Reidemeister torsion on the variety of characters

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Overview

- Goal
  - M oriented hyperbolic 3-manifold, $\mathrm{vol}(M) < \infty$, one cusp
  - $\rho\colon \pi_1 M \to \mathrm{SL}_2(\mathbb{C})$ a representation
  - The Reidemeister torsion $\tau(M, \rho)$ is invariant by conjugation of ρ, and defines a function

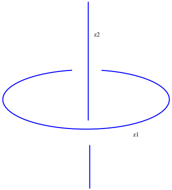
$$X(M) = \hom(\pi_1 M, \mathrm{SL}_2(\mathbb{C}) /\!/ \mathrm{SL}_2(\mathbb{C}) \dashrightarrow \mathbb{C}$$

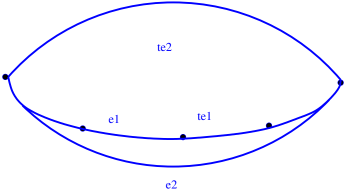
  that we want to study

- Plan of the talk:
  1. Tools: Reidemeister torsion, variety of characters, aciclicity
  2. The torsion function $X(M) \dashrightarrow \mathbb{C}$
  3. Compose the representation with $\mathrm{Sym}^n\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_{n+1}(\mathbb{C})$

Torsion: recall Lens spaces

$L(p,q) = S^3/\langle t \rangle.$ $\qquad (z_1, z_2) \overset{t}{\mapsto} (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i}{p} q} z_2).$





The lens $\tilde{e}_3 = \{0 \leq \theta_1 \leq \frac{2\pi}{p}\}$ is a fundamental domain for t

$$\begin{cases} \partial \tilde{e}_3 = (t-1)\tilde{e}_2 \\ \partial \tilde{e}_2 = (1 + t + \cdots + t^{p-1})\tilde{e}_1 \\ \partial \tilde{e}_1 = (t^r - 1)\tilde{e}_0 \end{cases} \qquad r\, q \equiv 1 \mod p$$

$$t \mapsto \xi \in \mathbb{C}, \quad \begin{matrix} \xi^p = 1 \\ \xi \neq 1 \end{matrix} \quad \begin{cases} \partial \tilde{e}_3 = (\xi - 1)\tilde{e}_2 \\ \partial \tilde{e}_2 = 0 \\ \partial \tilde{e}_1 = (\xi^r - 1)\tilde{e}_0 \end{cases} \qquad H_*(L(p,q), \xi) = 0$$

Def: (Reidemeister)$^{-1}$ $\tau(L(p,q), \xi) := |(\xi - 1)(\xi^r - 1)|^{-1}$

$$\{\tau(L(p,q),\xi)\}_{\substack{\xi^p=1 \\ \xi \neq 1}} = \{\tau(L(p,q'),\xi)\}_{\substack{\xi^p=1 \\ \xi \neq 1}} \Leftrightarrow q' = \pm q^{\pm 1} \mod p$$

Torsion of a CW-complex

- K compact CW-complex, $\rho\colon \pi_1 K \to \mathrm{SL_n}(\mathbb{C})$.

Def: $C_*(K,\rho) := \mathbb{C}^n_\rho \otimes_{\pi_1 K} C_*^{CW}(\tilde{K},\mathbb{Z})$

$$\left.\begin{array}{l} \{e^i_j\}_j \text{ } i\text{-cells of } K \\ \{v_k\}_k \text{ basis for } \mathbb{C}^n \end{array}\right\} \Rightarrow c_i = \{v_k \otimes \tilde{e}^i_j\}_{j,k} \text{ } \mathbb{C}\text{-basis for } C_i(K,\rho)$$

- $\partial_i\colon C_{i+1}(K,\rho) \to C_i(K,\rho))$.
- If $H_*(K,\rho)=0$, divide each basis into two $c_i = c'_i \sqcup c''_i$, so that $\mathrm{card}(c'_{i+1}) = \mathrm{card}(c''_i)$ and $\mathrm{minor}(\partial_i, c'_{i+1}, c''_i) \neq 0$ Notice that $c_n = c'_n$ and $c_0 = c''_0$.

