Characteristic classes of homological surface bundles and four-dimensional topology

Shigeyuki MORITA

based on jw/w Takuya SAKASAI and Masaaki SUZUKI

October 25, 2016

Contents

An "enlargement" $\mathcal{H}_{g,1}$ of the mapping class group (1)

Mapping class groups:

$$\mathcal{M}_g = \pi_0 \operatorname{Diff}^+ \Sigma_g, \quad \mathcal{M}_{g,1} = \pi_0 \operatorname{Diff}(\Sigma_g, D^2)$$

Another description:

$$\mathcal{M}_{g,1} = \{(\Sigma_{g,1} \times I, \varphi)\,;\, \varphi : \Sigma_{g,1} \overset{\text{rel } \partial}{\cong} \Sigma_{g,1} \times \{1\}\} \Big/ \text{isotopy}$$

An "enlargement" $\mathcal{H}_{g,1}$ of the mapping class group (1)

Mapping class groups:

$$\mathcal{M}_g = \pi_0 \operatorname{Diff}^+\Sigma_g, \quad \mathcal{M}_{g,1} = \pi_0 \operatorname{Diff}(\Sigma_g, D^2)$$

Another description:

$$\mathcal{M}_{g,1} = \{(\Sigma_{g,1} \times I, \varphi)\,;\, \varphi : \Sigma_{g,1} \overset{\text{rel } \partial}{\cong} \Sigma_{g,1} \times \{1\}\} \,/\,\text{isotopy}$$

Group of homology cobordism classes of homology cylinders:

Garoufalidis-Levine (based on Goussarov and Habiro):

$$\mathcal{H}_{g,1} = \{(\textbf{homology } \Sigma_{g,1} \times I, \varphi)\,;\, \varphi : \Sigma_{g,1} \overset{\text{rel } \partial}{\cong} \Sigma_{g,1} \times \{1\}\} \,/\,\text{homology cobordism}$$

An "enlargement" $\mathcal{H}_{g,1}$ of the mapping class group (2)

two versions:

$$\mathcal{H}_{g,1}^{\text{smooth}} \xrightarrow[\text{surjective}]{\text{big kernel, Freedman}} \mathcal{H}_{g,1}^{\text{top}}$$

enlargements of $\mathcal{M}_{g,1}$

An "enlargement" $\mathcal{H}_{g,1}$ of the mapping class group (2)

two versions:

$$\mathcal{H}_{g,1}^{\text{smooth}} \xrightarrow[\text{surjective}]{\text{big kernel, Freedman}} \mathcal{H}_{g,1}^{\text{top}}$$

enlargements of $\mathcal{M}_{g,1}$

$\boldsymbol{\Theta}^3 := \{\text{homology } 3\text{-spheres}\}/\text{smooth homology cobordism}$

infinite rank by Furuta, Fintushel-Stern

Define a group $\overline{\mathcal{H}}_{g,1}$ by the following central extension

$$0 \to \boldsymbol{\Theta}^3 = \mathcal{H}_{0,1}^{\text{smooth}} \to \mathcal{H}_{g,1}^{\text{smooth}} \to \overline{\mathcal{H}}_{g,1} \to 1$$

Problem

Study the Euler class

$$\chi(\mathcal{H}_{g,1}^{\textit{smooth}}) \in H^2(\overline{\mathcal{H}}_{g,1}; \mathbf{\Theta}^3)$$

Problem

Study the Euler class

$$\chi(\mathcal{H}_{g,1}^{\textit{smooth}}) \in H^2(\overline{\mathcal{H}}_{g,1}; \boldsymbol{\Theta}^3)$$

One of the foundational results of Freedman:

Theorem (Freedman)

Any homology 3-sphere bounds a contractible topological 4-manifold so that $\boldsymbol{\Theta}^3_{\textit{top}} = 0$

It follows that $\mathcal{H}_{g,1}^{\text{smooth}} \to \mathcal{H}_{g,1}^{\text{top}}$ factors through $\overline{\mathcal{H}}_{g,1}$

