Infinitely many corks with shadow complexity one

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Topology and Geometry of Low-dimensional Manifolds

The plan of talk

1. 4-manifolds and exotic pairs
   - Kirby diagram
   - Corks
2. Polyhedron and reconstruction of 4-manifold
   - Polyhedron
   - Shadows and 4-manifolds
3. Main result

※ In this talk we assume that manifolds are smooth.

§1 4-manifolds and exotic pairs

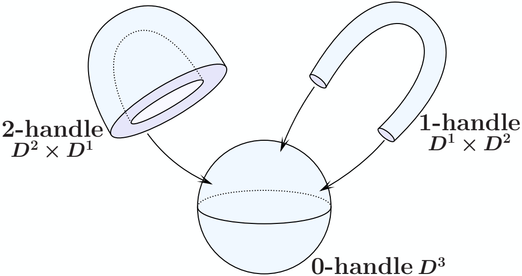
·**Kirby diagram**

·**Corks**

Handle decomposition

Definition

X : a compact n-dimensional manifold w/ ∂
An (**n-dimensional**)**k-handle** is a copy of $D^k \times D^{n-k}$, attached to ∂X along $\partial D^k \times D^{n-k}$ by an embedding $f : \partial D^k \times D^{n-k} \to \partial X$.



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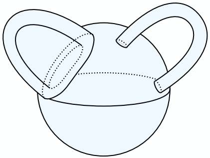
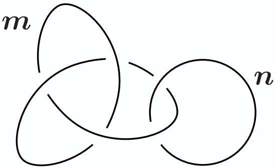


Kirby diagram(0- and 2-handle)

A **Kirby diagram** is a description of a handle decomposition of a 4-manifold by a knot/link diagram in $\mathbb{R}^3$.

- ∂(0-handle) $\cong S^3 = \mathbb{R}^3 \cup \{\infty\}$.
- An attaching region of a 2-handle is $S^1 \times D^2$.



Two 2-handles with **framing coefficients** m and n.

Kirby diagram(1-handle)

- An attaching region of a 1-handle is $D^3 \amalg D^3$.

1-handle.

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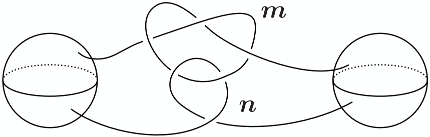

1- and 2-handles.
The 2-handles are attached along the 1-handle.

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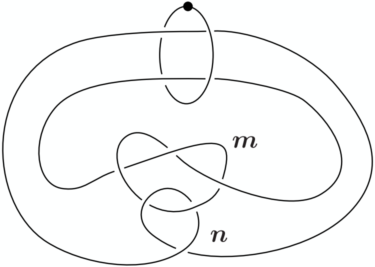

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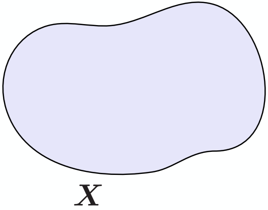
Definition

Two manifolds X and Y are said to be **exotic** if they are homeomorphic but not diffeomorphic.

Theorem (Akbulut-Matveyev, '98)

For any exotic pair (X, Y) of 1-connected closed 4-manifolds, Y is obtained from X by removing a contractible submanifold of codimension 0 and gluing it via an involution on the boundary.



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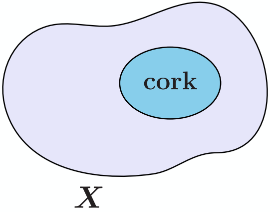

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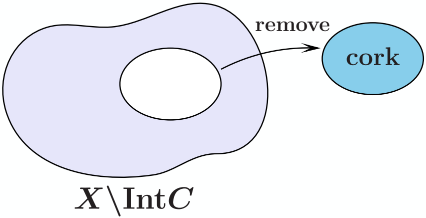

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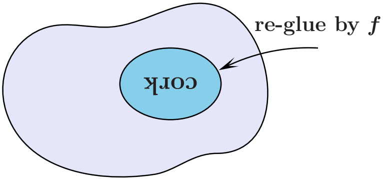

$(X \setminus \mathrm{Int}C) \cup_f C$

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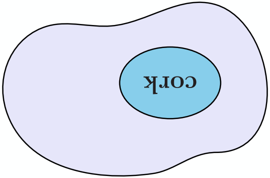


$$\overset{\text{diffeo.}}{\cong} \quad Y$$

$$(X \setminus \mathrm{Int}C) \cup_f C$$

Definition

A pair (C, f) of a contractible compact Stein surface C and an involution $f : \partial C \to \partial C$ is called a **cork** if f can extend to a self-homeomorphism of C but can not extend to any self-diffeomorphism of C.

