On simple closed curves and visualization of linearity for mapping class groups of surfaces

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Outline

Introduction

Mapping Class Group (MCG)

the group of isotopy classes of the orientation preserving homeomorphisms of an oriented surface.

(w/ some variants)

A fundamental problem is its linearity.

- A group is linear $\Leftrightarrow$ it admits a faithful finite dimensional linear representation over $^{\exists}$ field.
- A linear representation is faithful $\Leftrightarrow$ it is injenctive as a homomophism into the corresponding linear transformation group.

In particular, a group is said to be K-linear if it admits a faithful finite dimensional linear representation over a field K.

The purpose of this talk is

- to recall
  - the known results on the linearity problem on MCG of surfaces,
  - our previous visualization of linearity for MCG of 1-punctured surface,
- to derive a new linearity condition for MCG of closed surface, (NOT to claim the solution of the problem, unfortunately),
- and as a byproduct, to discuss the geometric intersection in arbitrary representation of MCG.

Notation

Σ_g : a closed oriented surrface of genus g

$\Sigma_{g,*}$: a pair of Σ_g and a fixed base point $* \in \Sigma_g$

$\Sigma_{g,n}$: a connected compact oriented surface of genus g with $n \geq 0$ boundary components

$\mathcal{M}_g$: MCG of Σ_g

$\mathcal{M}_{g,*}$: MCG of $\Sigma_{g,*}$
(homeo and isotopy are assumed to fix $*$)

$\mathcal{M}_{g,n}$: MCG of $\Sigma_{g,n}$
(homeo and isotopy are assumed to fix the boundary pointwise)

Known results on linearity

Classical. $\mathcal{M}_1 \cong \mathcal{M}_{1,*} \cong \mathrm{SL}(2,\mathbb{Z})$.

Therefore, $\mathcal{M}_1$ and $\mathcal{M}_{1,*}$ are $\mathbb{Q}$-linear.

For the genus 2 case:

Korkmaz ['00], Bigelow–Budney ['01]

$\mathcal{M}_2$ is linear.

Proved by the combination of:

- Artin's braid group B_n is linear (Bigelow ['01], Krammer ['02])
- the relation of B_n with MCG of $n+1$st punctured S^2,
- relation between $\mathcal{M}_2$ and MCG of 6th punctured S^2 (Birman–Hilden theory) □

(The same proof shows the hyperelliptic MCG of Σ_g for all $g \geq 3$ is linear.)

All the other cases are unknown.

The linearity of $\mathcal{M}_g$ for $g \geq 3$ and also the linearity of $\mathcal{M}_{g,*}$ for $g \geq 2$ seem to remain open.

Some difficulties of the problem

Non-existence of the Lawrence representation for $g \geq 1$:

H^1(configuration space of $\Sigma_g)^{\mathcal{M}_g}$ is at most $\mathbb{Z}/2\mathbb{Z}$ while the analogue for a punctured disk is $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$.

Hard to construct a candidate of faithful linear representations

- All the TQFT rep's: the image of a Dehn twist is of finite order
- Classification result of low dim. rep.'s over $\mathbb{C}$
- (Korkmaz) For genus $g \geq 3$, there are no faithful linear rep over $\mathbb{C}$ in dimensions $\leq 3g - 3$.

Subtleness of the problem

- Comparison with $\mathrm{Aut}\,(F_n)$
- Lattice in topological group vs Linearity

Motivation for Visualization

- Linearity problem seems quite subtle.
- If an ad-hoc way is good enough, the problem might be solved even today by somebody.
- However, any systematic approach seems missing.

So, we tried to rephrase the linearity problem for MCG in terms of proper MCG geometry/topology,

in the hope to find new interesting problems, and further hopefully a clue to the linearity problem itself.

Our previous result: a visualization of the linearity for $\mathcal{M}_{g,*}$

Let $g \geq 2$.

- $X_{\mathrm{GL}(n,K)} := \mathrm{Hom}\left(\pi_1(\Sigma_g, *), \mathrm{GL}(n, K)\right) / \mathrm{GL}(n, K)$ with K a field
- $\mathcal{M}_{g,*}$ naturally acts on $X_{\mathrm{GL}(n,K)}$, which descends to the action of $\mathcal{M}_g$ via the Birman exact sequence

$$1 \to \pi_1(\Sigma_g, *) \to \mathcal{M}_{g,*} \to \mathcal{M}_g \to 1$$

Theorem (K.)

