Let $K \subset S^3$ be a knot (oriented)

$X_K = S^3 -$ (open tub. nbhd. of K)

$\rho: \pi \to GL(n, R)$ i.e., R^n is a $\mathbb{Z}\pi$-module

$\phi: \pi \xrightarrow{ab} \pi/[\pi,\pi] \cong \mathbb{Z} = \langle t | \cdot \rangle$ i.e., $R[t^{\pm 1}]$ is a $\mathbb{Z}\pi$-module.

Then $R[t^{\pm 1}] \underset{R}{\otimes} R^n$ $(\cong R[t^{\pm 1}]^n)$ becomes an $(R[t^{\pm 1}], \mathbb{Z}\pi)$-bimodule

via the diagonal action of $g \in \pi$: $(f(t) \otimes x) \cdot g = f(t) \cdot \phi(g) \otimes x \cdot \rho(g)$.

The twisted Alexander module $A = A^i_{K,\rho}$ is defined by

$$A^i_{K,\rho} := H_i(X_K; R[t^{\pm 1}]^n)$$

Twisted Alexander polynomial (homological version of [Kirk-Livingston])

Given $\rho: \pi_1(X) \to GL(n, F)$ with F a field,

$A := H_1(X_K; F[t^{\pm 1}]^n)$ is a f.g. module over $F[t^{\pm 1}]$ = a PID!

$\therefore A \cong \underbrace{F[t^{\pm 1}]^r}_{\text{free part}} \oplus \underbrace{\left(\bigoplus_i F[t^{\pm 1}]/(f_i(t))\right)}_{\text{torsion part}}$ by the structure theorem over PID

Define the twisted Alexander polynomial by $\Delta_{K,\rho}(t) := \prod_i f_i(t)$

(up to units in $F[t^{\pm 1}]$)

Monic ?

[C. '01] Definition combining Kirk-Livingston and classical methods for elementary ideals (c.f. [Wada])

Suppose R is a Noetherian UFD. ($\Rightarrow$ so is $R[t^{\pm1}]$)

Then $A_{K,\rho} = H_1(X; R[t^{\pm1}]^n)$ is f.g. over $R[t^{\pm1}]$, and there is a presentation $R[t^{\pm1}]^b \xrightarrow{T} R[t^{\pm1}]^a \to A_{K,\rho} \to 0$.

Define $\mathcal{A}_{K,\rho}$:= the ideal of $R[t^{\pm1}]$ gen. by

$$\mathcal{D} = \{\det(T') \mid T' = a\times a \text{ submatrix of } T\}$$

$$\Delta_{K,\rho}(t) := \gcd(\mathcal{D})$$

The well-definedness is shown using "Tietze moves" for presentations (e.g. see [Cromwell-Fox])

Using $H_i(-)$ in place of $H_1(-)$, we define $\Delta^i_{K,\rho}(t)$.

Obstruction to being fibered

Theorem: If $K \subset S^3$ is fibered, then for any $\rho: \pi_1(X_K) \to GL(n,R)$,

1. $A_{K,\rho}$ is presented by a matrix of the form $tI - H$ over $R[t^{\pm1}]$
2. $\mathcal{A}_{K,\rho}$ is a principal ideal gen. by $\det(tI-H)$.
3. $\Delta_{K,\rho} = \Delta^1_{K,\rho}$ is monic (i.e. top & bottom coefficients are units in R) and $\deg \Delta^1_{K,\rho} - \deg \Delta^0_{K,\rho} = n(2g(K)-1)$.

Consequently, $A_{K,\rho}$ is annihilated by a monic polynomial.

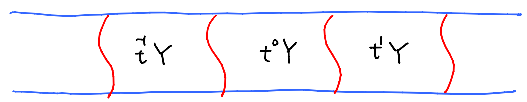
(proof) Let $X = X_K$ and $\bar X$ = infinite cyclic cover of X.

Then $A^\rho_K = H_1(X; R[t^{\pm1}]^n)$

$= H_1(\bar X; R^n)$ where R^n is a $\mathbb{Z}[\pi_1(\bar X)]$-module via $\pi_1(\bar X) \to \pi_1(X) \xrightarrow[\rho]{} R^n$

Let Σ = fiber surface, $h: \Sigma \xrightarrow{\cong} \Sigma$ monodromy,

$Y = (X \text{ cut along } \Sigma) \cong \Sigma \times [0,1]$.

