On two embedding theorems concerning right-angled Artin groups

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Right-angled Artin groups

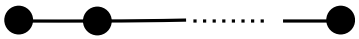
Γ: a finite (simplicial) graph
$V(\Gamma) = \{v_1, v_2, \cdots, v_n\}$: the vertex set of Γ
$E(\Gamma)$: the edge set of Γ

Definition

The right-angled Artin group (RAAG) on Γ is the group given by the following presentation:

$$G(\Gamma) = \langle\ v_1, v_2, \ldots, v_n \ \mid\ [v_i, v_j] = 1 \textit{ if } \{v_i, v_j\} \notin E(\Gamma)\ \rangle.$$

$G(\Gamma_1) \cong G(\Gamma_2)$ if and only if $\Gamma_1 \cong \Gamma_2$.



$\mathrm{P_n}$

P_n: the path graph consisting of n vertices

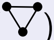
Example

$G(P_1) \cong \mathbb{Z}$

$G(P_1 \sqcup P_1 \sqcup P_1) \cong \mathbb{Z}^3$

$G(P_1 \sqcup P_2) \cong \mathbb{Z} \times F_2$

$G($  $) \cong \mathbb{Z}^2 * \mathbb{Z}$

$G($  $) \cong F_3$

Note: $G(\Gamma) = \langle\ v_1, v_2, \ldots, v_n \ \mid\ [v_i, v_j] = 1 \ \mathit{if}\ \{v_i, v_j\} \notin E(\Gamma)\ \rangle$

Motivation and main results

Problem (Crisp-Sageev-Sapir, 2008)

For given two finite graphs Λ and Γ, decide whether $G(\Lambda)$ can be embedded into $G(\Gamma)$.

The following is standard.

Proposition

Λ, Γ: *finite graphs*
If $\Lambda \leq \Gamma$, *then* $G(\Lambda) \hookrightarrow G(\Gamma)$.

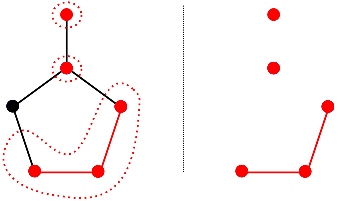
Proposition

Λ, Γ: *finite graphs*

If $\Lambda \leq \Gamma$, *then* $G(\Lambda) \hookrightarrow G(\Gamma)$.

A subgraph Λ of a graph Γ is said to be full if $E(\Lambda)$ contains every $e \in E(\Gamma)$ whose end points both lie in $V(\Lambda)$.

We denote by $\Lambda \leq \Gamma$ if Λ is a full subgraph of Γ.

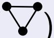


Proposition

Λ, Γ: *finite graphs*
If $\Lambda \leq \Gamma$, *then* $G(\Lambda) \hookrightarrow G(\Gamma)$.

In general, the converse implication "$G(\Lambda) \hookrightarrow G(\Gamma)$" $\Rightarrow$ "$\Lambda \leq \Gamma$" is false.

Example

$G($$) \cong F_3 \hookrightarrow F_2 \cong G(P_2)$.

So the following question naturally arises.

Question

Which finite graph Λ satisfies the following property $(*)$?
$(*)$ For any finite graph Γ, "$G(\Lambda) \hookrightarrow G(\Gamma)$" $\Rightarrow$ "$\Lambda \leq \Gamma$".

Question

Which finite graph Λ satisfies the following property $(*)$?
$(*)$ For any finite graph Γ, "$G(\Lambda) \hookrightarrow G(\Gamma)$" $\Rightarrow$ "$\Lambda \leq \Gamma$".

The following gives a complete answer to the above question.
A finite graph Λ is said to be a linear forest if each connected component of Λ is a path graph.

Theorem A (K.)

Let Λ be a finite graph.

(1) If Λ is a linear forest, then Λ has property $(*)$,
i.e., $\forall \Gamma$, if $G(\Lambda) \hookrightarrow G(\Gamma)$, then $\Lambda \leq \Gamma$.

(2) If Λ is not a linear forest, then Λ does not have property $(*)$,
i.e., $\exists \Gamma$ such that $G(\Lambda) \hookrightarrow G(\Gamma)$, though $\Lambda \not\leq \Gamma$.

