Exotic components in linear slices of quasi-Fuchsian groups

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Outline

S : ori. surface with $\chi(S) < 0$

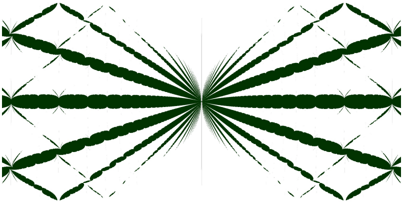
$$X(S) = \{\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}\}/\{\sim \text{conjugation}\} \quad \text{(character variety)}$$

$$\cup$$

$$AH(S) = \{[\rho] \in X(S) \mid \rho : \text{faithful, discrete}\}$$

By the celebrated Ending Lamination Theorem, $AH(S)$ is completely classified ($\exists$ explicit parametrization).

But the shape of $AH(S)$ in $X(S)$ is complicated (cf. bumping phenomena, non-local connectivity).



$AH(S)$ (shaded) in some slice of $X(S)$

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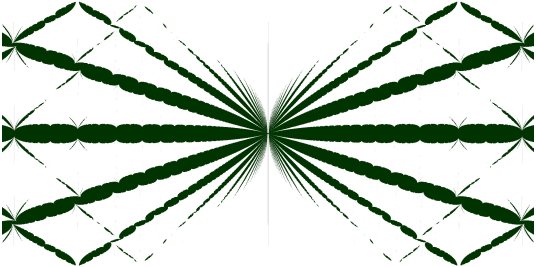


$AH(S)$ (shaded) in some slice of $X(S)$

Aim of this talk

- Try to understand the shape $AH(S)$ in $X(S)$ by taking slices.
- In particular, in terms of exotic projective structures.

Quick overview of Kleinian surface groups

$\mathbb{H}^3$: 3-dim hyperbolic space

$\mathrm{PSL}_2\mathbb{C}$ is isomorphic to the ori. pres. isometry group of $\mathbb{H}^3$.

For a surface S with $\chi(S) < 0$, let

$$X(S) = \{\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}\}/\{\sim \text{conj. by } \mathrm{PSL}_2\mathbb{C}\} \quad \text{(character variety)}$$

$$\cup$$

$$AH(S) = \{[\rho] \in X(S) \mid \text{faithful, } \rho(\pi_1(S)) \text{ is discrete}\}$$

(If S has punctures, we assume that reps are 'type-preserving'.)

If $\rho \in AH(S)$, $\mathbb{H}^3/\rho(\pi_1(S))$ is a hyp 3-mfd homotopy equiv. to S.

(Moreover, $\mathbb{H}^3/\rho(\pi_1(S))$ is homeo to $S \times (-1, 1)$ (Bonahon).)

Simple example : $\rho : \pi_1(S) \xrightarrow{\cong} \Gamma < \mathrm{PSL}_2(\mathbb{R})$ $\quad$ (Γ : Fuchsian group)

Quick overview of Kleinian surface groups

Simple example : $\rho : \pi_1(S) \xrightarrow{\cong} \Gamma < PSL_2(\mathbb{R})$ (Γ : Fuchsian group)

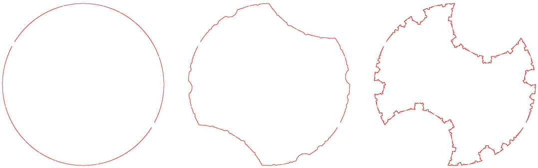
In this case, the limit set

$\Lambda = \{\text{accumulation pts of } \rho(\pi_1(S)) \cdot p \text{ at } \infty\} \subset \mathbb{C}P^1$ (for some $p \in \mathbb{H}^3$)

is a round circle.

$\rho \in AH(S)$ is called quasi-Fuchsian if the limit set Λ is homeo to a circle.

$$QF(S) = \{\rho \in AH(S) \mid \text{quasi-Fuchsian}\}$$



Anyway, known that

$$QF(S) = \mathrm{Int}(AH(S)).$$

Moreover,

$$\overline{QF(S)} = AH(S)$$

(Density Theorem).

Quick overview of Kleinian surface groups

By Ahlfors-Bers theorem,

$$QF(S) \cong T(S) \times T(S)$$

where $T(S)$ is the Teichmüller space of S.