$\bar X =$ 

$\Sigma\times[0,1] \xleftarrow[h]{} \Sigma \xrightarrow[id]{} \Sigma\times[0,1]$

$H_*(\bar X; R^n) \cong H_*(\Sigma; R^n)[t^{\pm1}]/tI - h_*$ where $h_*: H_*(\Sigma; R^n) \circlearrowleft$

$\Sigma \simeq \bigvee^{2g} S^1 \Rightarrow \cdots \to 0 \to \underset{\substack{\cong\\ R^{2ng}}}{C_1(\Sigma; R^n)} \xrightarrow{\partial_1} \underset{\substack{\| \\ C_0(\Sigma; \mathbb{Z}\pi_1\Sigma)\otimes_{\mathbb{Z}\pi_1\Sigma} R^n \cong R^n}}{C_0(\Sigma; R^n)}$

$(g = g(\Sigma))$

Therefore $H_1(\Sigma; R^n) = \operatorname{Ker}\partial_1 \subset R^{2ng}$ is R-free,

$\underset{\substack{\| \\ \deg \Delta^1_{K,\rho}}}{\operatorname{rank} H_1(\Sigma; R^n)} - \underset{\substack{\| \\ \deg \Delta^0_{K,\rho}}}{\operatorname{rank} H_0(\Sigma; R^n)} = n(2g-1)$

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Metabelian representations from finite cyclic covers

$N :=$ d-fold cyclic cover of X_K

$\downarrow$

X_K = exterior of $K \subset S^3$

$\phi: \pi_1(X_K) \to \mathbb{Z} = \langle t \mid \cdot\rangle$ induces $\psi: \pi_1(N) \to d\mathbb{Z} = \langle s \mid \cdot \rangle$ where $s = t^d$.

Given $\alpha: H_1(N) \to G$ = finite abelian group, $RG = R^{|G|}$ is a $\mathbb{Z}\pi_1(N)$-module.

{ It is very easy to choose α which are essentially different from ψ: e.g. $H_1(N) \to H_1(\text{d-fold branched cover}) \to G$ }

$\rightsquigarrow$ Twisted Alexander module $H_*(N; R[s^{\pm1}]^{|G|})$ is defined.

In fact, $RG \otimes_{\mathbb{Z}\pi_1 N} \mathbb{Z}\pi_1(X_K) = R^{|G|\cdot d}$ is a $\mathbb{Z}\pi_1(X_K)$-module and

$H_*(X_K; R^{|G|\cdot d}[t^{\pm1}]) \cong H_*(N; R^{|G|}[s^{\pm1}])$ over $R[s^{\pm1}]$.

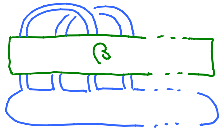
Example: Nonfibered knots with arbitrarily given nontrivial $\Delta_K(t)$ detected by twisted Alexander invariants

Suppose K_0 is a knot with $\Delta_K(t) \neq 1$.

$\Delta_{K_0}(t) \neq 1 \iff$ for some d, the d-fold branched cyclic cover M_0 of K_0 has nontrivial $H_1(M_0)$.
(infinitely many)

Choose $H_1(M_0) \overset{\alpha}{\twoheadrightarrow} \mathbb{Z}_r$, $r \geq 2$.

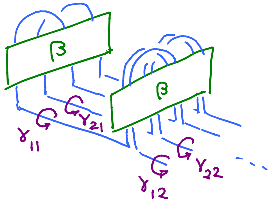
Given a Seifert surface F of K_0, M_0 is obtained as follows:



K_0

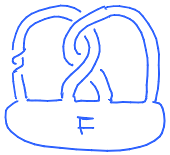
⇝ Akblute-Kirby

M_0 = surgery on



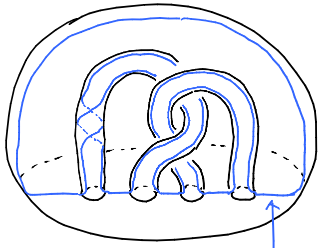
$H_1(M_0)$ is generated by $[\gamma_{ij}]$.

Key idea of Akbulut-Kirby:

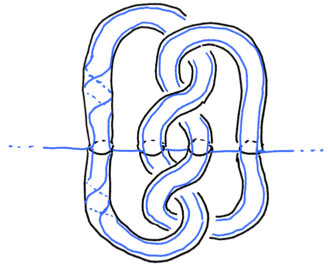


$\subset S^3$

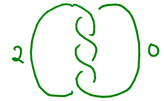
cut S^3 along F



take 2 copies and glue partly



⇝ by surgery (filling in the tori) we obtain 2-fold cyclic branched cover:



Claim: under $\alpha: H_1(M_0) \twoheadrightarrow \mathbb{Z}_r$, $\alpha[\gamma_{ij}] \neq \alpha[\gamma_{i(j+1)}]$ for some i, j.