Theorem A (K.)

Let Λ be a finite graph.

(1) If Λ is a linear forest, then
$\forall\Gamma$, the relation $G(\Lambda) \hookrightarrow G(\Gamma)$ implies the relation $\Lambda \leq \Gamma$.

(2) If Λ is not a linear forest, then
$\exists\Gamma$ such that $G(\Lambda) \hookrightarrow G(\Gamma)$, though $\Lambda \not\leq \Gamma$.

Application of Thm A(1) to concrete embedding problems

- $\neg(\mathbb{Z}^2 * \mathbb{Z} \hookrightarrow F_2 \times F_2 \times \cdots \times F_2)$.

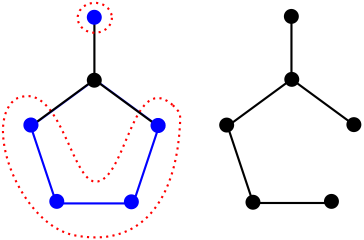
Proof) Suppose to the contrary that $\mathbb{Z}^2 * \mathbb{Z} \hookrightarrow F_2 \times F_2 \times \cdots \times F_2$.
Then since P_3 is a linear forest, Theorem A(1) implies
$P_3 \leq P_2 \sqcup P_2 \sqcup \cdots \sqcup P_2$, a contradiction. Q.E.D.
Note: $G(P_3) \cong \mathbb{Z}^2 * \mathbb{Z}$ and
$G(P_2 \sqcup P_2 \sqcup \cdots \sqcup P_2) \cong F_2 \times F_2 \times \cdots \times F_2$.

Similarly, we have $\neg(F_2 \times F_2 \times \cdots \times F_2 \hookrightarrow \mathbb{Z}^2 * \mathbb{Z})$.

Appl of Thm A(1) (cont'd).

- $\neg(G(\Lambda_1) \hookrightarrow G(\Lambda_2))$.

Proof) Suppose to the contrary that $G(\Lambda_1) \hookrightarrow G(\Lambda_2)$.
Then since $P_1 \sqcup P_4 \leq \Lambda_1$, we have $G(P_1 \sqcup P_4) \hookrightarrow G(\Lambda_1)$.
Hence, $G(P_1 \sqcup P_4) \hookrightarrow G(\Lambda_2)$.
This together with Theorem A(1) implies $P_1 \sqcup P_4 \leq \Lambda_2$, which is impossible. Q.E.D.



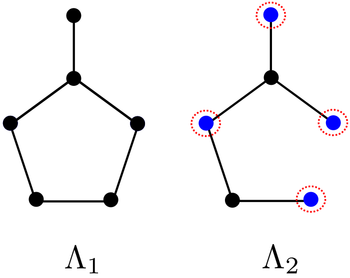
Λ_1 Λ_2

So Theorem A(1) is sometimes valid to decide whether the RAAG, on a graph which is not a linear forest, embeds into another RAAG.

Appl of Thm A(1) (cont'd).

- $\neg(G(\Lambda_2) \hookrightarrow G(\Lambda_1))$.

Proof) Use
a) $P_1 \sqcup P_1 \sqcup P_1 \sqcup P_1 \leq \Lambda_2$,
i.e., $G(P_1 \sqcup P_1 \sqcup P_1 \sqcup P_1) \hookrightarrow G(\Lambda_2)$ and
b) $P_1 \sqcup P_1 \sqcup P_1 \sqcup P_1 \not\leq \Lambda_1$,
i.e., $G(P_1 \sqcup P_1 \sqcup P_1 \sqcup P_1)$ cannot be embedded into $G(\Lambda_1)$. Q.E.D.



Thus we obtain $\neg(G(\Lambda_1) \hookrightarrow G(\Lambda_2))$ and $\neg(G(\Lambda_2) \hookrightarrow G(\Lambda_1))$.

Theorem A(1)

Let Λ be a finite graph.
If Λ is a linear forest, then
$\forall\Gamma$, *the relation* $G(\Lambda) \hookrightarrow G(\Gamma)$ *implies the relation* $\Lambda \leq \Gamma$.

