Motivic Lie algebra and cohomology of moduli spaces of graphs and curves

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based on jw/w Takuya SAKASAI and Masaaki SUZUKI

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Contents

Motivic Lie algebra (1)

Motivic Lie algebra (or fundamental Lie algebra)

$$\mathfrak{f} = \mathrm{FreeLie}\langle \sigma_3, \sigma_5, \sigma_7, \cdots \rangle \quad \text{(Soulé elements)}$$

plays important roles in number theory

$$\sigma_{2k+1} \sim H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_p(2k+1)) \cong \mathbb{Z}_p \quad (p\text{: prime})$$

Soulé p-adic regulator

Recently, it also appears in many branches of mathematics related to number theory, such as topology, mathematical physics,...

Appearance of $\mathfrak{f}$ in Johnson cokernel (1)

The mapping class group of $\Sigma_{g,1}$ relative to the boundary

Theorem (Dehn-Nielsen-Zieschang)

$\mathcal{M}_{g,1} \cong \{\varphi \in \mathrm{Aut}\, \pi_1 \Sigma_{g,1}; \varphi(\zeta) = \zeta\}$ $\quad \zeta$: *boundary*

"differentiate" $\Rightarrow$ "Lie algebra" of $\mathcal{M}_{g,1}$

$\mathfrak{h}_{g,1} = \{$symplectic derivation of $\mathrm{FreeLie}\,\langle H \rangle\}$

defined over $\mathbb{Z}$, but here we consider it over $\mathbb{Q}$

introduced by Johnson in his beautiful works on the Torelli

group during ($1979 \sim 1985$)

Appearance of $\mathfrak{f}$ in Johnson cokernel (2)

The action of $\mathcal{M}_{g,1}$ on the lower central series of $\pi_1\Sigma_{g,1}$

$\Rightarrow$ Johnson filtration $\{\mathcal{M}_{g,1}(k)\}_k$

$\Rightarrow$ embedding of Lie algebras (Johnson homomorphism)

$$\tau : \bigoplus_{k=1}^{\infty} \mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1) \overset{\subset}{\longrightarrow} \mathfrak{h}_{g,1}^{+}$$

Determination of $\operatorname{Im}\tau \subset \mathfrak{h}_{g,1}^{+}$: very important problem,

fundamental results :

$$\operatorname{Im}\tau(1) \cong \wedge^3 H \quad \text{(Johnson)}$$
$$\operatorname{Im}\tau \otimes \mathbb{Q} = \langle \wedge^3 H \otimes \mathbb{Q} \rangle \subset \mathfrak{h}_{g,1}^{+} \otimes \mathbb{Q} \quad \text{(Hain)}$$

Appearance of $\mathfrak{f}$ in Johnson cokernel (3)

Johnson cokernel $= \mathfrak{h}_{g,1}^{+}/\mathrm{Im}\,\tau$

Determination of $\mathrm{Im}\,\tau \subset \mathfrak{h}_{g,1}^{+}$ $=$

determination of Johnson cokernel

First, around the end of 1980's, a surjective homomorphism

$$\mathrm{trace}: \mathfrak{h}_{g,1}^{+} \twoheadrightarrow \bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}$$

was constructed such that it vanishes on $\mathrm{Im}\,\tau$ (M.)

I have (too optimistically) conjectured that $\wedge^3 H_{\mathbb{Q}}$ and the trace components $S^{2k+1} H_{\mathbb{Q}}$ will generate the Lie algebra $\mathfrak{h}_{g,1}^{+}$

Appearance of $\mathfrak{f}$ in Johnson cokernel (4)

It turned out that, this is not the case due to remarkable works of Conant-Kassabov-Vogtmann (hairy graphs) and Bartholdi

On the other hand, as for the Johnson cokernel, a series of results starting from the works of Nakamura as well as Matsumoto proving a certain conjecture of Oda:

arithmetic mapping class group:

$$1 \to \widehat{\mathcal{M}_g^1} \to \pi_1^{\mathrm{alg}}\left(\mathbf{M}_g^1/\mathbb{Q}\right) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1$$