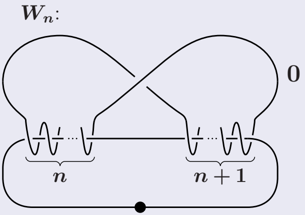
A real 4-dimensional manifold X is called a compact Stein surface

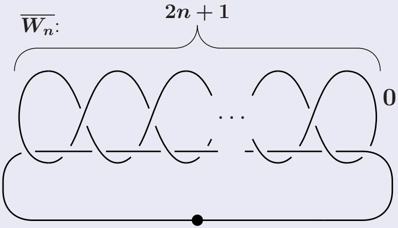
$\overset{\text{def}}{\iff}$ There exist a complex manifold W, a plurisubharmonic function $\varphi : W \to \mathbb{R}_{\geq 0}$ and its regular value r s.t. $\varphi^{-1}([0, r])$ is diffeomorphic to X.

Examples of corks

Theorem (Akbulut-Yasui, '08)

Let W_n and $\overline{W_n}$ be 4-manifolds given by the following Kirby diagrams. They are corks for $n \geq 1$.





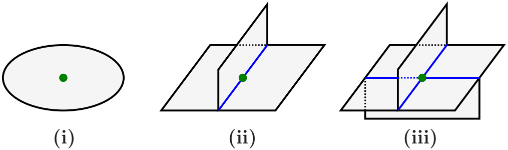
Application $\cdots$ Construction of exotic elliptic surfaces.
Counterexamples to Akbulut-Kirby conjecture.

§2 Polyhedron and reconstruction of 4-manifold

·Polyhedron

·Shadows and 4-manifolds

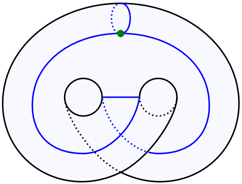
- An almost-special polyhedron is a compact topological space P s.t. a neighborhood of each point of P is one of the following :



- A point of type (iii) is called a **true vertex**.
- Each connected component of the set of points of type (i) is called a **region**.

If any regions of P are 2-disks and P has at least 1 true vertex, then P is called a **special polyhedron**.

Example : Abalone



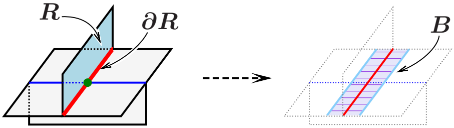
shadow

Definition

W : a compact oriented 4-manifold w/ ∂
$P \subset W$: an almost special polyhedron
We assume that W has a strongly deformation retraction onto P and P is proper and locally flat in W. Then we call P a **shadow** of W.

gleam

Let P be a special polyhedron and R be a region of P.



The band B is an imm. annulus or an imm. Möbius band in P s.t. its core is ∂R.

Definition

For each region R, we choose a (half) integer $gl(R)$ s.t.

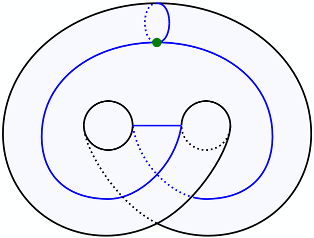
$$gl(R) \in \begin{cases} \mathbb{Z} & \text{if } B \text{ is an imm. annulus.} \\ \mathbb{Z} + \frac{1}{2} & \text{if } B \text{ is an imm. Möbius band.} \end{cases}$$

We call this value a **gleam**.

Turaev's reconstruction

Theorem (Turaev's reconstruction, '90s)

A 4-manifold W is reconstructed from a special polyhedron P and gleams on its regions in a unique way.