For some special linear forests, Theorem A(1) is known.

- $\Lambda = P_1 \sqcup P_1 \sqcup \cdots \sqcup P_1$ [Servatius, 1989]
- $\Lambda = P_3,\ P_4,\ P_2 \sqcup P_2$ [Kim-Koberda, 2013]

Theorem A(2)

Let Λ be a finite graph.
If Λ is not a linear forest, then
$\exists\Gamma$ such that $G(\Lambda) \hookrightarrow G(\Gamma)$, though $\Lambda \not\leq \Gamma$.

Theorem A(2) is known in the case Λ contains a cycle.

Theorem (Kim-Koberda, 2015)

Λ: a finite graph
Then there exists a finite tree T such that $G(\Lambda) \hookrightarrow G(T)$.

Hence, we have only to prove Theorem A(2) in the following case.

- Case: Λ is a forest containing a vertex of deg ≥ 3.

- Case: Λ is a forest containing a vertex of deg ≥ 3.

Today, instead of the proof of Theorem A(2) itself, I explain the proof of the following partial result of Theorem A(2).

Theorem B (K.)

T: a finite tree

Then there exists a finite tree T' satisfying the following.

(1) $G(T) \hookrightarrow G(T')$.

(2) $\deg_{\max}(T') \leq 3$, *where*
$\deg_{\max}(T') = \max\{m \mid m = \deg(v),\ v \in V(T')\}$.

(3) $|T'| \leq 2|T| - 4$.

Note that if $\deg_{\max}(T) > 3$, then we have $T \not\leq T'$.

By combining Kim-Koberda's embedding theorem and Theorem B, we have the following.

Theorem ([Kim-Koberda, 2015] + Thm B)

Λ: a finite graph
Then there exists a finite tree T such that $G(\Lambda) \hookrightarrow G(T)$ and $\deg_{\max}(T') \leq 3$.

[Wise, 2011],[Agol, 2014], [Kim-Koberda, 2015] + Thm B

Corollary

M: a complete hyperbolic 3-manifold with finite volume
Then $\pi_1(M)$ is virtually embedded into $G(T)$ for some finite tree T with $\deg_{\max}(T) \leq 3$.

Main results

Theorem A (K.)

Let Λ be a finite graph.

(1) *If Λ is a linear forest, then*
$\forall \Gamma$*, the relation* $G(\Lambda) \hookrightarrow G(\Gamma)$ *implies the relation* $\Lambda \leq \Gamma$*.*

(2) *If Λ is not a linear forest, then*
$\exists \Gamma$ *such that* $G(\Lambda) \hookrightarrow G(\Gamma)$*, though* $\Lambda \not\leq \Gamma$*.*

Theorem B (K.)

T: a finite tree

Then there exists a finite tree T' satisfying the following.

(1) $G(T) \hookrightarrow G(T')$.

(2) $\deg_{\max}(T') \leq 3$.

(3) $|T'| \leq 2|T| - 4$.

Moreover, we obtain the following as a consequence of Theorem A(1).

Theorem C (K.)

Λ: *a linear forest*
If $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$, *then* $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

This is a partial converse of the following embedding theorem.

Theorem (Koberda, 2012)

Λ: *a finite graph*
If $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$, *then* $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$

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Theorem A(1)

Λ: a linear forest
Γ: a finite graph
If $G(\Lambda) \hookrightarrow G(\Gamma)$, then $\Lambda \leq \Gamma$.

Sketch of proof.

Step 1. Prove $\Lambda \leq \overline{\Gamma^e}$, where $\overline{\Gamma^e}$ is a graph such that

- $V(\overline{\Gamma^e}) = \{g^{-1}ug \in G(\Gamma) \mid u \in V(\Gamma),\ g \in G(\Gamma)\}$.
- u^g and v^h span an edge $\Leftrightarrow$ u^g and v^h are not commutative.

Theorem (Casals-Ruiz, 2015)

For a forest Λ and a finite graph Γ, if $G(\Lambda) \hookrightarrow G(\Gamma)$, then $\Lambda \leq \overline{\Gamma^e}$.

Step 2. Prove that $\Lambda \leq \overline{\Gamma^e}$ implies $\Lambda \leq \Gamma$.