In particular, $QF(S)$ is homeo to $\mathbb{R}^{2(6g-6)}$ if S is closed, genus g.

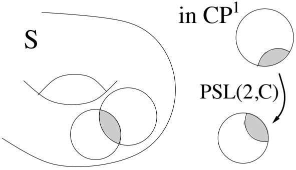
$$\begin{aligned}
X(S) &= \{\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}\}/\{\sim \text{conj. by } \mathrm{PSL}_2\mathbb{C}\} \\
&\cup \\
AH(S) &= \{[\rho] \in X(S) \mid \text{faithful, } \rho(\pi_1(S)) \text{ is discrete}\} \\
&\cup \ \text{open, dense} \\
QF(S) &= \{[\rho] \in AH(S) \mid \text{quasi-Fuchsian}\} \\
&\wr\!\| \\
T(S) \times T(S) &\cong \mathbb{R}^{2(6g-6)}
\end{aligned}$$

Complex projective structures

S : surface ($\chi(S) < 0$)

Definition

A complex projective structure or $\mathbb{C}P^1$-structure on S is a geometric structure locally modelled on $\mathbb{C}P^1$ with transition functions in $\mathrm{PSL}_2\mathbb{C}$.

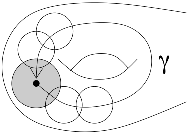


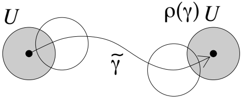
(If S has punctures, assume some boundary conditions.)

By analytic continuation, we have a pair of maps

$$D : \widetilde{S} \to \mathbb{C}P^1 \text{ (developing map)}, \quad \rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \text{ (holonomy)}$$

s.t. $D(\gamma \cdot x) = \rho(\gamma) \cdot D(x)$ ($\gamma \in \pi_1(S)$, $x \in \widetilde{S}$).





Conversely, the pair determines the $\mathbb{C}P^1$-str (mod $(D, \rho) \sim (gD, g\rho g^{-1})$).

Complex projective structures

Example (Fuchsian uniformization)

A hyperbolic str on S gives an identification $\widetilde{S} \cong \mathbb{H}^2$. Since $\mathbb{H}^2 \subset \mathbb{C}P^1$, this gives a $\mathbb{C}P^1$-str.

Similarly as Teichmüller space, we can define

$$P(S) = \{\text{marked } \mathbb{C}P^1\text{-structures on } S\}.$$

Two important maps :

- The holonomy gives a map

$$\text{hol} : P(S) \to X(S) = \text{Hom}(\pi_1(S), \text{PSL}_2\mathbb{C})/\text{conj.} : (D, \rho) \mapsto \rho$$

- Since Möbius transformations are holomorphic, a $\mathbb{C}P^1$-str defines a hol str (and the hyp. str. conformally equiv. to that).

$$P(S) \to T(S) = \text{Teichmüller space}$$

Bers slice

Each fiber of $P(S) \to T(S)$ is parametrized by

$$H^0(X, K_X^2) = \{\text{hol. quad. differentials}\}$$

via Schwarzian derivatives. In particular, if S is closed, genus g,

$$\dim_{\mathbb{R}} P(S) = \dim_{\mathbb{R}} T(S) + \dim_{\mathbb{R}} H^0(X, K_X^2) = 2(6g-6)$$

The set of $\mathbb{C}P^1$-strs with q-F holonomy in $H^0(X, K_X^2)$ is open.

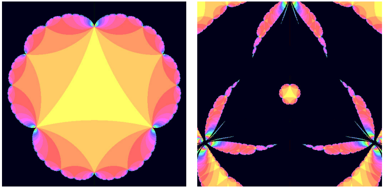


Image by Y. Yamashita

- $0 \in H^0(X, K_X^2)$ corresponds to the Fuchsian uniformization of X.
- The comp $\ni 0$ parametrizes $T(S)$. (This gives $T \times T \cong QF$.)

But there are many other components : exotic components.

We are interested in similar phenomena in another slice.

Goldman's classification

Let

$$Q_0 = \{\mathbb{C}P^1\text{-strs with q-F holonomy with inj. dev. map }\} \subset P(S).$$

Q_0 is a conn. comp. of $\mathrm{hol}^{-1}(QF(S)) = \{\mathbb{C}P^1\text{-strs with q-F holonomy}\}$.