An “enlargement” $\mathcal{H}_{g,1}$ of the mapping class group (4)

$$\boldsymbol{\Theta}^3 \to \mathcal{H}_{g,1}^{\text{smooth}} \to \overline{\mathcal{H}}_{g,1} \xrightarrow{\text{Freedman}} \mathcal{H}_{g,1}^{\text{top}}$$

Problem (about “Picard groups”)

Study the following homomorphisms ($g \geq 3$)

$$H^2(\mathcal{H}_{g,1}^{\text{top}}) \to H^2(\overline{\mathcal{H}}_{g,1}) \to H^2(\mathcal{H}_{g,1}^{\text{smooth}}) \to H^2(\mathcal{M}_{g,1}) \overset{\text{Harer}}{\cong} \mathbb{Z}$$

$$\Theta^3 \to \mathcal{H}_{g,1}^{\text{smooth}} \to \overline{\mathcal{H}}_{g,1} \xrightarrow{\text{Freedman}} \mathcal{H}_{g,1}^{\text{top}}$$

Problem (about "Picard groups")

Study the following homomorphisms ($g \geq 3$)

$$H^2(\mathcal{H}_{g,1}^{\text{top}}) \to H^2(\overline{\mathcal{H}}_{g,1}) \to H^2(\mathcal{H}_{g,1}^{\text{smooth}}) \to H^2(\mathcal{M}_{g,1}) \overset{\text{Harer}}{\cong} \mathbb{Z}$$

∞-rank? ∞-rank? $\cong$?

Representations of $\mathcal{H}_{g,1}$ (1)

Theorem (Dehn-Nielsen-Zieschang)

- $\mathcal{M}_g \cong \mathrm{Out}^+\pi_1\Sigma_g$ *(outer automorphism group)*
- $\mathcal{M}_{g,1} \cong \{\varphi \in \mathrm{Aut}\,\pi_1\Sigma_{g,1}; \varphi(\zeta) = \zeta\}$ ζ : *boundary curve*

"differentiate" $\Rightarrow$

Representations of $\mathcal{H}_{g,1}$ (1)

Theorem (Dehn-Nielsen-Zieschang)

- $\mathcal{M}_g \cong \mathrm{Out}^+ \pi_1 \Sigma_g$ *(outer automorphism group)*
- $\mathcal{M}_{g,1} \cong \{\varphi \in \mathrm{Aut}\, \pi_1 \Sigma_{g,1}; \varphi(\zeta) = \zeta\}$ ζ : *boundary curve*

"differentiate" $\Rightarrow$

Definition ("Lie algebra" of $\mathcal{M}_{g,1}$)

$\mathfrak{h}_{g,1} = \{$symplectic derivation of the free Lie algebra $\mathcal{L}(H_{\mathbb{Q}})\}$

$\mathfrak{h}_{g,1} = \bigoplus_{k=0}^{\infty} \mathfrak{h}_{g,1}(k)$: symplectic derivation Lie algebra of $\mathcal{L}(H_{\mathbb{Q}})$

very important in low dimensional topology

Representations of $\mathcal{H}_{g,1}$ (2)

Mal'cev nilpotent completion of $\pi_1\Sigma_{g,1}$:

$$\cdots \to N_{d+1} \to N_d \to \cdots \to N_1 = H_{\mathbb{Q}} \to 0 \quad (H_{\mathbb{Q}} = H_1(\Sigma_{g,1};\mathbb{Q}))$$

$\Rightarrow$ obtain a series of representations of $\mathcal{M}_{g,1}$:

$$\rho_\infty = \{\rho_d\}_d : \mathcal{M}_{g,1} \to \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d \quad (\rho_d : \mathcal{M}_{g,1} \to \mathrm{Aut}_0\, N_d)$$

Representations of $\mathcal{H}_{g,1}$ (2)

Mal'cev nilpotent completion of $\pi_1\Sigma_{g,1}$:

$$\cdots \to N_{d+1} \to N_d \to \cdots \to N_1 = H_{\mathbb{Q}} \to 0 \quad (H_{\mathbb{Q}} = H_1(\Sigma_{g,1};\mathbb{Q}))$$