Step 2. Prove that $\Lambda \leq \overline{\Gamma^e}$ implies $\Lambda \leq \Gamma$.

Use the "finiteness" of $\overline{\Gamma^e}$.

Theorem (Kim-Koberda, 2013)

If $\Lambda \leq \overline{\Gamma^e}$, then there exists a sequence of consecutive "co-doubles"

$$\Gamma = \Gamma_0 \leq \Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_n \leq \overline{\Gamma^e}$$

such that $\Gamma_i = \overline{D}(\Gamma_{i-1})$ and $\Lambda \leq \Gamma_n$.

Here, for a finite graph Δ,

$$\overline{D}(\Delta) := (D(\Delta^c))^c.$$

The operation c: "taking the complement graph".
The operation D: "taking the double graph along the star subgraph of a vertex"

Step 2. Prove that $\Lambda \leq \overline{\Gamma^e}$ implies $\Lambda \leq \Gamma$ (cont'd).

Use the "finiteness" of $\overline{\Gamma^e}$.

Theorem (Kim-Koberda, 2013)

If $\Lambda \leq \overline{\Gamma^e}$, then there exists a sequence of consecutive "co-doubles"

$$\Gamma = \Gamma_0 \leq \Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_n \leq \overline{\Gamma^e}$$

such that $\Gamma_i = \overline{D}(\Gamma_{i-1})$ and $\Lambda \leq \Gamma_n$.

Proposition (K.)

Λ: a linear forest, Δ: a finite graph
If $\Lambda \leq \overline{D}(\Delta)$, then $\Lambda \leq \Delta$.

Theorem A(1)

Λ: a linear forest, Γ: a finite graph
If $G(\Lambda) \hookrightarrow G(\Gamma)$, then $\Lambda \leq \Gamma$.

Theorem B

T: a finite tree

Then there exists a finite tree T' satisfying the following.

(1) $G(T) \hookrightarrow G(T')$.

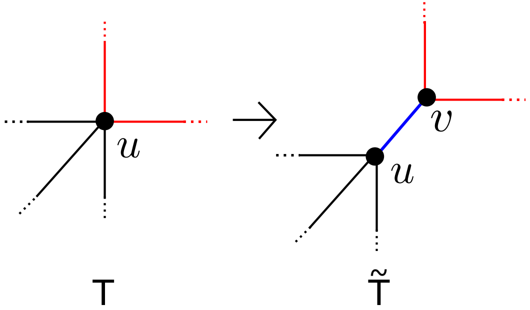
(2) $\deg_{\max}(T') \leq 3$.

(3) $|T'| \leq 2|T| - 4$.

- Sketch of proof.

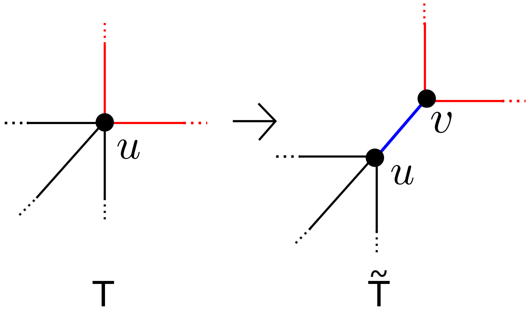
T: a finite tree with $\deg_{\max}(T) > 3$.
We would like to find a finite tree T' satisfying (1), (2) and (3)...

Pick a vertex u of deg > 3 in T. By splitting u as follows, we obtain the new finite tree $\tilde{T}$.



Note that, for the vertices u and v, we have

$$\deg(u, \tilde{T}) = \deg(u, T) - 1$$
$$\deg(v, \tilde{T}) = 3$$



Then we can prove $T \leq \overline{D}(\tilde{T})$, and so $G(T) \hookrightarrow G(\overline{D}(\tilde{T}))$.

Lemma (Kim-Koberda)

For any finite graph Γ, we have $G(\overline{D}(\tilde{\Gamma})) \hookrightarrow G(\Gamma)$.

Thus $G(T) \hookrightarrow G(\tilde{T})$.

By repeating this argument, we have a finite tree T' such that $G(T) \hookrightarrow G(T')$ and that T' consists only of the vertices of deg at most 3.