Def: $\tau(K,\rho) := \prod_{i=0}^{n-1} \mathrm{minor}(\partial_i, c'_{i+1}, c''_i)^{(-1)^{i+1}} \in \mathbb{C}^*/\{\pm 1\}$

- $\tau(K,\rho)$ is a combinatorial invariant (by cellular homeos and subdivision) and invariant of the conjugacy class of ρ
- I use the opposite convention from yesterday $1/\tau(K,\rho)$!!

Variety of representations

- Algebraic structure:
  $\pi_1 K = \langle \gamma_1, \ldots, \gamma_n \mid (r_i)_{i\in I} \rangle$ finitely generated.
  $$\begin{array}{rcl} \hom(\pi_1 K, \mathrm{SL}_2(\mathbb{C})) & \hookrightarrow & \mathrm{SL}_2(\mathbb{C}) \times \overset{(n)}{\cdots} \times \mathrm{SL}_2(\mathbb{C}) \subset \mathbb{C}^{4n} \\ \rho & \mapsto & (\rho(\gamma_1), \ldots, \rho(\gamma_n)) \end{array}$$
  The $(r_i)_{i\in I}$ yield polynomial equations.

$$\left.\begin{array}{c} \{e^i_j\}_j \text{ } i\text{-cells of } K \\ \{v_1, v_2\} \text{ basis for } \mathbb{C}^2 \end{array}\right\} \Rightarrow c_i = \{v_k \otimes \tilde{e}^i_j\}_{j,k} \ \mathbb{C}\text{-basis for } C_i(K,\rho)$$

- The coefficients of ∂ in the basis c_i are polynomial on ρ, thus
  $$\tau(K,\rho) = \prod_{i=0}^{n-1} \mathrm{minor}(\partial_i, c'_{i+1}, c''_i)^{(-1)^{i+1}} \in \mathbb{C}^*/\{\pm 1\}$$
  is a rational function $\hom(\pi_1 K, \mathrm{SL}_2(\mathbb{C})) \dashrightarrow \mathbb{C}^*$.
- If $\chi(K)$ is even, then there is no sign indeterminacy,
  (as $\dim \mathbb{C}^2$ is also even, the orderings of $\{v_1, v_2\}$ and of the cells e^i_j do not affect the determinant)

Variety of characters

Assume M oriented hyperbolic 3-manifold, $vol(M) < \infty$, with 1 cusp.

Def: $$X(M) = \hom(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))/\!/\,\mathrm{SL}_2(\mathbb{C})$$

Any polynomial/rational function $\hom(\pi_1 M, \mathrm{SL}_2(\mathbb{C})) \dashrightarrow \mathbb{C}$ invariant by conjugation induces a function $X(M) \dashrightarrow \mathbb{C}$.

- Distinguished component $X_0(M)$:
  component of $X(M)$ that contains the lift of the holonomy
  $\pi_1 M \to \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm \mathrm{Id}\}$
- Such a lift exits (Culler, Thurston)
  $\{\text{lifts of hol to } \mathrm{SL}_2(\mathbb{C})\} \leftrightarrow \{\text{spin structures on } M\}$
  because $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Frame}(\mathbb{H}^3)$ and $\mathrm{SL}_2(\mathbb{C}) \cong \mathrm{Spin}(\mathbb{H}^3)$
- Since M has one cusp, $X_0(M)$ is a $\mathbb{C}$-curve.
  (it contains lift of holonomy of Dehn fillings)

Question Before defining $\begin{array}{ccc} X_0(M) & \dashrightarrow & \mathbb{C} \\ [\rho] & \mapsto & \tau(M,\rho) \end{array}$, does $H_*(M,\rho) = 0$?

Acyclicity

Assume M oriented hyperbolic 3-manifold, $vol(M) < \infty$, with 1 cusp.

Lemma $\rho_0 : \pi_1 M \to \mathrm{SL}_2(\mathbb{C})$ lift of hol. $\Rightarrow H^*(M, \rho_0) = H_*(M, \rho_0) = 0$

Proof (Sketch) $\overline{M} \cong M \cup \partial \overline{M}$, $\partial \overline{M} \cong T^2$

- L^2 forms in $\Omega^*(M, E_\rho)$ are exact (Ragunathan, Garland, Matsushima-Murakami, ... 1960's):

$$H^1(\overline{M}, \partial \overline{M}, \rho) \xrightarrow{0} H^1(M, \rho) \to H^1(\partial \overline{M}, \rho)$$

- Up to conj. $\rho_0(\pi_1 \partial \overline{M}) \subset \pm \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ but $\rho_0(\pi_1 \partial \overline{M}) \not\subset \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$

$\Rightarrow$ $H^0(\partial \overline{M}, \rho_0) \cong (\mathbb{C}^2)^{\rho_0(\pi_1 \partial \overline{M})} = 0$
$\Rightarrow H^*(\partial \overline{M}, \rho_0) = 0$ (by Poincaré duality$+\chi(T^2) = 0$)
$\Rightarrow H^1(M, \rho_0) = 0$. (by the exact seq.)

