On the asymptotic behavior of the Reidemeister torsion for toroidal surgeries along twist knots

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Topology and Geometry of Low-dimensional Manifolds

Motivation

Purpose

Determine the asymptotic behavior of the sequcence given by R-torsion for **graph manifolds**.
(i.e., the order of growth and the limit of leading coefficient)

We need to know

- representation space for a graph manifold
- contribution of each Seifert piece in the asymptotic behavior of R-torsion

We focus on graph manifolds obtained by exceptional surgeries along a hyperbolic knot.

Graph manifolds

JSJ decomposition

Assume that a closed 3-manifold M is
connected orientable and *irreducible*.
We have the decomposition

$$M = M_1 \cup_{T^2} \ldots \cup_{T^2} M_k \quad \text{(JSJ decompo.)}$$

where each T^2 is imcompressible.

Graph manifold

M is called a *graph manifold*
if M is not Seifert fibered and the JSJ decomposition of M

$$M = M_1 \cup_{T^2} \ldots \cup_{T^2} M_k$$

has only Seifert fibered spaces M_i.

Exceptional surgery along a hyperbolic knot

Set

$E_K = S^3 \setminus \mathrm{Int}\, N(K)$: the knot exterior of a knot K

$m =$ a meridian $\subset \partial E_K$

$\ell =$ a preferred longitude $\subset \partial E_K$

p/q-surgery along K

We have

$$S^3_K(p/q) = E_K \cup_{p/q} D^2 \times S^1,$$

identifying $\partial D^2 \times \{*\} \sim p\, m + q\, \ell$ on $\partial E_K = T^2$.

Toroidal surgery

p/q-surgery is called *toroidal*
if K is hyperbolic & $\exists$incompressible $T^2 \subset S^3_K(p/q)$.

$S^3_K(p/q)$ is a graph manifold $\Rightarrow$ p/q-surgery must be toroidal.

Examples of toroidal surgery

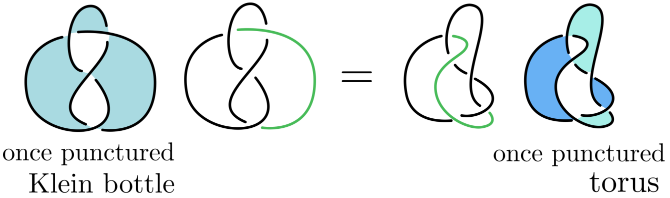
Assume that an incompressible surface $S \subset E_K$ is

- an once punctured Klein bottle or
- an once punctured torus.

If $D^2 \times \{*\} \sim p/q = \partial S$ then

- $\partial N(S \cup D^2 \times \{*\})$ or
  ($N(S \cup D^2 \times \{*\})$ is the twisted I-b'dle over Klein bottle)
- $S \cup D^2 \times \{*\}$

is an incompressible torus in $S^3_K(p/q)$.

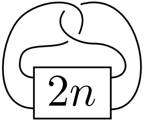


Toroidal surgeries along two–bridge hyperbolic knots

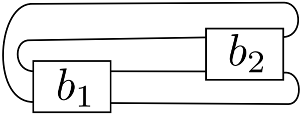
Classification by M. Brittenham and Y.-Q. Wu

Assume that K is a two–bridge knot.

(1) K = a twist knot $K[2n, \pm 2]$ and $p/q = 0/1$ or $p/q = \mp 4$;



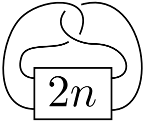
(2) $K = K[b_1, b_2](|b_1|, |b_2| > 2)$ and $p/q = 0/1$ (b_1 & b_2: even), $p/q = 2b_2/1$ (b_1:odd, b_2: even)



Toroidal surgeries yielding graph manifolds

Graph manifolds including torus knot exteriors (R. Patton, A. Clay, M. Teragaito)

(1) The twist knot $K[2n, \pm 2]$ & $p/q = \pm 4$ yields

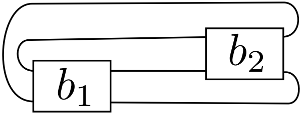


$$M = E_{T(2,2n+1)} \cup_{T^2} N(\text{Klein bottle}).$$

(2) The two–bridge knot $K[b_1, b_2]$ & $p/q = 2b_2/1$ yields

$$\begin{aligned} M &= E_{T(2,2b_1+1)} \cup_{T^2} N(\text{Klein bottle}) \cup_{T^2} \text{Cable space} \\ &= E_{T(2,2b_1+1)} \cup_{T^2} N(\text{Klein bottle}) \cup_A D^2 \times S^1 \end{aligned}$$

where A is an annulus.