2π-grafting

$c \subset S$: a simple closed curve
For $(D, \rho) \in Q_0$, we can change $D : \widetilde{S} \to \mathbb{C}P^1$ by inserting $\mathbb{C}P^1$ along each lift of c. This dose not change the holonomy ρ.

$$Q_c = \{2\pi\text{-grafting of } (D, \rho) \in Q_0\} \subset P(S)$$

$$\mathcal{ML}_{\mathbb{Z}}(S) = \{\text{disjoint union of scc's with } \mathbb{Z}_{\geq 0} \text{ weight}\}$$

The above operation can be generalized for $\mu \in \mathcal{ML}_{\mathbb{Z}}(S)$.

Theorem (Goldman (1987))

$$\mathrm{hol}^{-1}(QF(S)) = \bigsqcup_{\mu \in \mathcal{ML}_{\mathbb{Z}}(S)} Q_\mu \quad (Q_0\text{: standard, } Q_\mu\ (\mu \neq 0)\text{: exotic})$$

More on 2π-grafting

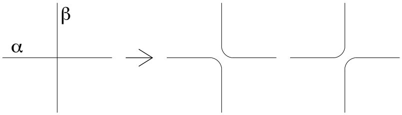
We have defined 2π-grafting for Q_0. This gives

$$Q_0 \xrightarrow{\cong} Q_\mu$$

We can also define 2π-grafting for Q_α along β ($\alpha, \beta \in \mathcal{ML}_\mathbb{Z}(S)$).

But if the intersection number $i(\alpha, \beta) \neq 0$, it depends on the choice of β in its isotopy class.

$$Q_\alpha \xrightarrow{\cong} Q_{(\alpha,\beta)_\sharp} \text{ or } Q_{(\alpha,\beta)_\flat}$$



(Kentaro Ito (2007), Calsamiglia-Deroin-Francaviglia (2014))

Linear slice

For $\gamma \in \pi_1(S)$ and $\rho \in X(S)$, $\rho(\gamma) \in \mathrm{PSL}_2\mathbb{C}$ acts on $\mathbb{H}^3$.

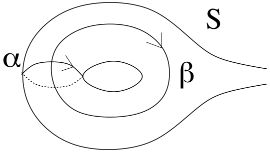
Define the complex length $X(S) \to \mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$ by

$$\lambda_\gamma(\rho) = (\text{translation length of } \rho(\gamma)) + \sqrt{-1}\,(\text{rotation angle of } \rho(\gamma)).$$

This is characterized by

$$\mathrm{tr}(\rho(\gamma)) = 2\cosh\left(\frac{\lambda_\gamma(\rho)}{2}\right).$$

From now on, we assume that S is a once punctured torus.
For convenience, fix $\alpha, \beta \in \pi_1(S)$ as in the figure.



In this case, $\dim_{\mathbb{C}} X(S) = 2$. For $\ell > 0$, define the linear slice by

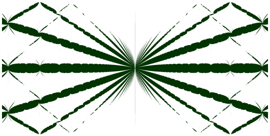
$$X(\ell) = \{\rho \in X(S) \mid \lambda_\alpha(\rho) \equiv \ell\}$$

Then $\dim_{\mathbb{C}} X(\ell) = 1$, so easy to visualize.

Complex Fenchel-Nielsen coordinates

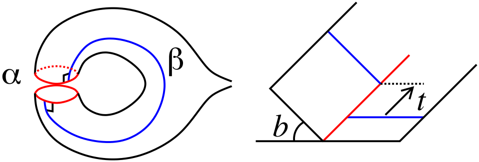
The complex Fenchel-Nielsen coordinates give a parametrization

$$\{\tau \in \mathbb{C} \mid -\pi < \mathrm{Im}(\tau) \leq \pi\} \xrightarrow{\cong} X(\ell)$$



$QF(S)$ in the linear slice $X(18.0)$.

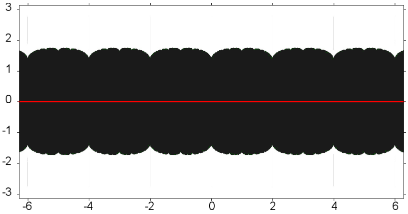
Geometrically speaking, if we let $\tau = t + \sqrt{-1}b$, the representation is obtained by twisting distance t and bending with angle b along α.