$\mathcal{M}_{g,*}$ is K-linear if and only if there exists some n such that the action of $\mathcal{M}_g$ on $X_{\mathrm{GL}(n,K)}$ has a global fixed point represented by a faithful representation of $\pi_1(\Sigma_g, *)$.

Key for the proof

- $\pi_1(\Sigma_g, *) \triangleleft \mathcal{M}_{g,*}$
- The conjugation action of $\mathcal{M}_{g,*}$ on $\pi_1(\Sigma_g, *)$ coincides with the natural action, which is faithful.

The proof does not work for closed surface since $\pi_1(\Sigma_g, *)$ is not contained in $\mathcal{M}_g$.

However, there is another geometric object contained in the MCG of a compact surface such that

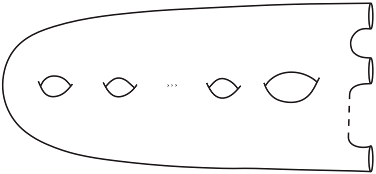
- the conjugation action of the MCG coincides with the natural action on it,
- the natural action on it is (almost) faithful.

I.e., the set $\mathcal{S}$ of the isotopy classes of (essential) simple closed curves on the underlying surface.

We can obtain another linearity condition for MCG in terms of $\mathcal{S}$.

Setting for compact surface case:

$\Sigma_{g,n}$: the compact connected oriented surface of genus $g \geq 1$ and $n \geq 0$ boundary components.



$\mathcal{M}_{g,n}$: MCG of $\Sigma_{g,n}$ (id on ∂)

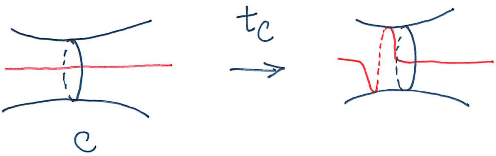
$\mathcal{S} = \mathcal{S}_{g,n}$: the set of isotopy classes of essential (unoriented) simple closed curves (SCC) on $\Sigma_{g,n}$

Here, essential means: not homotopic to a point nor parallel to any boudary component.

Note: $\mathcal{M}_{g,n}$ naturally acts on $\mathcal{S}$.

Dehn twist

For $C \in \mathcal{S}$, t_C denotes the (right-handed) Dehn twist $\in \mathcal{M}_{g,n}$:



Definition

We define a set mapping

$$\iota : \mathcal{S} \to \mathcal{M}_{g,n} \quad \text{by } \iota(C) := t_C \text{ for } C \in \mathcal{S}.$$

Fact

- ι is injective.
- For $f \in \mathcal{M}_{g,n}$, $f \cdot t_C \cdot f^{-1} = t_{f(C)}$, i.e.,

$$\iota(f(C)) = f \cdot \iota(C) \cdot f^{-1}.$$

Starting Lemma:

Lemma (K.)

For any group homomorphism $\varphi : \mathcal{M}_{g,n} \to G$,

$$\operatorname{Ker}\varphi \subset Z(\mathcal{M}_{g,n}) \quad \Leftrightarrow \quad \varphi \circ \iota \text{ is injective}$$

where Z denotes the center of the group.

Proof.

- $f \cdot t_C \cdot f^{-1} = t_{f(C)}$ $(f \in \mathcal{M}_{g,n})$
- $\operatorname{Ker}(\mathcal{M}_{g,n} \to \mathrm{Aut}(\mathcal{S})) = Z(\mathcal{M}_{g,n})$
- The action on $\mathcal{S}$ can detect $\mathcal{S}$. □

By making use of this lemma, we can "visualize" the linearity of $\mathcal{M}_{g,n}$, up to center.

To explain this, we introduce the following.

Module of curves

$K[\mathcal{S}]$: the vector space over K with basis $\mathcal{S}$

Definition (K.)

A module of curves (of type $\mathcal{S}$) is defnied as the pair of

- an $\mathcal{M}_{g,n}$-module M (over K),
- an $\mathcal{M}_{g,n}$-equivariant surjective homomorphism $p : K[\mathcal{S}] \to M$.

If p is clear, we will simply refer to M as a module of curves.

Module of curves (2)

We say a module of curves is of finite dimension if its dimension over K is finite.

- A module of curves is nothing but $K[\mathcal{S}]$ divided by skein type relations, i.e., some formal finite sums of finite numbers of SCCs.
- There is only one example of finite dimensional module of curves given, in terms of skein type relations (Luo['97]).

N.B. Not all $\mathcal{M}_{g,n}$-modules admit the structure of module of curves.