- In addition $H^0(M, \rho_0) \cong (\mathbb{C}^2)^{\rho_0(\pi_1 M)} = 0$.
  $+$ Euler characteristic & duality homology/cohomology. □

Cor: Acyclicity holds in $X_0(M)$ except for a finite set, by semicontinuity of (co-)homology.

The torsion function

Assume M or. hyp 3-manifold, $vol(M) < \infty$, with 1 cusp.

Def: $\mathbb{T}_M(\rho) = \begin{cases} \tau(M,\rho) & \text{if } H_*(M,\rho) = 0 \\ 0 & \text{if } H_*(M,\rho) \neq 0 \text{ and } [\rho] \text{ nontrivial} \end{cases}$

- The sign is well defined (since $\dim \mathbb{C}^2$ and $\chi(M)$ are even, ordering of cells and of basis for $\mathbb{C}^2$ do not change the sign)

Remark: $\forall [\rho] \in X_0(M)$ nontrivial, $H^0(M,\rho) \cong (\mathbb{C}^2)^{\rho(\pi_1 M)} = 0$

- $\mathbb{T}_M\colon \hom(\pi_1 M, SL_2(\mathbb{C})) - \{\rho \mid \operatorname{tr}\rho \equiv 2\} \to \mathbb{C}$ is algebraic
  Since M collapses to a 2-dim CW-complex.
  and $H_0(M,\rho) \cong H^0(M,\rho) = 0$ for ρ nontrivial,
  denominators in the def of torsion do not vanish.

Hence it defines an algebraic function $\mathbb{T}_M : X_0(M) - \{trivial\} \to \mathbb{C}$

Remark: When $\beta_1(M) = 1$, characters in $X_0(M)$ are nontrivial.
hence $\mathbb{T}_M : X_0(M) \to \mathbb{C}$ polynomial

Example

- $M = S^3 -$fig-8 knot.
  $\pi_1(M) = \langle a, b, m \mid mam^{-1} = ab, mbm^{-1} = bab \rangle$
- $X_0(M) = \{(x, y) \in \mathbb{C}^2 \mid x^2 - x - 1 = (x-1)y^2\}$
  $x = \mathrm{tr}_a$, $y = \mathrm{tr}_m = \mathrm{tr}_{ma} = \mathrm{tr}_{mb}$, $\mathrm{tr}_b = x/(x-1) = y^2 - 1 - x$
- Kitano (1994): $\mathbb{T}_M = 2 - 2y = 2 - 2\,\mathrm{tr}_m$
- Can define a twisted Alexander polynomial $\Delta_M([\rho], t)$, with $\Delta_M([\rho], 1) = \mathbb{T}_M([\rho])$:

$$\Delta_M(\cdot, t) = t^2 - 2t\,y + 1$$

Conj. (Dunfield, Friedl, and Jackson 2011)
For $M = S^3 -$hyp knot,
$\deg \Delta_M([\rho_0], t) = 2\,\mathrm{genus}(K)$ and
$\Delta_M([\rho_0], t)$ is monic iff M is fibered.

Branched coverings on the Fig-8 knot

- $M_n \to S^3$ n-fold cyclic branched covering, branched over the fig-eight knot K, $\Sigma_n = \widetilde{K} \subset M_n$ lift of branching locus.
- $\tau(M_n, \rho_n) = \tau(M_n - \Sigma_n, \rho_n)\tau(\Sigma_n, \rho_n)$
  $= \tau(M_n - \Sigma_n, \rho_n)\frac{1}{2(1-\cosh(\lambda(\Sigma_n))}$
  ($\rho_n \colon \pi_1 M_n \to \mathrm{SL}_2(\mathbb{C})$ lift of holonomy, $\lambda=$ complex length)
- Recall that $\Delta_M(y,t) = t^2 - 2yt + 1$. Fox formula:

$$\tau(M_n - \Sigma_n, \rho_n) = \prod_{k=0}^{n-1} \Delta(\pm 2\cos(\pi/n), e^{2\pi i \frac{k}{n}})$$