Linear slices of $QF(S)$

For each $\ell > 0$, we are interested in the shape of

$$QF(\ell) := QF(S) \cap X(\ell) \;\subset X(\ell)$$



$QF(2.0)$

- The Dehn twist along α acts on $X(\ell)$ as

$$\tau \mapsto \tau + \ell. \quad \text{(translation)}$$

- The real line $\{\tau \mid \mathrm{Im}(\tau) = 0\}$ corresponds to the Fuchsian representations satisfying $\lambda_\alpha = \ell$.
- By McMullen's disk convexity of $QF(S)$, $QF(\ell)$ is a union of (open) disks.

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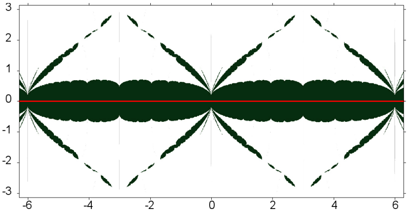

$QF(6.0)$

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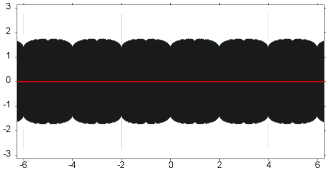
Linear slices of $QF(S)$

For any $\ell > 0$, there exists a unique standard component containing Fuchsian representations. As pictures suggest;

Theorem (Komori-Yamashita, 2012)

$QF(\ell)$ has only one component if ℓ is sufficiently small, has more than one component if ℓ is sufficiently large.



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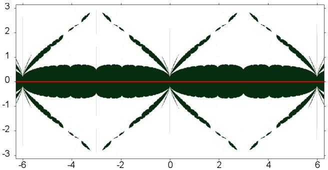


$QF(6.0)$

We will give another proof for the latter part. In fact, we characterize other components in terms of Goldman's classification.

We lift the slice $X(\ell) \subset X(S)$ to $P(S)$ by complex earthquake.

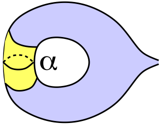
Grafting

(Remark : “grafting” here is similar but different from 2π-grafting before. In fact, “grafting” here changes the holonomy.)

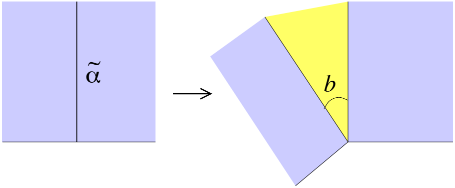
We can construct another $\mathbb{C}P^1$-str from a Fuchsian uniformization.

X : a hyp str on S, $\alpha \subset X$: a simple closed geodesic.

Let $\mathrm{Gr}_{b\cdot\alpha}(X)$ be the $\mathbb{C}P^1$-str obtained from X by inserting a height b annulus along α.

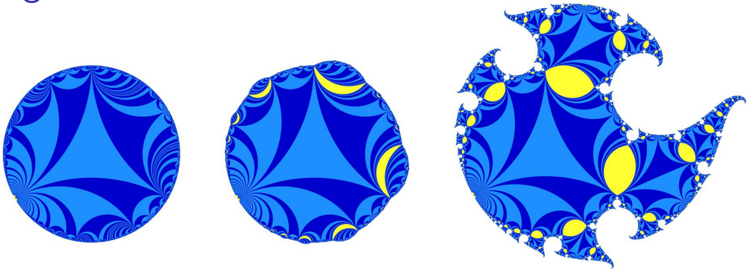


In the universal cover $\widetilde{X}$, the local picture looks like:



(By construction, $\mathrm{Gr}_{2\pi\cdot\alpha}(X)$ is obtained from X by 2π-grafting along α.)

Grafting



The grafting operation $\mathrm{Gr}_{b\cdot\alpha} : T(S) \to P(S)$ can be generalized for measured laminations. Let $\mathcal{ML}(S)$ be the set of measured laminations.