Reidemeister torsion for a CW–complex

Definition (R-trosion $\mathrm{Tor}(W;\rho)$)

W	:	a finite CW-complex,
$\rho\colon \pi_1(W) \to \mathrm{GL}_n(\mathbb{C})$	:	$\mathrm{GL}_n(\mathbb{C})$-representation of π_1
$C_*(W;\mathbb{C}^n_\rho)$	:	local system given by ρ
$= \mathbb{C}^n \otimes_\rho C_*(\widetilde{W};\mathbb{Z}[\pi_1])$		($\widetilde{W}$:universal cover)
		$v \otimes \gamma\sigma = \rho(\gamma)^{-1} v \otimes \sigma$

Under $H_*(W;\mathbb{C}^n_\rho) = 0$,

$$\mathrm{Tor}(W;\rho) := \prod_{i\geq 0} \det(\partial \boldsymbol{b}^{i+1} \cup \boldsymbol{b}^i / \boldsymbol{c}^i)^{(-1)^{i+1}}$$

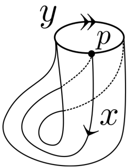
via the decomposition

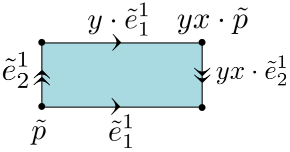
$$C_i(W;\mathbb{C}^n_\rho) = \mathrm{Ker}\,\partial_i \oplus (\text{a lift of } \mathrm{Im}\,\partial_i) = \mathrm{Im}\,\partial_{i+1} \oplus (\text{a lift of } \mathrm{Im}\,\partial_i)$$

R-torsion for the Klein bottle

$$\rho : \pi_1(Kb) = \langle x, y \mid yx = xy^{-1} \rangle \to \mathrm{SL}_2(\mathbb{C})$$

$$X := \rho(x) \quad \text{and} \quad Y := \rho(y) \quad s.t. \quad \mathrm{tr}\,\rho(y) \neq 2$$





$$0 \to C_2 \simeq \mathbb{C}^2 \xrightarrow{\partial_2} C_1 \simeq \mathbb{C}^2 \oplus \mathbb{C}^2 \xrightarrow{\partial_1} C_0 \simeq \mathbb{C}^2 \to 0$$

$$\partial_2 = \begin{pmatrix} \mathbf{1} - Y^{-1} \\ -(YX)^{-1} - \mathbf{1} \end{pmatrix}, \quad \partial_1 = \begin{pmatrix} X^{-1} - \mathbf{1} & Y^{-1} - \mathbf{1} \end{pmatrix}$$

$$\text{Then} \quad \mathrm{Tor}(Kb; \rho) = \frac{\det(\mathbf{1} - Y^{-1})}{\det(Y^{-1} - \mathbf{1})} = 1$$

Indeterminacy of R-torsion

R-torsion for general $\rho : \pi_1(W) \to \mathrm{GL}_n(\mathbb{C})$

$$\mathrm{Tor}(W;\rho) := \prod_{i\geq 0} \det(\partial \boldsymbol{b}^{i+1} \cup \boldsymbol{b}^i / \boldsymbol{c}^i)^{(-1)^{i+1}} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

is defined up to a factor $\pm \det(\rho(\gamma))$ $(\gamma \in \pi_1(W))$.

i.e. $\quad \mathrm{Tor}(W;\rho) \in \mathbb{C}^* / \{\pm \det(\rho(\gamma)) \mid \gamma \in \pi_1(W)\}$

For $\mathrm{SL}_{2N}(\mathbb{C})$-representations $\rho : \pi_1(W) \to \mathrm{SL}_n(\mathbb{C})$

$\mathrm{Tor}(W;\rho) \in \mathbb{C}$ has no indeterminacy,

i.e. $\quad \mathrm{Tor}(W;\rho) \in \mathbb{C}^*$.

Sequence of R-torsion for $\mathrm{SL}_{2N}(\mathbb{C})$-reps.