- Hence, since $\lambda(\Sigma_n) = \sqrt{3}\pi/n + O(1/n^3)$:

$$\lim_{n\to\infty} \frac{\log|\tau(M_n, \rho_n)|}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log|\Delta(\pm 2, e^{2\pi i \frac{k}{n}})|$$

$$= \frac{1}{2\pi}\int_{|z|=1} \log|\Delta(\pm 2, z)| = \frac{1}{2\pi}\int_{|z|=1} \log|z^2 \mp 4z + 1| = \log(2+\sqrt{3})$$

Dehn fillings

Assume M oriented hyperbolic 3-manifold, $vol(M) < \infty$, with 1 cusp.

- Choose a frame for $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^2$.
  $M_{p/q}$ Dehn filling (with filling meridian $\pm(p, q) \in \mathbb{Z}^2$).
  It is hyperbolic for $p^2 + q^2$ large, and
  a lift of its holonomy $[\rho_{p/q}|_{\pi_1 M}] \to [\rho_0]$, holonomy of M.
- $\tau(M_{p/q}, \rho_{p/q})$ is a topological invariant of the spin mfld $M_{p/q}$

Thm: $\tau(M_{p/q}) = \mathbb{T}_M(\rho_{p/q}|_M) \frac{1}{2(1-\cosh(\lambda(\gamma_{p/q})))}$

where $\lambda(\gamma_{p/q}) \in \mathbb{C}$ complex length of $\gamma_{p/q}$ soul of filling torus

Proof: Mayer-Vietoris to the pair $(\overline{M}, D^2 \times S^1)$ & $\tau(T^2) = 1$ □

Cor: $|\tau(M_{p/q}, \rho_{p/q})|$ is dense in $[\frac{1}{4}|\mathbb{T}_M(\rho_0)|, +\infty)$

(Because $\mathrm{Re}(\lambda(\gamma_{p/q})) \to 0$ as $p^2 + q^2 \to \infty$
but $\mathrm{Im}(\lambda(\gamma_{p/q}))$ is dense in $\mathbb{R}/2\pi\mathbb{Z}$)

- $\tau(M_{p/q}, \rho_{p/q})$ distinguishes spin structures
  (for $M = S^3 -$fig 8, $\mathbb{T}_M(y) = 2 - 2y$ and $\mathbb{T}_M(2) \neq \mathbb{T}_M(-2)$) .

Representations of $\mathsf{SL}_2(\mathbb{C})$

- $\mathsf{Sym}^n : \mathsf{SL}_2(\mathbb{C}) \to \mathsf{SL}_{n+1}(\mathbb{C})$ irreducible
  $\mathbb{C}^{n+1} = \mathsf{Sym}^n(\mathbb{C}^2) = \{$homog. polynomials on $\mathbb{C}^2$ of deg $n\}$
  If $\mathbb{C}^2 = \langle v_1, v_2 \rangle$, then $\mathsf{Sym}^n(\mathbb{C}^2) = \langle v_1^n, v_1^{n-1} v_2, \cdots, v_2^n \rangle$.

Aim Want to define $\mathbb{T}_M^{n+1}([\rho]) = \tau(M, \mathsf{Sym}^n \circ \rho)$

- For ρ_0 lift of holonomy, $H^*(M, \mathsf{Sym}^n \circ \rho_0) = 0$ iff $n+1$ even.
  Can define $\mathbb{T}_M^{n+1}$.
- For $n+1$ odd, $H^i(M, \mathsf{Sym}^n \circ \rho_0)) = \mathbb{C}$ for $i = 1, 2$,
  and there are natural choices of basis for $H^i(M, \mathsf{Sym}^n \circ \rho_0)$,
  depending on peripheral elements $1 \neq \gamma \in \pi_1 T^2$.

$$\begin{cases} \mathbb{T}_M^{n+1} : X_0(M) \dashrightarrow \mathbb{C} & \text{for } n+1 \text{ even} \\ \mathbb{T}_{M,\gamma}^{n+1} : X_0(M) \dashrightarrow \mathbb{C} & \text{for } n+1 \text{ odd, } 1 \neq \gamma \in \pi_1 T^2 \end{cases}$$

- Can study its domain, Dehn fillings, twisted polynomials, etc...