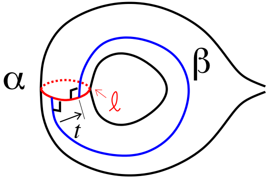
Theorem (Thurston, Kamishima-Tan)

$$\begin{aligned} \mathrm{Gr} : \mathcal{ML}(S) \times T(S) &\to P(S) \\ (\mu, X) &\mapsto \mathrm{Gr}_\mu(X) \end{aligned}$$

is a homeomorphism (Thurston coordinates).

Complex Earthquake

Let $\overline{\mathbb{H}} = \{\tau = t + \sqrt{-1}b \in \mathbb{C} \mid b \geq 0\}$. Fix $\ell > 0$.

Let $\mathrm{tw}_{t\cdot\alpha}(X_\ell) = \Big($ 

$\Big) \in T(S)$.

Define $\mathrm{Eq} : \overline{\mathbb{H}} \to P(S)$ by

$$\mathrm{Eq}(t + \sqrt{-1}b) = \mathrm{Gr}_{b\cdot\alpha}(\mathrm{tw}_{t\cdot\alpha}(X_\ell)) \in P(S)$$

By Thurston coordinates, we can regard $\overline{\mathbb{H}} \subset P(S)$.

Simply denote the image of $\overline{\mathbb{H}}$ by $\mathrm{Eq}(\ell)$.

Complex Earthquake

By construction, hol is the natural projection:

$$
\begin{array}{ccc}
P(S) & \xrightarrow{\text{hol}} & X(S) \\
\cup & & \cup \\
\text{Eq}(\ell) & \to & X(\ell) \\
\| & & \| \\
\{\tau \mid \text{Im}(\tau) \geq 0\} & & \{\tau \mid -\pi < \text{Im}(\tau) \leq \pi\} \\
\Cup & & \Cup \\
\tau & \mapsto & \tau \mod 2\pi\sqrt{-1}
\end{array}
$$

We are interested in

$$QF(\ell) := QF(S) \cap X(\ell) \ \subset X(\ell),$$

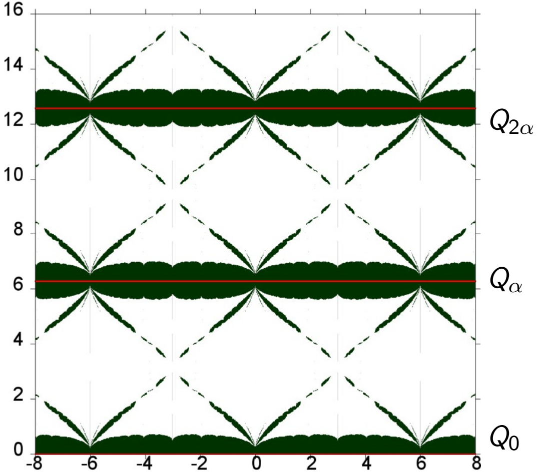
so consider

$$
\begin{aligned}
\text{hol}^{-1}(QF(\ell)) &= \text{hol}^{-1}(X(\ell) \cap QF(S)) \\
&= \text{Eq}(\ell) \cap \text{hol}^{-1}(QF(S)).
\end{aligned}
$$

$\mathrm{hol}^{-1}(QF(S))$ in $\mathrm{Eq}(\ell)$

By Goldman's Theorem, we have

$$\mathrm{Eq}(\ell) \cap \mathrm{hol}^{-1}(QF(S)) = \bigsqcup_{\mu \in \mathcal{ML}_{\mathbb{Z}}(S)} \mathrm{Eq}(\ell) \cap Q_\mu .$$



Eq(6.0)

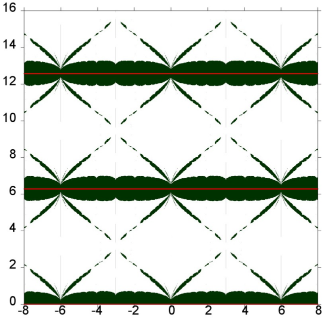
Each component belongs to some Q_μ.

Complex Earthquake

hol maps each component of $\mathrm{Eq}(\ell) \cap Q_\mu$ into a comp of $QF(\ell)$. Thus if

$$\mathrm{Eq}(\ell) \cap Q_\mu \neq \emptyset$$

for some $\mu \notin \{0, \alpha, 2\alpha, \cdots\}$, $QF(\ell)$ has a comp other than the standard one. Moreover,



Prop (K.)

$$Eq(\ell) \cap \mathrm{hol}^{-1}(\text{std comp}) = \bigsqcup_{k \geq 0} Eq(\ell) \cap Q_{k \cdot \alpha}$$

for any $\ell > 0$.