"$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ should appear in the Sp-invariant part of the Johnson cokernel"

until the definitive work of Brown in 2010, it is now known that there exists an embedding

$$\mathfrak{f} \subset \text{Johnson cokernel} : (\mathfrak{h}_{g,1}^{+}/\mathrm{Im}\,\tau)^{\mathrm{Sp}}$$

Appearance of $\mathfrak{f}$ in Johnson cokernel (6)

These results are based on the fundamental theories of

Grothendieck, Drinfeld, Ihara, Deligne

After this, there appeared many important results about the

Johnson cokernel:

- Enomoto-Satoh map (at present, the best result)
- Conant-Kassabov (Hopf algebra)
- Sakasai-Suzuki-M. : general theory for the structure of $(\mathfrak{h}_{g,1})^{\mathrm{Sp}}$+canonical metric on the space of Sp-invariant

Appearance of $\mathfrak{j}$ in Johnson cokernel (7)

tensors $\Rightarrow$ determined $\mathrm{Im}\,\tau \otimes \mathbb{Q}$ completely up to degree 6

Kawazumi-Kuno (Lie bialgebra)

However, both the problems of determining

Johnson cokernel and a system of generators for $\mathfrak{h}_{g,1}^{+}$

remain completely open (despite of many known results...)

Appearance of $\mathfrak{f}$ in $H^*(\mathrm{M}_g)$ (moduli space of curves) (1)

Theorem (Chan-Galatius-Payne)

There exists a surjection

$$H^{4g-6}(\mathbf{M}_g;\mathbb{Q}) \to \mathfrak{grt}(2g) \supset \mathfrak{f}(2g)$$

$$\Rightarrow H^{4g-6}(\mathbf{M}_g;\mathbb{Q}) \neq 0 \text{ for } g = 3,5 \text{ or } g \geq 7 \text{ and}$$

$$\dim H^{4g-6}(\mathbf{M}_g;\mathbb{Q}) > \beta^g + \text{constant} \quad \text{for any } \beta < \beta_0$$

where $\beta_0 = 1.3247\ldots$ *is the real root of* $t^3 - t - 1 = 0$

all the above cohomology classes are unstable classes

(beyond Harer stable range) and before this result,

only the following two unstable classes were known:

Appearance of $\mathfrak{f}$ in $H^*(\mathrm{M}_g)$ (moduli space of curves) (2)

Looijenga: $H^6(\mathbf{M}_3;\mathbb{Q}) \cong \mathbb{Q}$ and Tommasi: $H^5(\mathbf{M}_4;\mathbb{Q}) \cong \mathbb{Q}$

On the other hand

Theorem (Harer)

For any $g \geq 2$

$$\mathrm{vcd}(\mathcal{M}_g) = 4g-5 \ \Rightarrow H^k(\mathbf{M}_g;\mathbb{Q}) = 0 \text{ for all } k > 4g-5$$

Theorem (Sakasai-Suzuki-M., Church-Farb-Putman)

$$H^{4g-5}(\mathbf{M}_g;\mathbb{Q}) = 0$$

Problem

Construct explicit cocycles for the classes in $H^{4g-6}(\mathbf{M}_g;\mathbb{Q})$ *guaranteed by the above result of Chan-Galatius-Payne*

Kontsevich graph homology (1)

Lie version of Kontsevich graph homology

Theorem (Kontsevich, Lie version)

There exists an isomorphism

$$PH^k_c(\widehat{\mathfrak{h}}_{\infty,1})_{2n} \cong H_{2n-k}(\mathrm{Out}\, F_{n+1}; \mathbb{Q}) \quad (n \geq 1)$$

$\widehat{\mathfrak{h}}_{\infty,1}$: completion of $\quad \mathfrak{h}_{\infty,1} = \lim_{g\to\infty} \mathfrak{h}_{g,1}$

$$\bigoplus_{n\geq 2} H_*(\mathrm{Out}\, F_n; \mathbb{Q}) \;\Leftrightarrow\; PH^*_c(\widehat{\mathfrak{h}}_{\infty,1})$$

equivalent!