Representations of $\mathsf{SL}_2(\mathbb{C})$ *(continued)*

- $\mathbb{T}_M^{n+1}([\rho]) = \tau(M, \mathrm{Sym}^n \circ \rho)$, with $\mathrm{Sym}^n \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_{n+1}(\mathbb{C})$

$$\begin{cases} \mathbb{T}_M^{n+1} \colon X_0(M) \dashrightarrow \mathbb{C} & \text{for } n+1 \text{ even} \\ \mathbb{T}_{M,\gamma}^{n+1} \colon X_0(M) \dashrightarrow \mathbb{C} & \text{for } n+1 \text{ odd, } 1 \neq \gamma \in \pi_1 T^2 \end{cases}$$

- $M_{p/q}$ Dehn filing

$$|\tau(M_{p/q}, \mathrm{Sym}^n \circ \rho_{p/q})| \begin{cases} \text{dense in } \left[\frac{1}{2^{n+1}} |\mathbb{T}_M^{n+1}(\rho_0)|, \infty\right) \text{ for } n+1 \text{ even} \\ \text{goes to } \infty \text{ as } p^2 + q^2 \to \infty \text{ for } n+1 \text{ odd} \end{cases}$$

- (Menal-Ferrer-P, based on Müller's work)

$$\lim_{k\to\infty} \frac{\log |\mathbb{T}_M^{2k+2}([\rho_0])|}{(2k+2)^2} = \lim_{k\to\infty} \frac{\log |\mathbb{T}_{M,\gamma}^{2k+1}([\rho_0])|}{(2k+1)^2} = \frac{1}{4\pi} \mathrm{vol}(M)$$

- For $n+1 = 3$, $\mathrm{Sym}^2 = \mathrm{Ad}$ (1/(yesterday's torsion))

Conj $K \subset S^3$ hyperbolic knot. $\langle K \rangle_N =$ Kashaev invariant, $N \in \mathbb{N}$

$$\langle K \rangle_N = e^{\mathrm{CS} + i\,\mathrm{V}} \frac{1}{\sqrt{2\pi\, i\, \tau}} N^{3/2} (1 + O(\frac{1}{N}))$$

where $\mathrm{CS} + i\,\mathrm{V} = (\text{Chern-Simons} + i\,\text{Volume})(S^3 - K)$

$\tau = \tau(S^3 - K, \mathrm{Ad} \circ \mathrm{hol}, \text{meridian})$

Examples of torsion for higher representations

- $M = S^3 - \text{fig-8}$, $y = \mathrm{tr}_m$.
- $n+1$ even:

$$\begin{aligned}
\mathbb{T}^2_M &= 2\,(1-y) \\
\mathbb{T}^4_M &= -\left(y^2 - 2\,y - 2\right)^2 \\
\mathbb{T}^6_M &= 2(y-1)\left(y^8 + 2y^7 - 13y^6 - 20y^5 + 49y^4 + 48y^3 - 33y^2 - 18y - 18\right) \\
\mathbb{T}^8_M &= -\,(y-1)^2\left(2\,y^7 - 4\,y^6 - 21\,y^5 + 19\,y^4 + 57\,y^3 + 13\,y^2 - 18\,y - 6\right)^2 \\
\mathbb{T}^{10}_M &= 2\,(y-1)\left(y^{12} + 2\,y^{11} - 13\,y^{10} - 13\,y^9 + 27\,y^8 - y^7 + 95\,y^6 + 90\,y^5\right. \\
&\qquad \left. -148\,y^4 - 74\,y^3 + 61\,y^2 + 12\,y - 6\right)^2 .
\end{aligned}$$

- $n+1$ odd:

  l =longitude, m =meridian

$$\begin{aligned}
\mathbb{T}^3_{M,l} &= \pm(5 - 2y^2), \\
\mathbb{T}^3_{M,m} &= \pm\tfrac{1}{2}\sqrt{(y^2-1)(y^2-5)} = \pm(2\,x + 1 - y^2)/2 \\
\mathbb{T}^5_{M,l} &= \pm 4(1 - 6y^2 + y^4),
\end{aligned}$$

THANKS FOR YOUR ATTENTION