$\Rightarrow$ obtain a series of representations of $\mathcal{M}_{g,1}$:

$$\rho_\infty = \{\rho_d\}_d : \mathcal{M}_{g,1} \to \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d \quad (\rho_d : \mathcal{M}_{g,1} \to \mathrm{Aut}_0\, N_d)$$

associated embedding of Lie algebras:

$$\tau : \bigoplus_{d=1}^{\infty} \mathcal{M}_{g,1}(d)/\mathcal{M}_{g,1}(d+1) \overset{\text{small}}{\subset} \mathfrak{h}_{g,1}^{+} \overset{\text{ideal}}{\subset} \mathfrak{h}_{g,1}$$

$\mathcal{M}_{g,1}(d) := \mathrm{Ker}\, \rho_d$ Johnson filtration

Representations of $\mathcal{H}_{g,1}$ (3)

Stallings' theorem $\Rightarrow$

Theorem (Garoufalidis-Levine, Habegger)

There exists a homomorphism

$$\tilde{\rho}_\infty : \mathcal{H}_{g,1}^{\mathrm{top}} \to \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d$$

which extends ρ_∞, each finite factor $\tilde{\rho}_d : \mathcal{H}_{g,1}^{\mathrm{top}} \to \mathrm{Aut}_0\, N_d$ is surjective over $\mathbb{Z}$ for any $d \geq 1$

Representations of $\mathcal{H}_{g,1}$ (3)

Stallings' theorem $\Rightarrow$

Theorem (Garoufalidis-Levine, Habegger)

There exists a homomorphism

$$\tilde{\rho}_\infty : \mathcal{H}_{g,1}^{\mathrm{top}} \longrightarrow \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d$$

which extends ρ_∞, each finite factor $\tilde{\rho}_d : \mathcal{H}_{g,1}^{\mathrm{top}} \to \mathrm{Aut}_0\, N_d$ is surjective over $\mathbb{Z}$ for any $d \geq 1$

$$\begin{array}{ccc} \mathcal{M}_{g,1}(d) & \xrightarrow[\text{image small}]{\tau_d} & \mathfrak{h}_{g,1}(d) \\ \cap\downarrow & & \| \\ \mathcal{H}_{g,1}^{\mathrm{top}}(d) & \xrightarrow[\text{surjective}]{\tilde{\tau}_d} & \mathfrak{h}_{g,1}(d) \end{array}$$

Representations of $\mathcal{H}_{g,1}$ (4)

$$\begin{array}{ccccc} & & \mathcal{M}_{g,1} & \xrightarrow[\text{injective}]{\rho_\infty} & \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d \\ & & \cap\downarrow & & \| \\ \mathcal{H}_{g,1}^{\mathrm{smooth}} & \xrightarrow{\text{surjective}} & \mathcal{H}_{g,1}^{\mathrm{top}} & \xrightarrow{\tilde{\rho}_\infty} & \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d \end{array}$$

$\Rightarrow$ obtain

$$\begin{array}{ccccc} & & \mathcal{M}_{g,1} & \xrightarrow[\text{injective}]{\rho_\infty} & \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d \\ & & \cap\downarrow & & \| \\ \mathcal{H}_{g,1}^{\text{smooth}} & \xrightarrow{\text{surjective}} & \mathcal{H}_{g,1}^{\text{top}} & \xrightarrow{\tilde{\rho}_\infty} & \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d \end{array}$$

$\Rightarrow$ obtain

$$\tilde{\rho}_\infty^* : \lim_{d\to\infty} H^*(\mathrm{Aut}_0\, N_d) \to H^*(\mathcal{H}_{g,1}^{\text{top}};\mathbb{Q})$$

$$\tilde{\rho}_\infty^* : \lim_{g\to\infty} \lim_{d\to\infty} H^*(\mathrm{Aut}_0\, N_d) \to H^*(\mathcal{H}_{g,1}^{\text{top}};\mathbb{Q})$$

Representations of $\mathcal{H}_{g,1}$ (5)

$\mathrm{Aut}_0\, N_d$ is a linear algebraic group and we have

$$\mathrm{Aut}_0\, N_d \cong \mathrm{IAut}_0\, N_d \rtimes \mathrm{Sp}(2g, \mathbb{Q})$$

$$\mathrm{Lie}(\mathrm{IAut}_0\, N_d) \cong \mathfrak{h}_{g,1}^+[d] \quad \text{(truncated)}$$