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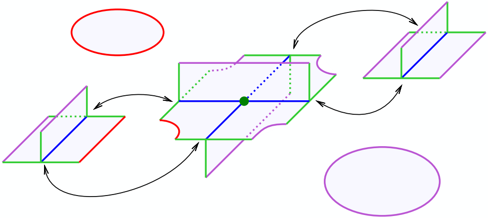

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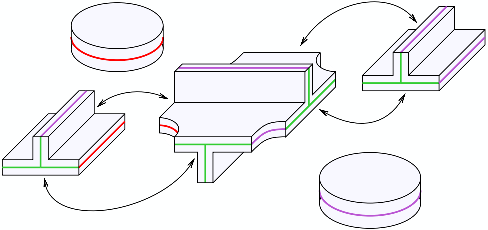

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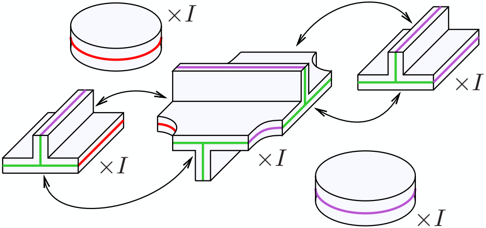

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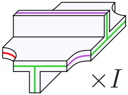
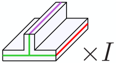
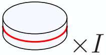


(true vertex) $\longleftrightarrow$ 0-handle



(edge) $\longleftrightarrow$ 1-handle(attached along $D^3 \amalg D^3$)

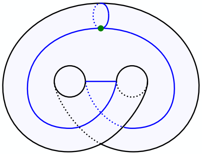


(region) $\longleftrightarrow$ 2-handle(attached along $S^1 \times D^2$)

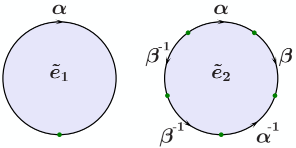
contractible special polyhedra

We want to construct corks from special polyhedra(shadows).

- no true vertex There is no such a polyhedron.
- one true vertex There are just 2 special polyhedra A and $\widetilde{A}$ shown in the following[Ikeda, '71] :



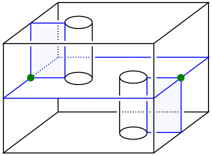
A



$\widetilde{A}$

- two true vertices

  e.g. Bing's house



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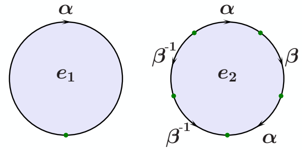

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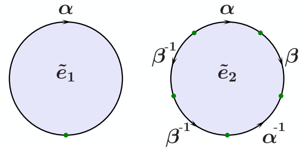

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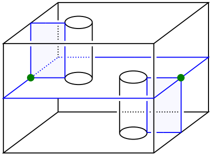
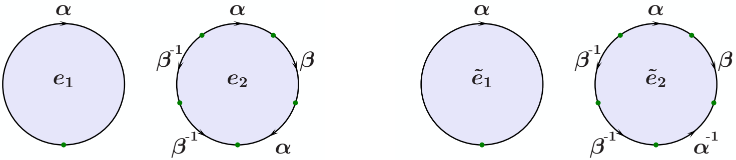


4-manifolds from A and $\widetilde{A}$



$gl(e_1) = m,\ gl(e_2) = n$ $\qquad$ $gl(\tilde{e}_1) = m,\ gl(\tilde{e}_2) = n - \frac{1}{2}$

$\downarrow$ Turaev's reconstruction $\downarrow$

$A(m, n)$ $\qquad$ $\widetilde{A}(m, n - \frac{1}{2})$

§3 Main result

Main theorem

Definition

W : a compact oriented 4-manifold w/ ∂
The **special shadow complexity** $sc^{sp}(W)$ of W is defined by

$$sc^{sp}(W) = \min_{\substack{P \text{ is a special} \\ \text{shadow of } W.}} \sharp\{\text{true vertices of } P\}$$

Theorem (N.)

Consider the family $\{\widetilde{A}(m, -\frac{3}{2})\}_{m<0}$ of 4-manifolds. Then the following hold :

(1) $sc^{sp}(\widetilde{A}(m, -\frac{3}{2})) = 1.$

(2) They are mutually not homeomorphic.

(3) They are corks.

We prove by the following two lemmas.

Lemma A

Let m and n be integers.

(1) $\lambda(\partial A(m,n)) = -2m$. Therefore $A(m,n)$ and $A(m',n)$ are not homeomorphic unless $m = m'$.

(2) $\lambda(\partial \widetilde{A}(m, n-\frac{1}{2})) = 2m$. Therefore $\widetilde{A}(m, n-\frac{1}{2})$ and $\widetilde{A}(m', n-\frac{1}{2})$ are not homeomorphic unless $m = m'$.