Sequence of indeuced $\mathrm{SL}_n(\mathbb{C})$-representations

An $\mathrm{SL}_2(\mathbb{C})$-representation $\rho : \pi_1(W) \to \mathrm{SL}_2(\mathbb{C})$ induces

$$\rho_n = \sigma_n \circ \rho : \pi_1(W) \xrightarrow{\rho} \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\sigma_n} \mathrm{SL}_n(\mathbb{C})$$

for $\forall n \in \mathbb{N}$.
Here σ_n is given by the action of $\mathrm{SL}_2(\mathbb{C})$ on

$$V_n = \{p(x,y) \,|\, \text{homog.}, \deg p(x,y) = n-1\} \quad \text{as}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p(x,y) = p(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}) = p(dx - by, -cx + ay)$$

Sequence of R-torsion

For $\rho : \pi_1(W) \to \mathrm{SL}_2(\mathbb{C})$, there exists a sequence

$$\mathrm{Tor}(W;\rho_2) = \mathrm{Tor}(W;\rho), \mathrm{Tor}(W;\rho_4), \ldots \mathrm{Tor}(W;\rho_{2N}), \ldots \in \mathbb{C}^*$$

Asymptotic behavior for a Hyperbolic manifold

W. Müller, P. Menal–Ferrer & J. Porti

M: a hyperbolic 3-manifold of finite volume

$$\lim_{N\to\infty} \frac{\log |\mathrm{Tor}(M; \sigma_N \circ \mathrm{hol})|}{N^2} = \frac{\mathrm{Vol}(M)}{4\pi}$$

where $\mathrm{Vol}(M)$: hyperbolic volume of M.

Remark

$\mathrm{Tor}(M; \sigma_N \circ \mathrm{hol})$ is the inverse in their conventions.

Previous work on the asymptotics of R-torsion

Asymptotic behavior for a Seifert fibered space (Y)

M: a Seifert fibered space with m exceptional fibers

$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{(2N)^2} = 0$$

$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{2N} = \log|\mathrm{Tor}(\text{regular fiber};\rho)|^{-\chi'}$$

$$\rho_{2N} = \sigma_{2N}\circ\rho : \pi_1(M) \xrightarrow{\rho} \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\sigma_{2N}} \mathrm{SL}_{2N}(\mathbb{C})$$

$$s.t. \quad \text{regular fiber} \mapsto -\mathbf{1} \mapsto -\mathbf{1}_{2N},$$

g : the genus of the base orbifold,

$2\lambda_j$: the order of the $\mathrm{SL}_2(\mathbb{C})$-matrix corresponding to j-th exceptional fiber

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$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{(2N)^2} = 0$$

$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{2N} = -\left(2 - 2g - \sum_{j=1}^{m} \frac{\lambda_j - 1}{\lambda_j}\right) \log 2$$

$$\rho_{2N} = \sigma_{2N} \circ \rho : \pi_1(M) \xrightarrow{\rho} \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\sigma_{2N}} \mathrm{SL}_{2N}(\mathbb{C})$$

$$s.t. \quad \text{regular fiber} \mapsto -\mathbf{1} \mapsto -\mathbf{1}_{2N},$$

g : the genus of the base orbifold,

$2\lambda_j$: the order of the $\mathrm{SL}_2(\mathbb{C})$-matrix corresponding to j-th exceptional fiber

Asymptotic behavior for a torus knot exterior

$T(p, q)$: the torus knot of type (p, q)

There exists

$$\rho : \pi_1(E_{T(p,q)}) \to \mathrm{SL}_2(\mathbb{C}) \text{ irreducible} \,\&\, \rho(\text{regular fiber}) = -\mathbf{1}.$$

The asymptotic behavior of R-torsion

$$\lim_{N\to\infty} \frac{|\mathrm{Tor}(E_{T(p,q)}; \rho_{2N})|}{2N} = \left(1 - \frac{1}{p'} - \frac{1}{q'}\right) \log 2$$

where p' and q' are divisors of p and q respectivily.
In particular,

$$\text{the maximum of } \lim_{N\to\infty} \frac{|\mathrm{Tor}(E_{T(p,q)}; \rho_{2N})|}{2N} = \left(1 - \frac{1}{p} - \frac{1}{q}\right) \log 2$$

Main results

$M = S^3_K(4)$ for $K = K_{[2n,-2]}$ $(n \neq 0, -1)$.