Proposition

$$\lim_{g\to\infty} \lim_{d\to\infty} H^*(\mathrm{Aut}_0\, N_d) \cong H_c^*(\hat{\mathfrak{h}}_{\infty,1}^+)^{\mathrm{Sp}} \otimes H^*(\mathrm{Sp}(2\infty, \mathbb{Q}); \mathbb{Q})$$

$\widehat{\mathfrak{h}}_{\infty,1}^+$: completion of $\mathfrak{h}_{\infty,1}^+ = \lim_{g\to\infty} \mathfrak{h}_{g,1}^+$

Representations of $\mathcal{H}_{g,1}$ (5)

$\mathrm{Aut}_0\, N_d$ is a linear algebraic group and we have

$$\mathrm{Aut}_0\, N_d \cong \mathrm{IAut}_0\, N_d \rtimes \mathrm{Sp}(2g, \mathbb{Q})$$

$$\mathrm{Lie}(\mathrm{IAut}_0\, N_d) \cong \mathfrak{h}^+_{g,1}[d] \quad \text{(truncated)}$$

Proposition

$$\lim_{g\to\infty} \lim_{d\to\infty} H^*(\mathrm{Aut}_0\, N_d) \cong H^*_c(\hat{\mathfrak{h}}^+_{\infty,1})^{\mathrm{Sp}} \otimes H^*(\mathrm{Sp}(2\infty, \mathbb{Q}); \mathbb{Q})$$

$\widehat{\mathfrak{h}}^+_{\infty,1}$: completion of $\mathfrak{h}^+_{\infty,1} = \lim_{g\to\infty} \mathfrak{h}^+_{g,1}$

$\Rightarrow$ obtain

$$\tilde{\rho}^*_\infty : H^*_c(\hat{\mathfrak{h}}^+_{\infty,1})^{\mathrm{Sp}} \otimes H^*(\mathrm{Sp}(2\infty, \mathbb{Q}); \mathbb{Q}) \to H^*(\mathcal{H}^{\mathrm{top}}_{g,1}; \mathbb{Q})$$

Lie version of Kontsevich graph homology

By using theory of Outer Space due to Culler and Vogtmann:

Theorem (Kontsevich, Lie version)

$$PH_c^k(\widehat{\mathfrak{h}}_{\infty,1}^+)_{2n}^{\mathrm{Sp}} \cong H_{2n-k}(\mathrm{Out}\, F_{n+1}; \mathbb{Q}) \Rightarrow$$

$$H_c^*(\widehat{\mathfrak{h}}_{\infty,1}^+)^{\mathrm{Sp}} \cong \Lambda \left[\bigoplus_{n\geq 2} H_*(\mathrm{Out}\, F_n; \mathbb{Q}) \right]$$

Λ: free associative algebra

$\mathrm{degree}\,(x) = 2n - 2 - k \;(x \in H_k(\mathrm{Out}\, F_n; \mathbb{Q}))$

Kontsevich's theorem and homology of $\mathrm{Out}\, F_n$ (2)

$$\bigoplus_{n\geq 2} H_{2n-3}(\mathrm{Out}\, F_n;\mathbb{Q}) \;\Leftrightarrow\; PH_c^1(\widehat{\mathfrak{h}}_{\infty,1}) \;\overset{\text{dual}}{\Leftrightarrow}\; H_1(\mathfrak{h}^+_{\infty,1})_{\mathrm{Sp}}$$

Culler-Vogtmann: $\mathrm{vcd}(\mathrm{Out}\, F_n) = 2n-3$

Problem

What are the generators: $H_1(\mathfrak{h}^+_{\infty,1})$ for the Lie algebra $\mathfrak{h}^+_{\infty,1}$?