Remark

In this argument, we do not need the assumption that T is a tree. However, to deduce the assertion (3), we need the assumption.

Theorem B

T: a finite tree

Then there exists a finite tree T' satisfying the following.

(1) $G(T) \hookrightarrow G(T')$.

(2) $\deg_{\max}(T') \leq 3$.

(3) $|T'| \leq 2|T| - 4$.

Remark: (3) is best possible.

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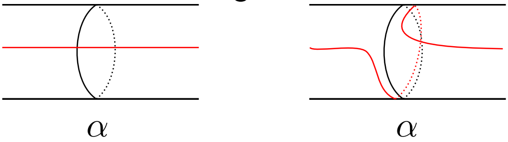
The mapping class groups of surfaces

$\Sigma_{g,n}$: the orientable compact surface of genus g with n punctures
We assume $\chi(\Sigma_{g,n}) < 0$.
The mapping class group of $\Sigma_{g,n}$ is defined as follows.

$$\mathcal{M}(\Sigma_{g,n}) := \pi_0(Homeo^+(\Sigma_{g,n}))$$

α: an essential simple loop on $\Sigma_{g,n}$
T_α: the Dehn twist along α



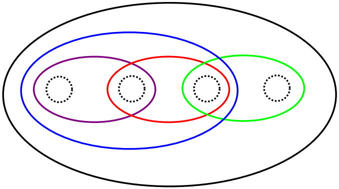
Theorem (Dehn-Lickorish)

Dehn twists on $\Sigma_{g,n}$ generate $\mathcal{M}(\Sigma_{g,n})$.

The complement graph of the curve graph of $\Sigma_{g,n}$

The complement graph of the curve graph $\mathcal{C}^c(\Sigma_{g,n})$ is a graph such that

- $V(\mathcal{C}^c(\Sigma_{g,n})) = \{\text{isotopy classes of esls on } \Sigma_{g,n}\}$
- esls α, β span an edge iff α, β CANNOT be realized disjointly.



Theorem A(1) implies Theorem C

Theorem A(1)

Λ: a linear forest
Γ: a finite graph
If $G(\Lambda) \hookrightarrow G(\Gamma)$, then $\Lambda \leq \Gamma$.

Theorem C

Λ: a linear forest
If $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$, then $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

The embedding theorem due to Koberda

Theorem (Koberda, 2012)

Λ: a finite graph
Then the following hold.

(1) *If $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$, $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$.*
(2) *There exists a compact surface Σ such that $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma)$.*

The following lemma follows from Koberda's embedding theorem.

Lemma (Koberda)

Λ: a finite graph
If $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$, then there exists a finite full subgraph $\Gamma \leq \mathcal{C}^c(\Sigma_{g,n})$ such that $G(\Lambda) \hookrightarrow G(\Gamma)$.

Lemma (Koberda)

Λ: a finite graph
If $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$, then there exists a finite full subgraph $\Gamma \leq \mathcal{C}^c(\Sigma_{g,n})$ such that $G(\Lambda) \hookrightarrow G(\Gamma)$.

Theorem C

Λ: a linear forest
If $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$, then $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

Proof.

Λ: a linear forest
Suppose $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$.
Then $\exists \Gamma \leq \mathcal{C}^c(\Sigma_{g,n})$: a finite full subgraph; $G(\Lambda) \hookrightarrow G(\Gamma)$.
Theorem A(1) now implies $\Lambda \leq \Gamma(\leq \mathcal{C}^c(\Sigma_{g,n}))$, as desired. □

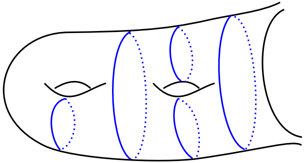
Theorem ([Koberda, 2012] + Thm C)

Λ: a linear forest
Then $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$ if and only if $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

We can regard the above theorem as a generalization of the following classical result.

Theorem (Birman-Lubotzky-McCarthy, 1983)

The maximum rank of free abelian subgroup of $\mathcal{M}(\Sigma_{g,n})$ is bounded by the number of simple closed curves needed in the pants-decomposition of $\Sigma_{g,n}$ ($= 3g + n - 3 =: \xi$).