$M =$ Exterior of $T(2, 2n+1) \cup$ twisted I–b'dle over Klein bottle

Theorem (A. T. Tran and Y.)

Every irreducible $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ is induced by metabelian representation of $\pi_1(E_K)$.

$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{2N} = \frac{1}{r}(\log|\Delta_{T(2,2n+1)}(-1)| - \log 2)$$

where $r > 1$ is a divisor of $|\Delta_K(-1)|$.
In particular,

$$\text{the minimum of } \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{2N} = \frac{1}{|\Delta_K(-1)|}(\log|\Delta_{T(2,2n+1)}(-1)| - \log 2).$$

Our approach

Set $M = M_1 \cup M_2$ where

$$\begin{aligned} M_1 &= E_{T(2,2n+1)} \text{ torus knot exteriror} \\ M_2 &= \text{twisted I-bundle over the Klein bottle} \end{aligned}$$

Then

$$\mathrm{Tor}(M; \rho_{2N}) = \mathrm{Tor}(M_1; \rho_{2N}) \cdot \mathrm{Tor}(M_2; \rho_{2N})$$

Contribution of each Seifert piece

- $\rho|_{\pi_1(M_1)}$: abelian and $\rho(\text{regular fiber}) \neq -\mathbf{1}$;
- $\rho|_{\pi_1(M_2)}$: irreducible and $\mathrm{Tor}(M_2; \rho_{2N}) = 1$ $(\forall N)$

Hence

$$\lim_{N\to\infty} \frac{\log |\mathrm{Tor}(M; \rho_{2N})|}{2N} = \lim_{N\to\infty} \frac{\log |\mathrm{Tor}(M_1; \rho_{2N})|}{2N}$$

Surgery and Representations

Induced representation $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$

M: resulting manifold by 4-surgery along K

$$\begin{array}{ccc} \pi_1(E_K) & \xrightarrow{\ \rho\ } & \mathrm{SL}_2(\mathbb{C}) \\ \downarrow & \nearrow_{\bar{\rho}} & \\ \pi_1(M) = \pi_1(E_K) \,/\, \langle\!\langle m^4\ell \rangle\!\rangle & & \end{array}$$

Therefore

$$\rho(m^4\ell) = \mathbf{1} \Leftrightarrow \bar{\rho} \text{ is induced}$$

Representation space $R(M) = \{\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})\}$

$$R^{\mathrm{irr}}(M) = \{\rho \in R^{\mathrm{irr}}(E_K) \,|\, \rho(m^4\ell) = \mathbf{1}\}$$

Here “irr” means irreducible representations.

Equivalent condition for $\rho \in R^{\mathrm{irr}}(E_K)$ ($K = K_{[2n,-2]}$)

$$\rho(m^4\ell) = \mathbf{1} \Leftrightarrow \operatorname{tr}\rho(m) = 0$$

($\Leftarrow$) For any two–bridge knot K, by F. Nagasato & Y.

$$\operatorname{tr}\rho(m) = 0 \Leftrightarrow \rho(m) \overset{\text{conj.}}{\sim} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho(\ell) = \mathbf{1}$$

$$\Rightarrow \rho(m^4\ell) = \mathbf{1}$$

($\Rightarrow$) $\mathcal{M}^{\pm 1}$, $\mathcal{L}^{\pm 1}$: eigenvalues of $\rho(m)$, $\rho(\ell)$.
From $\mathcal{M}^4\mathcal{L} = 1$ and the recursive formula of A-polynomial $A(\mathcal{M}, \mathcal{L}) = 0$ by J. Hoste & P. Shanahan, one can see that

$$A(\mathcal{M}, \mathcal{M}^{-4}) = \begin{cases} \mathcal{M}^{-8n+3}(\mathcal{M} + \mathcal{M}^{-1})^{2n-1} & (n > 0) \\ \mathcal{M}^{-8|n|}(\mathcal{M} + \mathcal{M}^{-1})^{2|n|} & (n < 0). \end{cases}$$

Hence $\operatorname{tr}\rho(m) = \mathcal{M} + \mathcal{M}^{-1} = 0$.