Recall.

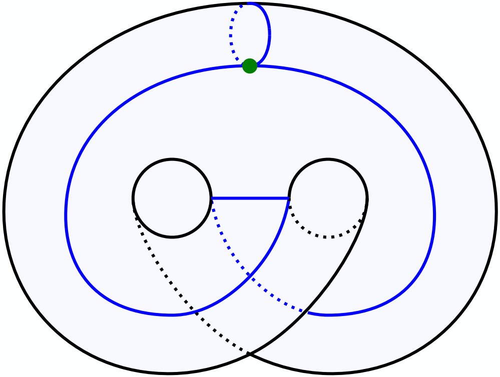
- $\lambda : \{\mathbb{Z}\text{HS}^3\} \to \mathbb{Z}$: Casson invariant is a topological invariant.
- Any contractible manifold is bounded by a homology sphere.

Lemma B

The manifold $\widetilde{A}(m, -\frac{3}{2})$ is a cork if $m < 0$.

Reconstruction of (A, gl)

First we describe Kirby diagrams of $A(m,n)$ and $\widetilde{A}(m, n-\frac{1}{2})$.



Abalone A.

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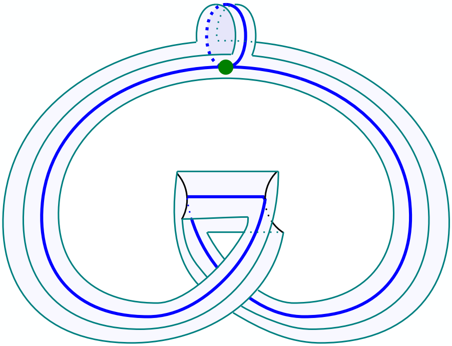

$A \setminus \{\text{region parts}\}$.

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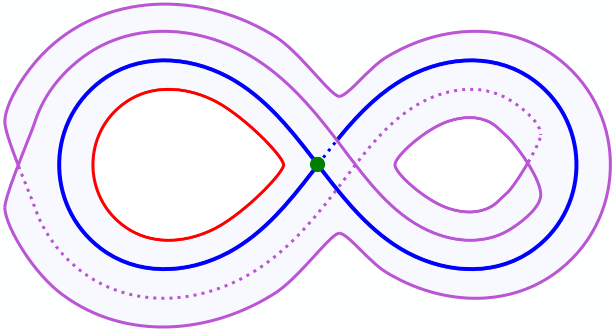

A subpolyhedron consisting of one true vertex and two edges.

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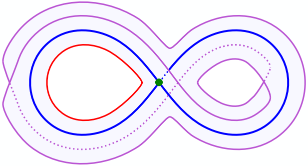

true vertex $\longleftrightarrow$ 0-handle
edge $\longleftrightarrow$ 1-handle

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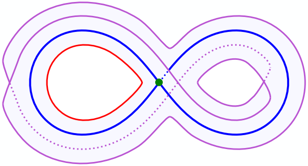

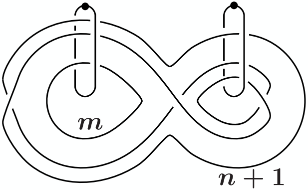

true vertex $\longleftrightarrow$ 0-handle
edge $\longleftrightarrow$ 1-handle
region $\longleftrightarrow$ 2-handle

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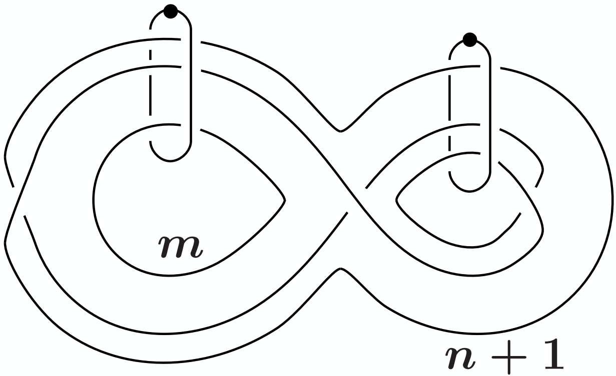