Theorem (BLM)

The maximum rank of free abelian subgroup of $\mathcal{M}(\Sigma_{g,n})$ is bounded by the number of simple closed curves needed in the pants-decomposition of $\Sigma_{g,n}$.

In our terminology, Birman-Lubotzky-McCarthy's obstruction can be translated as follows.

Theorem (BLM in our terminology)

Λ: the disjoint union of finitely many copies of P_1
Then $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$ if and only if $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

If Λ is the disjoint union of finitely many copies of P_1, then $G(\Lambda) \cong \mathbb{Z}^{|\Lambda|}$.
Moreover, $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$ means disjointly represented simple closed curves on $\Sigma_{g,n}$.

Theorem ([Koberda, 2012] + Thm C)

Λ: *a linear forest*
Then $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$ *if and only if* $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

Hence, our obstruction theorem generalizes Birman-Lubotzky-McCarty's.

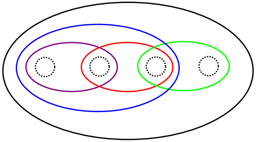
Theorem (BLM in our terminology)

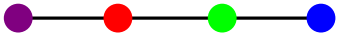
Λ: *the disjoint union of finitely many copies of* P_1
Then $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$ *if and only if* $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

Linear chains on surfaces

$L_m = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$: a set of esls on $\Sigma_{g,n}$
$L_m \subset \Sigma_{g,n}$ is said to be a linear chain

$\overset{def}{\Leftrightarrow}$
- α_i and α_{i+1} cannot be realized disjointly.
- α_i and α_j $(i+2 \leq j)$ can be realized disjointly.





$$L(\Sigma_{g,n}) := \max\{m \mid L_m \subset \Sigma_{g,n}\text{: a linear chain}\}$$
$$= \max\{m \mid P_m \leq \mathcal{C}^c(\Sigma_{g,n})\}$$
$$= \max\{m \mid G(P_m) \hookrightarrow \mathcal{M}(\Sigma_{g,n})\}$$

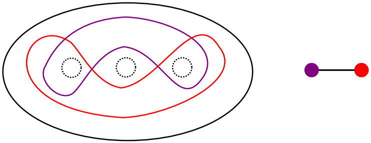
$L(\Sigma_{g,n})) = ???$

Proposition

If $g = 0$, we have the following.

$$L(\Sigma_{0,n})) = \begin{cases} 2 & n = 4 \\ n-1 & n \geq 5 \end{cases}$$

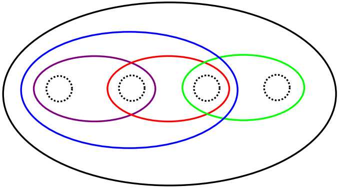
- $L(\Sigma_{0,4})) = 2.$

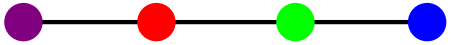


This picture shows $L(\Sigma_{0,4})) \geq 2$.
To see $L(\Sigma_{0,4})) = 2$, suppose to the contrary that $L(\Sigma_{0,4})) \geq 3$.
Then there exists a linear chain $L_3 = \{\alpha_1, \alpha_2, \alpha_3\} \subset \Sigma_{0,4}$.
We may assume that α_3 and α_1 is disjointly represented.
Then α_3 divide $\Sigma_{0,4}$ into two surfaces, $\Sigma_{0,3}$ and $\Sigma_{0,2}$, not containing an esl, though α_1 must be contained in either $\Sigma_{0,3}$ or $\Sigma_{0,2}$.

- $\underline{L(\Sigma_{0,5})) = 4.}$

For $L(\Sigma_{0,n})) \geq 4$, see the picture below.





Suppose to the contrary that $L(\Sigma_{0,n})) \geq 5$.
Then there exists a linear chain of length 5, $L_5 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, on $\Sigma_{0,5}$.
We may assume that α_5 and $\alpha_1 \cup \alpha_2 \cup \alpha_3$ are disjointly represented.
Since α_5 is a separating curve, α_5 divide $\Sigma_{0,5}$ into $\Sigma_{0,4}$ and $\Sigma_{0,2}$.
Hence, the linear chain $L_3 := \{\alpha_1, \alpha_2, \alpha_3\}$ is contained in either $\Sigma_{0,4}$.
However, this is impossible.