Kontsevich's theorem and homology of $\mathrm{Out}\, F_n$ (2)

$$\bigoplus_{n\geq 2} H_{2n-3}(\mathrm{Out}\, F_n;\mathbb{Q}) \;\Leftrightarrow\; PH^1_c(\widehat{\mathfrak{h}}_{\infty,1}) \;\overset{\text{dual}}{\Leftrightarrow}\; H_1(\mathfrak{h}^+_{\infty,1})_{\mathrm{Sp}}$$

Culler-Vogtmann: $\mathrm{vcd}(\mathrm{Out}\, F_n) = 2n-3$

Problem

What are the generators: $H_1(\mathfrak{h}^+_{\infty,1})$ for the Lie algebra $\mathfrak{h}^+_{\infty,1}$?

$$\bigoplus_{n\geq 2} H_{2n-4}(\mathrm{Out}\, F_n;\mathbb{Q}) \;\Leftrightarrow\; PH^2_c(\widehat{\mathfrak{h}}_{\infty,1})$$

Problem

What is the second cohomology of the Lie algebra $\mathfrak{h}_{\infty,1}$?

Kontsevich's theorem and homology of $\mathrm{Out}\, F_n$ (3)

Cohomology of $\mathrm{Out}\, F_n$ and $H_1(\mathfrak{h}_{\infty,1}), H_c^2(\hat{\mathfrak{h}}_{\infty,1})$

Generators for $\mathfrak{h}^+_{g,1}$ $(= H_1(\mathfrak{h}^+_{g,1}))$:

$\wedge^3 H_{\mathbb{Q}} = \mathfrak{h}_{g,1}(1)$ Johnson

Kontsevich's theorem and homology of $\mathrm{Out}\,F_n$ (3)

Cohomology of $\mathrm{Out}\,F_n$ and $H_1(\mathfrak{h}_{\infty,1}), H_c^2(\hat{\mathfrak{h}}_{\infty,1})$

Generators for $\mathfrak{h}_{g,1}^+$ $(= H_1(\mathfrak{h}_{g,1}^+))$:

$\wedge^3 H_{\mathbb{Q}} = \mathfrak{h}_{g,1}(1)$ Johnson

traces: $\bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}$ Morita

Cohomology of $\mathrm{Out}\, F_n$ and $H_1(\mathfrak{h}_{\infty,1}), H_c^2(\hat{\mathfrak{h}}_{\infty,1})$

Generators for $\mathfrak{h}_{g,1}^+$ $(= H_1(\mathfrak{h}_{g,1}^+))$:

$\wedge^3 H_{\mathbb{Q}} = \mathfrak{h}_{g,1}(1)$ Johnson

traces: $\bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}$ Morita

Theorem (Conant-Kassabov-Vogtmann)

$H_1(\mathfrak{h}_{g,1}^+) \cong \wedge^3 H_{\mathbb{Q}}$ (*Johnson,* 0*-loop*)
$\oplus \left(\oplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}\right)$ (*M., trace maps:* 1*-loop*)
$\oplus \left(\oplus_{k=1}^{\infty} [2k+1, 1]_{\mathrm{Sp}} \oplus\right.$ *other part*$)$ (2*-loops*)
$\oplus$ *non-trivial ?* ($3, 4, \ldots$*-loops*) *?* : *deep question*

Theorem (Bartholdi)

$$H_k(\mathrm{Out}\, F_7; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} \ (k = 0, 8, 11) \\ 0 \ (\textit{otherwise}) \end{cases}$$

$$\overset{\text{Kontsevich}}{\Rightarrow} \quad H^1_c(\hat{\mathfrak{h}}^+_{\infty,1})^{\mathrm{Sp}}_{12} \cong \mathbb{Q}$$

Sakasai-Suzuki-M. have given a direct proof of this fact

without using Kontsevich's theorem, and furthermore

Kontsevich's theorem and homology of $\mathrm{Out}\, F_n$ (4)

Theorem (Bartholdi)

$$H_k(\mathrm{Out}\, F_7; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} \ (k = 0, 8, 11) \\ 0 \ (\textit{otherwise}) \end{cases}$$

$$\overset{\text{Kontsevich}}{\Rightarrow} \quad H^1_c(\hat{\mathfrak{h}}^+_{\infty,1})^{\mathrm{Sp}}_{12} \cong \mathbb{Q}$$