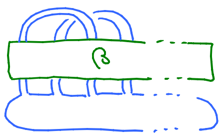
($\Rightarrow \alpha([\gamma_{ij}] - [\gamma_{i(j+1)}])$ has order $n > 1$)

In fact, $[\gamma_{ij}]$ gives the presentation matrix
$$\begin{bmatrix} V+V^T & -V^T & & \\ -V & V+V^T & -V^T & \\ & -V & V+V^T & \\ & & & \ddots \end{bmatrix}$$
where V = Seifert matrix for F

If $\alpha[\gamma_{ij}]$ is a const x_i, then $x = (x_i)$ satisfies $\begin{cases} x(V+V^T) - xV^T = 0 \\ -xV + x(V+V^T) = 0 \end{cases}$

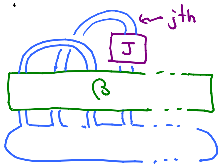
$\therefore x(V - V^T) = 0 \Rightarrow x = 0 \Rightarrow \alpha$ is trivial $\Rightarrow\Leftarrow$!

Now consider



K_0

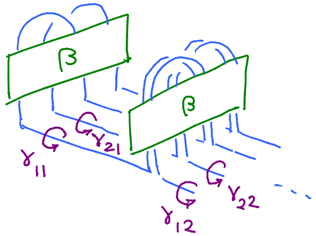
tie in a knot J



K

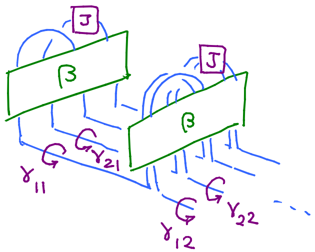
K_0 and K have the same Seifert matrix ($\Rightarrow \Delta_{K_0}(t) = \Delta_K(t)$.)

d-fold branched covers look like:



M_0

⇝



M

Consider the representation

$$\rho: \pi_1 N \to \pi_1 M \to H_1 M \cong H_1 M_0 \overset{\alpha}{\twoheadrightarrow} \mathbb{Z}_r \subset GL(\mathbb{Z}[\mathbb{Z}_r])$$

N = d-fold cyclic cover for K; regular repr'n

$\rho \otimes \psi$ makes $\mathcal{V} := \mathbb{Z}[\mathbb{Z}_r] \otimes_{\mathbb{Z}} \mathbb{Z}[s^{\pm 1}] \cong \mathbb{Z}[\mathbb{Z}_r \times \langle s | \rangle]$ a $\mathbb{Z}\pi_1(N)$-module.

Then, by Mayer-Vietoris arguments,

$H_1(N;\mho) \supset H_1(M_J) \otimes_{\mathbb{Z}} \mathbb{Z}[s^{\pm 1}]$ where M_J = n-fold branched cover for J

since (meridean of J) $= \gamma_{ij} - \gamma_{i(j+1)} \overset{\alpha}{\longmapsto}$ order n elt $\in \mathbb{Z}_r$

If $H_1(M_J) \neq 0$ (for $\forall n$), then $H_1(N;\mho)$ is never annihilated by any monic polynomial !

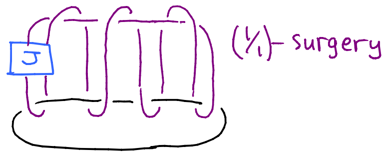
e.g. J has Seifert matrix $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \Rightarrow H_1(M_J) \neq 0 \ \forall n \geq 1$.

Summarizing, we have proven :

Theorem : For any knot K_0 with $\Delta_{K_0}(t) \neq 1$,
$\exists$ infinitely many knots K which have the same Seifert matrix as K_0
but have twisted Alexander polynomials of nonfibered knots.

Remark : ① This construction illustrates that there are many knots with non-torsion twisted Alexander modules
$(\Rightarrow \Delta^1_{K,\rho}(t) = 0)$

② Similar but more sophisticated construction of representations produces examples with $\Delta_K(t) = 1$ having the same properties.
i.e., non-torsion twisted Alexander module which is not annihilated by any monic poly.

$K =$ 

Subsequent works include :

[Goda-Kitano-Morifuji '05]
Reformulation of monicness for fibered knots using Reidemeister torsion and Wada's approach

[Friedl-Kim '06]
Lower bounds for genus / Thurston norm and monic / degree property of TAP for fibered 3-manifolds

[Friedl-Vidussi '07-]
Symplectic structures on $S^1 \times M^3$ and monic / degree property of TAP
Sufficiency of TAP obstruction to being fibered :
TAP detects fibered knots and 3-manifolds

Thank You !