By an inductive argument, we yield:

Proposition

If $g = 0$, we have the following.

$$L(\Sigma_{0,n}) = \begin{cases} 2 & n = 4 \\ n-1 & n \geq 5 \end{cases}$$

Since $L(\Sigma_{0,6}) = 5$, $G(P_6)$ cannot be embedded into $\mathcal{M}(\Sigma_{0,6})$.

Further studies (1/4)

Question

$L(\Sigma_{g,n}) = \max\{m \mid G(P_m) \hookrightarrow \mathcal{M}(\Sigma_{g,n})\} = ???$

If either genus g or the number of punctures n is equal to 0,

$$L(\Sigma_{0,n}) = n - 1 \ (n \geq 5)$$
$$L(\Sigma_{g,0}) \underset{?}{=} 2g + 1 \ (g \geq 2).$$

In general,

$$L(\Sigma_{g,n}) \fallingdotseq -\chi(\Sigma_{g,n})?$$

More precisely,

$$|L(\Sigma_{g,n}) - |\chi(\Sigma_{g,n})|\ | \leq 3?$$

Further studies (2/4)

Theorem (Kim-Koberda, 2014)

(1) Λ: a finite graph
If $\xi(\Sigma_{g,n}) < 3$, then $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$ if and only if $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

(2) If $\xi(\Sigma_{g,n}) > 3$, then there exist a finite graph Λ such that $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$ but $\Lambda \not\leq \mathcal{C}^c(\Sigma_{g,n})$.

Kim-Koberda said, "we do not know how to resolve the case $\xi = 3$".
Since $L(\Sigma_{0,6}) = 5$, $G(P_6)$ cannot be embedded into $\mathcal{M}(\Sigma_{0,6})$.
(I think) studying unembeddability is valid to resolve the case $\xi = 3$...

Further studies (3/4)

C_n: the cyclic graph on n vertices
Theorem A(1) directly implies that, for any finite graph Γ, if $G(C_5) \hookrightarrow G(\Gamma)$, then $P_4 \leq \Gamma$.

Conjecture (Casals-Ruiz)

Γ: a finite graph
Then $G(\Gamma)$ contains the fund group of a closed hyp surface if and only if $G(\Gamma)$ contains $G(C_n^c)$ for some $n \geq 5$.

Note: $C_5^c = C_5$.

Theorem (Servatius-Droms-Servatius)

For any $n \geq 5$, $G(C_n^c)$ contains the fund group of a closed hyp surface.

Conjecture (Casals-Ruiz)

Γ*: a finite graph*
Then $G(\Gamma)$ *contains the fund group of a closed hyp surface if and only if* $G(\Gamma)$ *contains* $G(C_n^c)$ *for some* $n \geq 5$*.*

At this time, we have no counter-example of the "only if" part.
However, for example, which RAAG contains $G(C_5)$?
$G(C_5) \hookrightarrow G(P_8)$ (Casals-Ruiz) and $\neg(G(C_5) \hookrightarrow G(P_4))$ (Droms).
A concrete problem: we do not know whether $G(C_5)$ embeds into $G(P_n)$ for $5 \leq \forall n \leq 7$.

Further studies (4/4)

Theorem ([Kim-Koberda, 2015] + Thm B)

Λ: a finite graph
Then there exists a finite tree T such that $G(\Lambda) \hookrightarrow G(T)$ and $\deg_{\max}(T') \leq 3$.

Question (Lee, 2016)

For any finite graph Λ, is it possible that $G(\Lambda) \hookrightarrow G(P_n)$ for some n?

(I think) it's only a matter of time...

Theorem (Casals-Ruiz, 2015)

For a forest Λ and a finite graph Γ, if $G(\Lambda) \hookrightarrow G(\Gamma)$, then $\Lambda \leq \overline{\Gamma^e}$.

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• T. Katayama, 'An obstruction to the existence of embeddings between RAAGs', in preparation.

It's almost complete...

Thank you very much for your attention!