The subset $R^{\mathrm{irr}}(M)$ in $R^{\mathrm{irr}}(E_K)$

$R^{\mathrm{irr}}(M)$ consists of metabelian representations

$K = K[2n, -2]$ and $M = S^3_K(4)$

$$\begin{aligned} R^{\mathrm{irr}}(M) &= \{\rho \in R^{\mathrm{irr}}(E_K) \mid \rho(m)^4 \rho(\ell) = \mathbf{1}\} \\ &= \{\rho \in R^{\mathrm{irr}}(E_K) \mid \operatorname{tr} \rho(m) = 0\} \\ &= \{\rho \in R^{\mathrm{irr}}(E_K) \mid \rho\text{: metabelian}\} \end{aligned}$$

Definition of metabelian representation

$\rho : \pi_1(E_K) \to \mathrm{SL}_2(\mathbb{C})$ is called *metabelian*

if $\rho([\pi_1(E_K), \pi_1(E_K)]) \subset \mathrm{SL}_2(\mathbb{C})$: abelian.

i.e., if $\gamma_1, \gamma_2 \in \pi_1(E_K)$ are null–homologous,

then $\rho(\gamma_1)$ and $\rho(\gamma_2)$ are commutative.

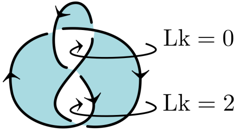
Restrictions of an $\mathrm{SL}_2(\mathbb{C})$-representation

Restriction to π_1(Torus knot exterior)

Since $\gamma \subset E_{T(2,2n+1)} \subset S^3 \setminus$ Klein bottle,
(∂ Klein bottle $= K[2n,-2]$)

$$\alpha(\gamma) = \mathrm{Lk}(\gamma, K[2n,-2]) \in 2\mathbb{Z}$$

$$\begin{aligned}\rho(\gamma) &= \rho(m^{\alpha(\gamma)} \cdot m^{-\alpha(\gamma)}\gamma) \\ &= \pm\mathbf{1} \cdot \rho(m^{-\alpha(\gamma)}\gamma) \\ &\in \rho([\pi_1(K_{[2n,-2]}), \pi_1(K[2n,-2]]))\end{aligned}$$



once punctured Klein bottle

Then $\rho|_{\pi_1(E_{T(2,2n+1)})}$ is abelian.

Restriction to π_1(twisted I-b'dle over Klein bottle)

$\rho|_{\pi_1(M_2)}$ is irreducible from the irreducibility of ρ.

R-torsion for $2N$-dim representations

Multiplicativity of R-torsion

In $\mathrm{Tor}(M;\rho_{2N}) = \mathrm{Tor}(M_1;\rho_{2N}) \cdot \mathrm{Tor}(M_2;\rho_{2N})$,

$$\mathrm{Tor}(M_1;\rho_{2N}) = \frac{\prod_{k=1}^{N} \Delta_{T(2,2n+1)}(\zeta^{2k-1})\Delta_{3_1}(\zeta^{-2k+1})}{\prod_{k=1}^{N}(\zeta^{2k-1}-1)(\zeta^{-2k+1}-1)}$$
$$\mathrm{Tor}(M_2;\rho_{2N}) = 1$$

where
M_1 : torus knot exterior, M_2 : twisted I-b'dle over Klein bottle and
$\zeta^{\pm 1}$: the eigenvalues of $\rho(\mu)$ for a meridian in $M_1 = E_{T(2,2n+1)}$.

Remark

$\exists$divisor r of $|\Delta_{K[2n,-2]}(-1)|$ s.t. the order of $\rho(\mu)$ is given by $2r$.

i.e., ζ is a $2r$-th root of unity.

The asymptotic behavior for a graph manifold

Theorem (the limit of leading coefficient)

$$
\begin{aligned}
&\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{2N} \\
&= \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M_1;\rho_{2N})|}{2N} + \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M_2;\rho_{2N})|}{2N} \\
&= \lim_{N\to\infty} \frac{\log|\prod_{k=1}^{N} \Delta_{T(2,2n+1)}(\zeta^{2k-1})\Delta_{3_1}(\zeta^{-2k+1})|}{2N} \\
&\quad - \lim_{N\to\infty} \frac{\log|\prod_{k=1}^{N}(\zeta^{2k-1}-1)(\zeta^{-2k+1}-1)|}{2N} \\
&= \frac{1}{r}\log|\Delta_{T(2,2n+1)}(-1)| - \frac{1}{r}\log 2
\end{aligned}
$$

Note

$$
\Delta_{T(2,q)}(t) = \frac{t^q+1}{t+1}
$$