Sakasai-Suzuki-M. have given a direct proof of this fact

without using Kontsevich's theorem, and furthermore

Theorem (Massuyeau-Sakasai)

(i) $\mathcal{H}_{g,1} \overset{\textit{homo.}}{\rightarrow} \hat{H}_1(\mathfrak{h}^+_{g,1}) \rtimes \mathrm{Sp}(2g, \mathbb{Z})$ *with dense image*

(ii) $H_1(\mathcal{H}_{g,1}; \mathbb{Q}) \neq 0$ (*sharp contrast with:* $\mathcal{M}_g$ *is perfect* ($g \geq 3$))

Kontsevich's theorem and homology of $\mathrm{Out}\, F_n$ (5)

Construction of elements of $H_c^2(\hat{\mathfrak{h}}_{\infty,1})$

$$\text{trace maps}: \mathfrak{h}_{g,1}^{+} \to \bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}},\ H^2(S^{2k+1} H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q} \Rightarrow$$

$$\mathbf{t}_{2k+1} \in H_c^2(\widehat{\mathfrak{h}}_{\infty,1})_{4k+2} \overset{K.}{\cong} H_{4k}(\mathrm{Out}\, F_{2k+2}; \mathbb{Q})$$

$\mu_k \in H_{4k}(\mathrm{Out}\, F_{2k+2}; \mathbb{Q})$ $(k = 1, 2, \ldots)$ Morita classes

Construction of elements of $H_c^2(\hat{\mathfrak{h}}_{\infty,1})$

trace maps : $\mathfrak{h}_{g,1}^+ \to \bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}, \ H^2(S^{2k+1} H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q} \Rightarrow$

$$\mathbf{t}_{2k+1} \in H_c^2(\widehat{\mathfrak{h}}_{\infty,1})_{4k+2} \overset{K.}{\cong} H_{4k}(\mathrm{Out}\, F_{2k+2}; \mathbb{Q})$$

$\mu_k \in H_{4k}(\mathrm{Out}\, F_{2k+2}; \mathbb{Q}) \ (k = 1, 2, \ldots)$ Morita classes

Theorem (non-triviality of μ_k)

$\mu_1 \neq 0 \in H_4(\mathrm{Out}\, F_4; \mathbb{Q})$ (*M. 1999*)

$\mu_2 \neq 0 \in H_8(\mathrm{Out}\, F_6; \mathbb{Q})$ (*Conant-Vogtmann 2004*)

$\mu_3 \neq 0 \in H_{12}(\mathrm{Out}\, F_8; \mathbb{Q})$ (*Gray 2011*)

Kontsevich's theorem and homology of $\mathrm{Out}\, F_n$ (6)

$H_*(\mathrm{Out}\, F_n; \mathbb{Q})$ computed for $n \leq 7$: only four non-trivial parts

$$H_4(\mathrm{Out}\, F_4; \mathbb{Q}) \cong \mathbb{Q} \quad \text{(Hatcher-Vogtmann)}$$

$$H_8(\mathrm{Out}\, F_6; \mathbb{Q}) \cong \mathbb{Q} \quad \text{(Ohashi)}$$

$$H_{11}(\mathrm{Out}\, F_7; \mathbb{Q}) \cong H_8(\mathrm{Out}\, F_7; \mathbb{Q}) \cong \mathbb{Q} \quad \text{(Bartholdi)}$$

Kontsevich's theorem and homology of $\mathrm{Out}\, F_n$ (6)

$H_*(\mathrm{Out}\, F_n; \mathbb{Q})$ computed for $n \leq 7$: only four non-trivial parts

$$H_4(\mathrm{Out}\, F_4; \mathbb{Q}) \cong \mathbb{Q} \quad \text{(Hatcher-Vogtmann)}$$
$$H_8(\mathrm{Out}\, F_6; \mathbb{Q}) \cong \mathbb{Q} \quad \text{(Ohashi)}$$
$$H_{11}(\mathrm{Out}\, F_7; \mathbb{Q}) \cong H_8(\mathrm{Out}\, F_7; \mathbb{Q}) \cong \mathbb{Q} \quad \text{(Bartholdi)}$$