Kontsevich graph homology (2)

associative version of Kontsevich graph homology

Theorem (Kontsevich, associative version)

There exists an isomorphism

$$PH_c^k(\widehat{\mathfrak{a}}_\infty)_{2n} \cong \bigoplus_{2g-2+m=n, m>0} H_{2n-k}(\mathbf{M}_g^m;\mathbb{Q})^{\mathfrak{S}_m} \quad (n \geq 1)$$

$\widehat{\mathfrak{a}}_\infty$: completion of $\mathfrak{a}_\infty = \lim_{g\to\infty} \mathfrak{a}_g$

$$\bigoplus_{2g-2+m=n, m>0} H_{2n-k}(\mathbf{M}_g^m;\mathbb{Q})^{\mathfrak{S}_m} \Leftrightarrow PH_c^*(\widehat{\mathfrak{a}}_\infty)$$

equivalent!

Kontsevich graph homology (3)

associative version of Kontsevich graph homology

Theorem (Kontsevich, associative version)

There exists an isomorphism

$$PH_k(\mathfrak{a}_\infty)_{2n} \cong \bigoplus_{2g-2+m=n, m>0} H^{2n-k}(\mathbf{M}_g^m;\mathbb{Q})^{\mathfrak{S}_m} \quad (n \geq 1)$$

$$\bigoplus_{g\geq 0, m>0} H^{4g-6+2m}(\mathbf{M}_g^m;\mathbb{Q})^{\mathfrak{S}_m} \Leftrightarrow PH_2(\mathfrak{a}_\infty)$$

equivalent!

Disproof of a conjecture of Kontsevich (1)

Theorem [Chan-Galatius-Payne] +

Theorem [Kontsevich, associative] $\Rightarrow$

$$\dim H_2(\mathfrak{a}_\infty) = \infty$$

This disproves (!) the associative case of the following conjecture

Conjecture (Kontsevich)

For any k, the k-th homology group of each of the infinite dimensional Lie algebras

$$\mathfrak{c}_\infty,\ \mathfrak{a}_\infty,\ \mathfrak{l}_\infty \quad (\textit{commutative, associative, lie})$$

is finite dimensional.

Disproof of a conjecture of Kontsevich (2)

Kontsevich mentioned that

$$\dim H_2(\mathfrak{c}_\infty) = 1$$

On the other hand, I have constructed a series of elements

$$\mathfrak{t}_{2k+1} \in H^2(\mathfrak{h}_{g,1})_{4k+2} \ (k = 1, 2, \ldots)$$

by making use of the trace map

$$\mathrm{trace} : \mathfrak{h}_{g,1} \to S^{2k+1} H_{\mathbb{Q}}$$

and conjectured that all of these classes are non-trivial

Disproof of a conjecture of Kontsevich (3)

Since

$$\mathfrak{h}_{\infty,1} = \mathfrak{l}_\infty \quad \text{(the LHS appeared before the RHS)},$$

the above conjecture (non-triviality of the classes $\mathfrak{t}_{2k+1}$) implies

$$\dim H_2(\mathfrak{l}_\infty) = \infty$$

which would disprove the lie version of Kontsevich's conjecture

Since the structure of $\mathfrak{l}_\infty$ seems much richer that that of $\mathfrak{a}_\infty$

Theorem [Chan-Galatius-Payne] should be a strong supporting

evidence for the non-trivialities of the following classes

Disproof of a conjecture of Kontsevich (4)

$$H^2(\mathfrak{h}_{\infty,1})_{4k+2} \ni \mathbf{t}_{2k+1} \;\Leftrightarrow\; \mu_k \in H_{4k}(\mathrm{Out}F_{2k+2};\mathbb{Q})$$

Morita classes

At present, only the first three classes

$$\mu_1, \mu_2, \mu_3$$

are known to be non-trivial

due to M., Conant-Vogtmann, Gray:

Theorem (non-triviality of μ_k)

$\mu_2 \neq 0 \in H_8(\mathrm{Out}\, F_6; \mathbb{Q})$ (*Conant-Vogtmann 2004*)

$\mu_3 \neq 0 \in H_{12}(\mathrm{Out}\, F_8; \mathbb{Q})$ (*Gray 2011*)

very interesting general property of the classes μ_k:

Theorem (Conant-Hatcher-Kassabov-Vogtmann, 2015)

The class μ_k can be represented by the fundamental cycle of a certain abelian subgroup $\mathbb{Z}^{4k} \subset \mathrm{Out}\, F_{2k+2}$

It is now known that

there exists an embedding

$$\mathfrak{f} \subset \text{Johnson cokernel} : (\mathfrak{h}^+_{g,1}/\mathrm{Im}\,\tau)^{\mathrm{Sp}}$$

I thought first that the Galois image might serve as new

generators for $\mathfrak{h}^+_{g,1}$, namely they will survive in the abelianization

$$H_1(\mathfrak{h}^+_{g,1})$$

but soon I became to conjecture that they should be

represented by brackets of the trace components:

$$\mathfrak{h}_{g,1}(2k+1) \supset (\text{unique by Nakamura}) S^{2k+1}H_{\mathbb{Q}} \xrightarrow[\text{trace}]{\sim} S^{2k+1}H_{\mathbb{Q}}$$

Conjecture

$(\wedge^2 S^{2k+1}H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q} \ni 1 \overset{[\ ,\ ]}{\longmapsto} \sigma_{2k+1}^{\mathrm{top}} \in \mathfrak{h}_{g,1}(4k+2)$ *Galois image ?*

unsolved, but Hain told that he has some progress concerning

the above conjecture in his joint work with Brown

Conjecture

The bracket operation

$$\sum_{i=1}^{2k} \mathfrak{h}_{g,1}(i) \otimes \mathfrak{h}_{g,1}(4k+2-i) \xrightarrow{[\ ,\ ]} \mathfrak{h}_{g,1}(4k+2)$$

hits the element $\sigma_{4k+2}^{\mathrm{top}} \in \mathfrak{h}_{g,1}(4k+2)$

If this conjecture is true $\Rightarrow$ $\mathfrak{t}_{2k+1} \neq 0 \Rightarrow \mu_k \neq 0$

Morita classes and their conjectural relation with $\mathfrak{f}$ (4)

If the above two conjectures are true,

then we can say that the Galois images $\subset (\mathfrak{h}_{g,1}/\mathrm{Im}\,\tau)^{\mathrm{Sp}}$

and the classes μ_k are very closely related

$2k+1$	1	3	5	$\cdots$
weight $(4k+2)$	2	6	10	$\cdots$
generators of $\mathfrak{h}_{g,1}$, $\sqrt{\text{Galois}}$	$\Lambda^3 H/H$	$S^3 H$	$S^5 H$	$\cdots$
period	$\zeta(1)$	$\zeta(3)$	$\zeta(5)$	$\cdots$
Soulé (Galois image)		σ_3	σ_5	$\cdots$
$H^2(\mathfrak{h}_{\infty,1})_{4k+2}$	e_1	$\mathbf{t}_3$	$\mathbf{t}_5$	$\cdots$
$H_{4k}(\mathrm{Out}\, F_{2k+2})$		μ_1	μ_2	$\cdots$
$H^{8k-2}(\mathbf{M}_{2k+1})$		$H^6(\mathbf{M}_3)$	$H^{14}(\mathbf{M}_5)$	$\cdots$

$\mathfrak{h}_{g,1}(2k+1) \supset S^{2k+1} H_{\mathbb{Q}}$ (trace component)

$(\wedge^2 S^{2k+1} H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q} \xrightarrow{[\ ,\ ]} \sigma^{\mathrm{top}}_{2k+1} \in \mathfrak{h}_{g,1}(4k+2)$ Galois image ?

$$\mapsto \mathbf{t}_{2k+1} \in H^2(\mathfrak{h}_{g,1})_{4k+2} \cong \mu_k \in H_{4k}(\mathrm{Out}\, F_{2k+2})$$

$$\mapsto \tilde{\mathbf{t}}_{2k+1} \in H^2(\mathcal{H}^{\mathrm{top}}_{g,1})_{4k+2} \to H^2(\mathcal{H}^{\mathrm{smooth}}_{g,1})_{4k+2}$$