E.g., any $\mathcal{M}_g$-equivariant homomorphism $K[\mathcal{S}] \to H_1(\Sigma_g; K)$ must be zero, if $\mathrm{char}(K) \neq 2$.

Any linear rep. of $\mathcal{M}_{g,n}$ induces a module of curves.

V: a finite dimensional vector space over K
$\rho : \mathcal{M}_{g,n} \to \mathrm{GL}(V)$: a given linear representation
$\mathrm{End}\,(V)$ is naturally an $\mathcal{M}_{g,n}$-module by

$$f_* X = \rho(f) X \rho(f)^{-1} \quad (f \in \mathcal{M}_{g,n},\, X \in \mathrm{End}\,(V)).$$

Definition

Let $M_\rho := \mathrm{Spann}_K(\rho \circ \iota(\mathcal{S})) \subset \mathrm{End}\,(V)$.

M_ρ is preserved under the $\mathcal{M}_{g,n}$-action

$$\left(\because \quad \begin{aligned} f_*(\rho \circ \iota(C)) &= \rho(f)\rho(t_C)\rho(f)^{-1} = \rho(f t_C f^{-1}) \\ &= \rho(t_{f(C)}) = \rho \circ \iota(f(C)) \end{aligned} \right)$$

M_ρ receives a structure of fin. dim. module of curves with

$$p_\rho : K[\mathcal{S}] \to M_\rho, \quad p_\rho(C) := \rho \circ \iota(C).$$

Visualization for closed surface

The construction of the module of curves associated to a linear representation, together with the Starting Lemma, implies:

Theorem (K.)

Let $g \geq 1$ and $n \geq 0$. Then $\mathcal{M}_{g,n}$ admits a finite dimensional linear representation over K with kernel $\subset Z(\mathcal{M}_{g,n})$ if and only if it has a finite dimensional module of curves $p : K[\mathcal{S}] \to M$ such that $p|_{\mathcal{S}}$ is an injection.

Since $Z(\mathcal{M}_{g,n}) = 1$ for $g \geq 3$ and $n = 0$, we have

Corollary (K.)

For $g \geq 3$, $\mathcal{M}_g$ is K-linear if and only if it admits a finite dimensional module of curves $p : K[\mathcal{S}] \to M$ such that $p|_{\mathcal{S}}$ is injective.

Problems

For a module of curves

$$p : K[\mathcal{S}] \to M,$$

- When does M have finite dimensions over K?
- When is $\operatorname{Ker} p$ finitely generated as an $\mathcal{M}_{g,n}$-module?

Geometric intersection in some representations (Recall)

As for some linear representations of MCG such as Burau rep., Lawrence–Krammer rep., and the Magnus rep. of Torelli group,

Faithfulness critetia known (up to center):

whether or not the MCG-invariant Blanchfield type pairing of the representation space detects the geometric intersenction among certain types of curves in the underlying surface.

Merit: each curve corresponds directly to a point in rep. space; ($\Rightarrow$ rather easy to check the criterion).

- seems to depend on the explicit descriptions of representation.

We show here that

we may use our Starting Lemma

to obtain a faithfulness criterion which is rather indirect but applicable for any group hom. of MCG

by replacing the Blanchfield type pairing with commutator of Dehn twists

Geometric intersection (definition)

$\Sigma_{g,n}$: the compact connected oriented surface of genus $g \geq 1$ and $n \geq 0$ boundary components.

$\mathcal{M}_{g,n}$: MCG of $\Sigma_{g,n}$ (id on ∂)

$\mathcal{S}$: the set of isotopy classes of essential SCC on $\Sigma_{g,n}$

Definition

$i_{\text{geom}} : \mathcal{S} \times \mathcal{S} \to \mathbb{Z}_{\geq 0}$ is defined for $a, b \in \mathcal{S}$ by

$$i_{\text{geom}}(a, b) = \min \# |\alpha \cap \beta|$$

where α and β vary the representing curves of a and b, respectively.

We say a and b has geometric intersection if $i_{\text{geom}}(a, b) \neq 0$.
Note: any $a \in \mathcal{S}$ does not have geometric intersection with itself.

Basic fact (e.g., Farb–Margalit's book)

For $c_1,\ c_2 \in \mathcal{S}$,

$$i_{\mathrm{geom}}(c_1, c_2) = 0 \quad \Leftrightarrow \quad [t_{c_1}, t_{c_2}] = 1.$$

Proof.