Conjecture (very difficult and important)

$\mu_k \neq 0$ *for all* k $\quad \left(\Rightarrow H_c^2(\hat{\mathfrak{h}}_{\infty,1}) \supset \mathbb{Q}\langle e_1, \mathbf{t}_3, \mathbf{t}_5, \cdots \rangle\right)$

Theorem (Conant-Hatcher-Kassabov-Vogtmann)

The class μ_k *is supported on certain subgroup* $\mathbb{Z}^{4k} \subset \mathrm{Out}\, F_{2k+2}$

CKV new generators $\Rightarrow$ more classes in $H_c^2(\hat{\mathfrak{h}}_{\infty,1})$

Many odd dimensional cohomology classes exist:

Theorem (Sakasai-Suzuki-M.)

The integral Euler characteristics of $\mathrm{Out}\, F_n$ is given by

$$e(\mathrm{Out}\, F_n) = 1, 1, 2, 1, 2, 1, 1, -21, -124, -1202 \ (n = 2, 3, \ldots, 11)$$

The unique explicit one is: $H_{11}(\mathrm{Out}\, F_7; \mathbb{Q}) \cong \mathbb{Q}$ (Bartholdi)

Problem

Construct non-trivial odd dim. homology classes of $\mathrm{Out}\, F_n$

Kontsevich's theorem and homology of $\mathrm{Out}\, F_n$ (8)

Conjectural geometric meaning of the classes

$$\mu_k \in H_{4k}(\mathrm{Out}\, F_{2k+2}; \mathbb{Q})$$

Kontsevich's theorem and homology of $\mathrm{Out}\, F_n$ (8)

Conjectural geometric meaning of the classes

$$\mu_k \in H_{4k}(\mathrm{Out}\, F_{2k+2}; \mathbb{Q})$$

secondary classes associated with the difference between

two reasons for the vanishing of Borel regulator classes

$$\beta_k \in H^{4k+1}(\mathrm{GL}(N, \mathbb{Z}); \mathbb{R})$$

(1) $\beta_k = 0 \in H^{4k+1}(\mathrm{Out}\, F_N; \mathbb{R})$ (Igusa, Galatius)

(2) $\beta_k = 0 \in H^{4k+1}(\mathrm{GL}(N_k^*, \mathbb{Z}); \mathbb{R})$ critical $N_k^* \stackrel{?}{=} 2k+2$, yes for $k = 1$ (Lee-Szczarba) and $k = 2$ (E.Vincent-Gangl-Soulé)

Theorem (Bismut-Lott, Lee, Franke)

$$\beta_k = 0 \in H^{4k+1}(\mathrm{GL}(2k+1, \mathbb{Z}); \mathbb{R})$$

$\tilde{\rho}_\infty$ on H^* yields many stable cohomology classes of $\mathcal{H}_{g,1}^{\mathrm{top}}$

$$H_c^*(\hat{\mathfrak{h}}_{g,1}^+)^{\mathrm{Sp}} \overset{Kontsevich}{\cong} \Lambda\left[\bigoplus_{n\geq 2} H_*(\mathrm{Out}\, F_n;\mathbb{Q})\right] \Rightarrow$$

Characteristic classes of homological surface bundles (1)

$\tilde{\rho}_\infty$ on H^* yields many stable cohomology classes of $\mathcal{H}_{g,1}^{\text{top}}$

$$H_c^*(\hat{\mathfrak{h}}_{g,1}^+)^{\mathrm{Sp}} \overset{Kontsevich}{\cong} \Lambda \left[\bigoplus_{n\geq 2} H_*(\mathrm{Out}\, F_n; \mathbb{Q}) \right] \Rightarrow$$

Theorem (Sakasai-Suzuki-M.)

$$\tilde{\rho}_\infty^* : \Lambda \left[\bigoplus_{n\geq 2} H_*(\mathrm{Out}\, F_n; \mathbb{Q}) \right] \otimes H^*(\mathrm{Sp}(2\infty, \mathbb{Q})) \to H^*(\mathcal{H}_{g,1}^{\text{top}}; \mathbb{Q})$$