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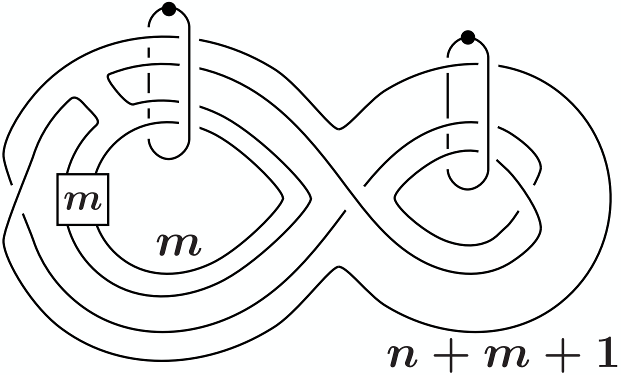

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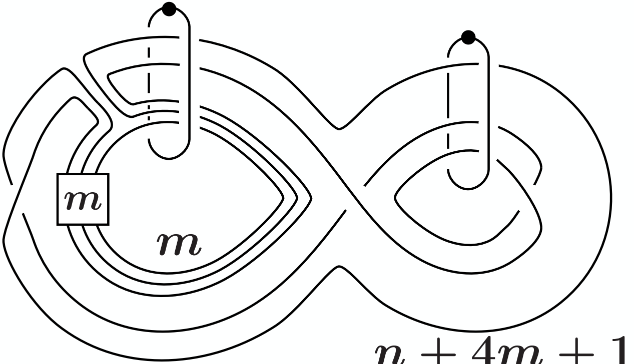

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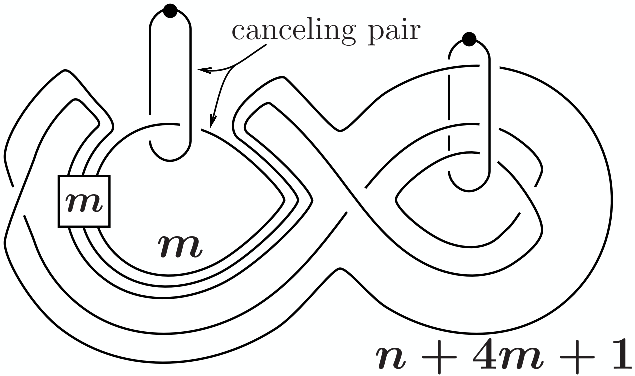

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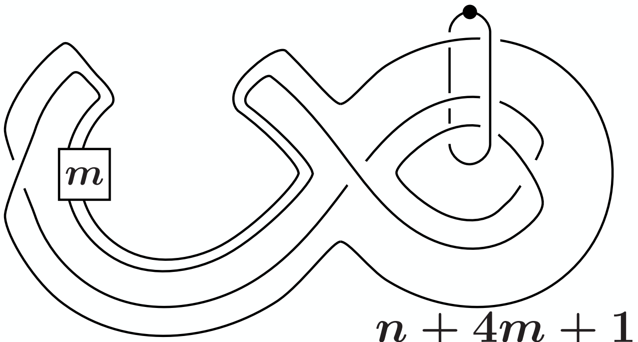

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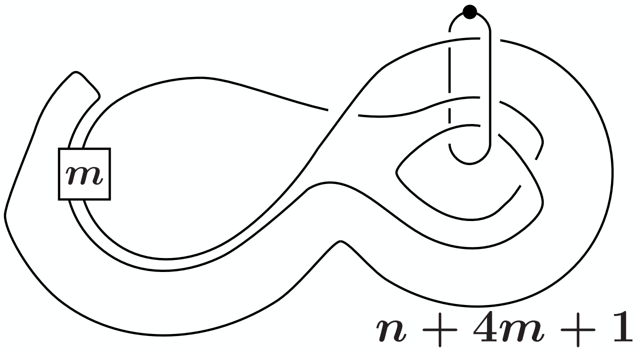

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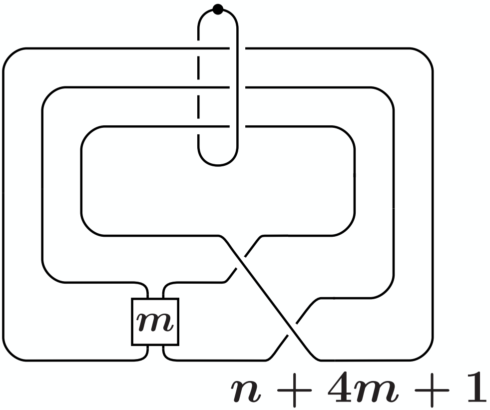

Lemma A (again)

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(1) $\lambda(\partial A(m,n)) = -2m$. Therefore $A(m,n)$ and $A(m',n)$ are not homeomorphic unless $m = m'$.