($\Rightarrow$) Trivial.

($\Leftarrow$) Suppose $[t_{c_1}, t_{c_2}] = 1$.

Since $ft_cf^{-1} = t_{f(c)}$ in general, $[t_{c_1}, t_{c_2}] = t_{t_{c_1}(c_2)} \cdot t_{c_2}^{-1} = 1$, i.e.,

$$t_{t_{c_1}(c_2)} = t_{c_2}$$

$\Rightarrow t_{c_1}(c_2) = c_2$ ($\because\ \iota : \mathcal{S} \to \mathcal{M}_{g,n}$ is injective)

$\Rightarrow i_{\mathrm{geom}}(c_1, c_2) = 0$ ($\because\ i_{\mathrm{geom}}(t_{c_1}(c_2), c_2) = i_{\mathrm{geom}}(c_1, c_2)^2$) □

In view of this, we have

Theorem (K.)

For any $\rho : \mathcal{M}_{g,n} \to G$,

$$(\rho \textit{ detects the geometric intersection in } \mathcal{S}) \quad \Leftrightarrow \quad \operatorname{Ker}\rho \subset Z(\mathcal{M}_{g,n})$$

Proof

($\Rightarrow$) Suppose $[t_{c_1}, t_{c_2}] = 1$ for those satisfying $\rho([t_{c_1}, t_{c_2}]) = 1$.

For $c_1 \neq c_2$, $\exists d \in \mathcal{S}$ s.t.

$$i_{\text{geom}}(c_1, d) = 0 \quad \& \quad i_{\text{geom}}(c_2, d) \neq 0.$$

Then, $\rho([t_{c_1}, t_{c_2}]) = 1$ while $\rho([t_{c_1}, t_{c_2}]) \neq 1 \Rightarrow \rho(t_{c_1}) \neq \rho(t_{c_2})$,
I.e., $\iota \circ \rho : \mathcal{S} \to G$ is injective.

$\Rightarrow$ Starting Lemma implies $\operatorname{Ker}\rho \subset Z(\mathcal{M}_{g,n})$.

Proof (cont')

($\Leftarrow$) Suppose Ker $\rho \subset Z(\mathcal{M}_{g,n})$.

If $\rho([t_{c_1}, t_{c_2}]) = 1$,

$$t_{t_{c_1}(c_2)} \cdot t_{c_2}^{-1} = [t_{c_1}, t_{c_2}] \in \text{Ker}\,\rho \subset Z(\mathcal{M}_{g,n})$$

Since $Z(\mathcal{M}_{g,n}) = \text{Ker}\,(\mathcal{M}_{g,n} \to \mathrm{Aut}\,(\mathcal{S}))$,

$$(t_{t_{c_1}(c_2)})_* = (t_{c_2})_* \quad \text{on } \mathcal{S}.$$

$\Rightarrow t_{c_1}(c_2) = c_2$ in $\mathcal{S}$.

$\Rightarrow [t_{c_1}, t_{c_2}] = t_{c_1}(c_2) \cdot t_{c_2}^{-1} = 1$.

I.e., ρ detects the geometric intersection in $\mathcal{S}$. □

Refinement

- After Brendle–Margalit: $\mathrm{Ker}\left(\mathcal{M}_{g,n} \to \mathrm{Aut}\left(\mathcal{S}_{\mathrm{sep}}\right)\right) = Z(\mathcal{M}_{g,n})$

- For $c_1 \neq c_2 \in \mathcal{S}$, $\exists d \in \mathcal{S}$ s.t. $i_{\mathrm{geom}}(c_i, d) = \delta_{i,2}$; Furthermore,
  - if c_1, $c_2 \in \mathcal{S}^{\mathrm{nonsep}}$, d can be also chosen in $\mathcal{S}^{\mathrm{nonsep}}$;
  - if c_1, $c_2 \in \mathcal{S}_{\mathrm{sep}}$, d can be also chosen in $\mathcal{S}_{\mathrm{sep}}$.

These imply that the previous theorem holds true

w/ $\mathcal{S}$ replaced by $\mathcal{S}^{\mathrm{nonsep}}/\mathcal{S}_{\mathrm{sep}}$.

In particular, we have

Theorem (K.)