Corollary

The MMM-classes are defined already in $H^*(\mathcal{H}_{g,1}^{\text{top}}, \mathbb{Q})$

Comparison with the case of the mapping class group:

The image of the homomorphism

$$\rho_\infty^* : \Lambda \left[\bigoplus_{n \geq 2} H_*(\mathrm{Out}\, F_n; \mathbb{Q}) \right] \otimes H^*(\mathrm{Sp}(2\infty, \mathbb{Z}); \mathbb{Q})$$
$$\rightarrow H^*(\mathcal{M}_{g,1}; \mathbb{Q})$$

consists of stable classes $\overset{\mathrm{Madsen-Weiss}}{\Rightarrow}$

$\mathrm{Im}\, \rho_\infty^* = \mathcal{R}^*(\mathcal{M}_{g,1}) = \langle \mathrm{MMM} - \mathrm{classes} \rangle$ (tautological algebra)

$\Rightarrow \rho_\infty^*$ has a big kernel

Comparison with higher dimensional cases:

Theorem (Berglund-Madsen)

For any d : *odd* ≥ 3

$$\Lambda\left[\bigoplus_{n\geq 2} H_*(\mathrm{Out}\, F_n;\mathbb{Q})\right]^{(d)} \otimes H^*(\mathrm{Sp}(2\infty,\mathbb{Z});\mathbb{Q})$$

isomorphism

$$\cong \lim_{g\to\infty} H^*(\mathrm{Baut}_\partial(\sharp_g(S^d\times S^d)\setminus \mathrm{Int}\, D^{2d});\mathbb{Q})$$

$$\mathrm{degree}\,(x) = 2nd-2-k \;\; (x\in H_k(\mathrm{Out}\, F_n;\mathbb{Q}))$$

Definition (most important characteristic classes)

$$\tilde{\mathbf{t}}_{2k+1} = \tilde{\rho}_\infty^*(\mu_k) \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}), H^2(\mathcal{H}_{g,1}^{\text{top}};\mathbb{Q}) \quad (k=1,2,\ldots)$$

most important classes coming from $H^2(S^{2k+1}H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q}$

candidates for $\chi(\mathcal{H}_{g,1}^{\text{smooth}}) \in H^2(\overline{\mathcal{H}}_{g,1};\mathbf{\Theta}^3)$

group version of $\mathbf{t}_{2k+1} \in H_c^2(\hat{\mathfrak{h}}_{\infty,1})$ defined earlier geometrical meaning of the classes $\tilde{\mathbf{t}}_{2k+1} \in H^2(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q})$:

Intersection numbers of higher and higher Massey products (using works of Kitano, Garoufalidis-Levine) geometrical meaning of the classes $\tilde{\mathbf{t}}_{2k+1} \in H^2(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q})$:

Intersection numbers of higher and higher Massey products (using works of Kitano, Garoufalidis-Levine)

Conjecture

In the central extension

$$0 \to \mathbf{\Theta}^3 \to \mathcal{H}_{g,1}^{\mathrm{smooth}} \to \overline{\mathcal{H}}_{g,1} \to 1$$

$\mathbf{\Theta}^3$ *"transgresses" to the classes* $\tilde{\mathbf{t}}_{2k+1} \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \Rightarrow$

$$\tilde{\mathbf{t}}_{2k+1} \neq 0 \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}),\, H^2(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q}) \quad \textit{and}$$
$$\tilde{\mathbf{t}}_{2k+1} = 0 \in H^2(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})$$

$\rho_\infty^*(3e_1 - c_1) = 0 \in H^2(\mathcal{M}_{g,1};\mathbb{Q}) \Rightarrow$ secondary invariant:

Casson invariant λ for homology 3-spheres

Prospect

$\rho_\infty^*(3e_1 - c_1) = 0 \in H^2(\mathcal{M}_{g,1};\mathbb{Q}) \Rightarrow$ secondary invariant:

Casson invariant λ for homology 3-spheres

If Conjecture is true $\Rightarrow$ obtain homomorphisms

$$\nu_k : \boldsymbol{\Theta}^3 \to \mathbb{Q} \quad (k = 1, 2, \ldots)$$

homology cobordism invariants