(2) $\lambda(\partial \widetilde{A}(m, n-\frac{1}{2})) = 2m$. Therefore $\widetilde{A}(m, n-\frac{1}{2})$ and $\widetilde{A}(m', n-\frac{1}{2})$ are not homeomorphic unless $m = m'$.

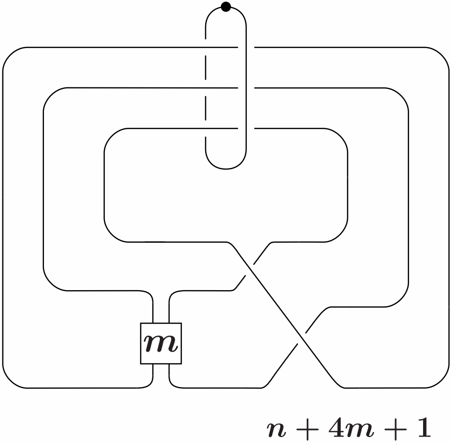
Proof : We describe surgery diagrams of $\partial A(m,n)$ and $\widetilde{A}(m', n-\frac{1}{2})$ and calculate their Casson invariants by using the surgery formula.

Theorem (Casson)

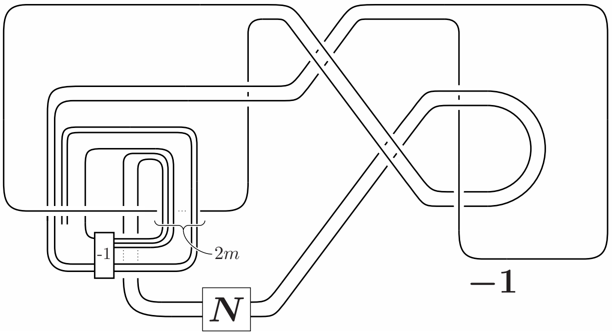
For any integer-homology sphere Σ and knot $K \subset \Sigma$, the following holds

$$\lambda(\Sigma + \frac{1}{m} \cdot K) = \lambda(\Sigma) + \frac{m}{2} \Delta''_{K\subset\Sigma}(1).$$

Proof(1/2) A surgery diagram of $\partial A(m,n)$



Proof(1/2) A surgery diagram of $\partial A(m,n)$



This knot is a **ribbon knot**. Calculate its Alexander polynomial by using the way in [1].

$$\Delta_K(t) = t^{m+1} - t^m - t + 3 - t^{-1} - t^{-m} + t^{-m-1}.$$

[1] H. Terasaka, *On null-equivalent knots*, Osaka Math. J. 11 (1959), 95-113.

証明 (2/2) Calculate the Casson invariant

By the Surgery formula :

$$\begin{aligned}\lambda(\partial A(m,n)) &= \lambda(S^3) + \frac{-1}{2}\Delta_K''(1)\\ &= 0 - \frac{1}{2}\cdot 4m\\ &= -2m.\end{aligned}$$

We can prove (2) similarly to (1). □

Lemma B (again)

The manifold $\widetilde{A}(m, -\frac{3}{2})$ is a cork if $m < 0$.

Theorem (Akbulut-Karakurt '12)

Let C be a compact oriented 4-manifold w/ ∂ whose Kirby diagram is given by a dotted circle K_1 and a 0-framed unknot K_2. C is a cork if the following hold :

(1) K_1 and K_2 are symmetric.

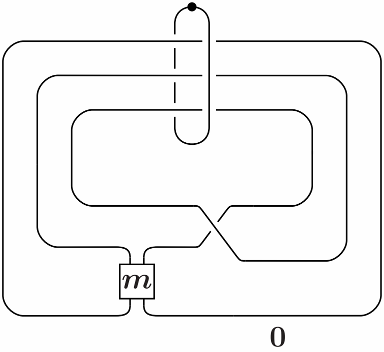
(2) $lk(K_1, K_2) = \pm 1$.

(3) The diagram satisfies the condition of Stein handlebody.

Remark.