Let $\mathcal{S}_{sep} \neq \phi$, *i.e.*, $n \geq 4 - 2g$;
$H < \mathcal{M}_{g,n}$: *a subgp containing* $\iota(\mathcal{S}_{sep})$;
$\rho : H \to G$ *arbitrary group homomorphism.*

Then

$$(\rho \text{ detects geometric intersection in } \mathcal{S}_{sep}) \quad \Leftrightarrow \quad \operatorname{Ker} \rho \subset Z(\mathcal{M}_{g,n}).$$

Remark

1) This theorem seems to explain partially the significance of the work by Suzuki on a constuction of non-trivial kernel of the Magnus representation of the Torelli subgroup of $\mathcal{M}_{g,1}$ associated to the abelianization $\pi_1(\Sigma_{g,1}) \to H_1(\Sigma_{g,1}; \mathbb{Z})$.

2) This theorem is true for $g = 0$ (the proof is easier).

The Johnson filtration (notation)

Let $g \geq 1$ and $n = 1$ (for simplicity).

$\Gamma := \pi_1(\Sigma_{g,1}, *)$ with $* \in \partial\Sigma_{g,1}$ fixed.

$\{\Gamma_k\}_{k\geq 1}$: the lower central series of Γ,

defined by $\Gamma_1 = \Gamma$ and $\Gamma_{k+1} = [\Gamma_k, \Gamma]$ for $k \geq 1$.

$N_k := \Gamma/\Gamma_{k+1}$

$\rho_k : \mathcal{M}_{g,1} \to \mathrm{Aut}\,(N_k)$

$\mathcal{M}(k) := \mathrm{Ker}\,\rho_k$; ($\mathcal{M}(1)$: Torelli gp,

$\mathcal{M}(2)$: the Johnson kernel, ...)

$\{\mathcal{M}(k)\}_{k\geq 1}$: the Johnson filtration of $\mathcal{M}_{g,1}$.

Fact

- $\bigcap_{k=1}^{\infty} \mathcal{M}(k) = \{1\}$ *(Johnson)*
- $\mathcal{M}(k) \not\subset Z(\mathcal{M}_{g,n})$ *for* $\forall k \geq 1$ *(easy to check)*

This implies the following:

Geometric intersection in Johnson filtration

Corollary (K.)

For $\forall k \geq 1$, ρ_k does not detect the geometric intersection in $\mathcal{S}/\mathcal{S}^{nonsep}/\mathcal{S}_{sep}$, while the totality of $\{\rho_k\}_{k\geq 1}$ does.

$\Rightarrow$ one may consider:

Geometric intersection in Johnson filtration (Cont')

Definition

For $c_1,\ c_2 \in \mathcal{S}$,

$$i_{\mathrm{JF}}(c_1, c_2) := \begin{cases} 1 & \text{if } [t_{c_1}, t_{c_2}] \notin \mathcal{M}(1); \\ k+1 & \text{if } [t_{c_1}, t_{c_2}] \in \mathcal{M}(k) \text{ and } [t_{c_1}, t_{c_2}] \notin \mathcal{M}(k+1); \\ 0 & \text{if } [t_{c_1}, t_{c_2}] \in \mathcal{M}(k) \text{ for all } k \geq 1. \end{cases}$$

Remark

- i_{JF} is unbounded.
- $i_{\mathrm{JF}}(c_1, c_2) = 0 \Leftrightarrow i_{\mathrm{geom}}(c_1, c_2) = 0$
- $i_{\mathrm{JF}}(c_1, c_2) \geq 2 \iff i_{\mathrm{geom}}(c_1, c_2) \neq 0 \ \&\ \langle \vec{c_1}, \vec{c_2} \rangle = 0$.
  In particular, if $\langle \vec{c_1}, \vec{c_2} \rangle \neq 0$, then $i_{\mathrm{JF}}(c_1, c_2) = 1$.

Problem

Study further properties of i_{JF}.

Summary

- $\mathcal{M}_g$ is K-linear if and only if there exists a finite dimensional module of curves
$$p : K[\mathcal{S}] \to M$$
such that $p|_{\mathcal{S}}$ is injective.

$$\begin{pmatrix}\text{When is a module of curves finite dimensional, or has} \\ \text{a finite MCG-generators, in general?}\end{pmatrix}$$

- Focusing on $i_{\mathrm{geom}}(c_1, c_2) = 0 \Leftrightarrow [t_{c_1}, t_{c_2}] = 1$, the injectivity condition can be obtained for any homomorphism of $\mathcal{M}_{g,n}$ in terms of detection of geometric intersection.

$$\begin{pmatrix}\text{Geometric intersection in Johnson filtration} \\ \text{—how quantitive?}\end{pmatrix}$$