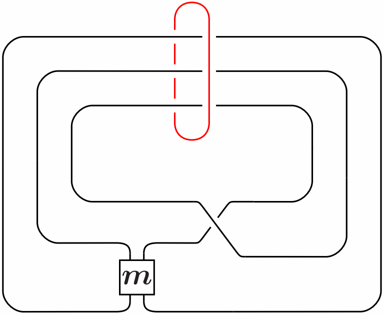
Gomph showed a necessary and sufficient condition for that a 4-dimensional handlebody is a compact Stein surface.

Proof : (1)symmetry and (2)linking number

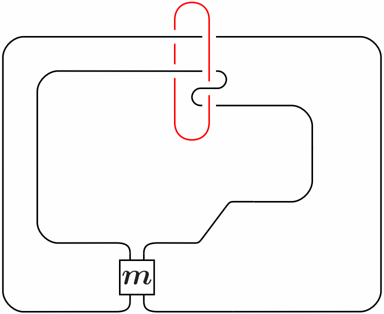


A Kirby diagram of $\widetilde{A}(m, -\frac{3}{2})$

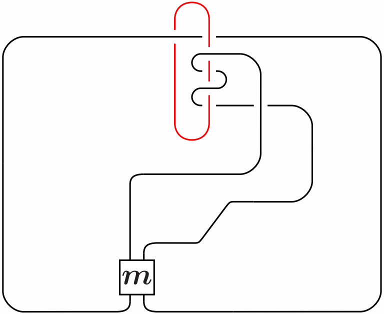
Proof : (1)symmetry and (2)linking number



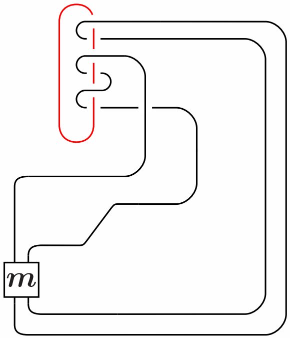
Proof : (1)symmetry and (2)linking number



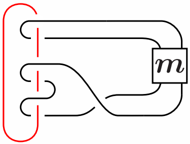
Proof : (1)symmetry and (2)linking number



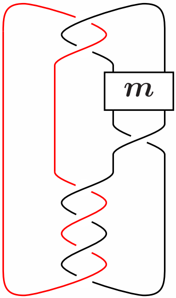
Proof : (1)symmetry and (2)linking number



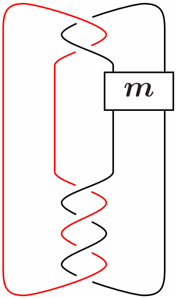
Proof : (1)symmetry and (2)linking number



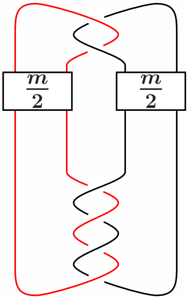
Proof : (1)symmetry and (2)linking number



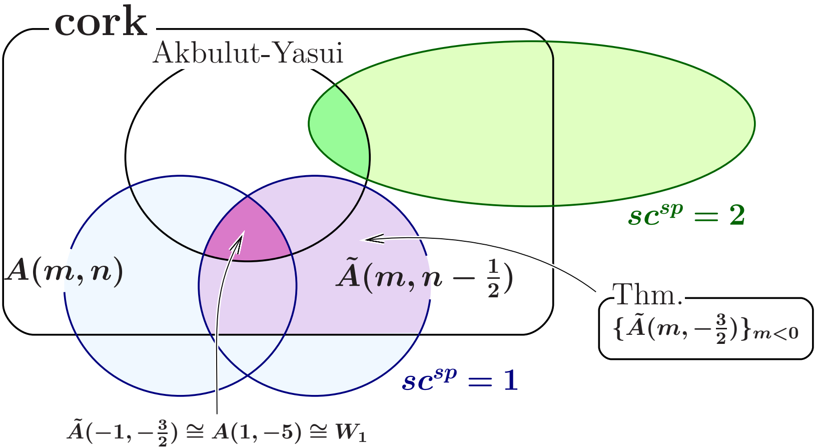
Proof : (1)symmetry and (2)linking number



Proof : (1)symmetry and (2)linking number



Summary


cork
Akbulut-Yasui
$A(m,n)$
$\tilde{A}(m, n - \frac{1}{2})$
$sc^{sp} = 2$
$sc^{sp} = 1$
Thm.
$\{\tilde{A}(m, -\frac{3}{2})\}_{m<0}$
$\tilde{A}(-1, -\frac{3}{2}) \cong A(1,-5) \cong W_1$