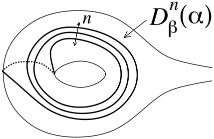
Existence of exotic components in $\mathrm{Eq}(\ell)$

We need to find $\mu \notin \{0, \alpha, 2\alpha, \cdots\}$ s.t. $\mathrm{Eq}(\ell) \cap Q_\mu \neq \emptyset$ for sufficiently large $\ell > 0$. Consider the case $\mu = \beta$.

Let D_β be the Dehn twist along β. Fix $X \in \mathcal{T}(S)$.

Consider a sequence in $P(S) \cong \mathcal{ML}(S) \times \mathcal{T}(S)$

$$\left(\frac{2\pi}{n} D_\beta^n(\alpha),\, X \right)$$



which converges to $(2\pi\beta,\, X) \in Q_\beta$ as $n \to \infty$.

Thus $(\frac{2\pi}{n} D_\beta^n(\alpha),\, X) \in Q_\beta$ for large n.

Apply D_β^{-n}, then $(\frac{2\pi}{n}\alpha,\, D_\beta^{-n}(X)) \in Q_\beta$ for large n.

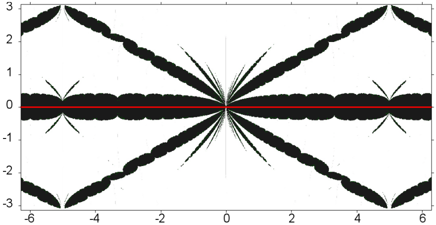
But if we let $\ell = \ell_\alpha(D_\beta^{-n}(X))$, $(\frac{2\pi}{n}\alpha,\, D_\beta^{-n}(X)) \in \mathrm{Eq}(\ell)$.

Final Remarks

- $\ell_\alpha(D_\beta^{-n}(X))$ is getting longer as $n \to \infty$, but $\ell_\beta(D_\beta^{-n}(X))$ is constant. Thus the Fenchel-Nielsen twist of $D_\beta^{-n}(X)$ w.r.t. α is relatively small. So Q_β is near the origin (probably with 'bumping').
- For $k \in \mathbb{N}$, we can show $\mathrm{Eq}(\ell) \cap Q_{k\cdot\beta} \neq \emptyset$ similarly for large ℓ by considering

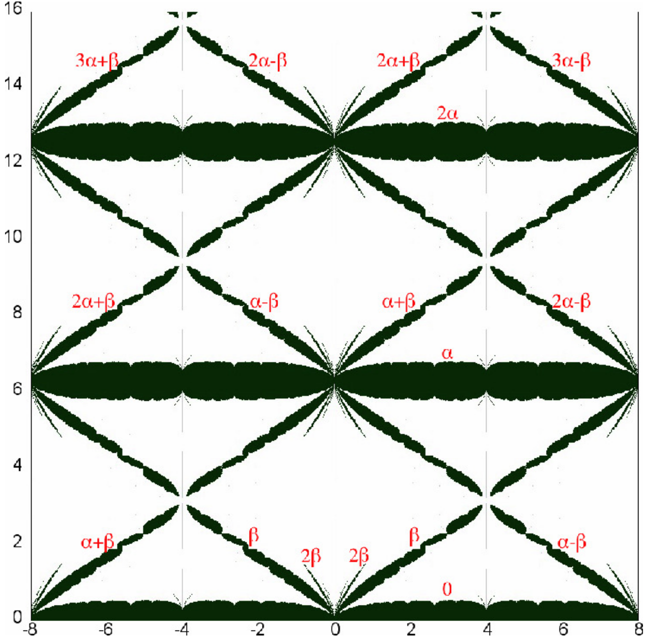
$$\left(\frac{2\pi k}{n} D_\beta^n(\alpha),\, X\right) \xrightarrow{n\to\infty} (2\pi k\beta,\, X) \in Q_{k\cdot\beta}.$$

$\ell = 10.0$

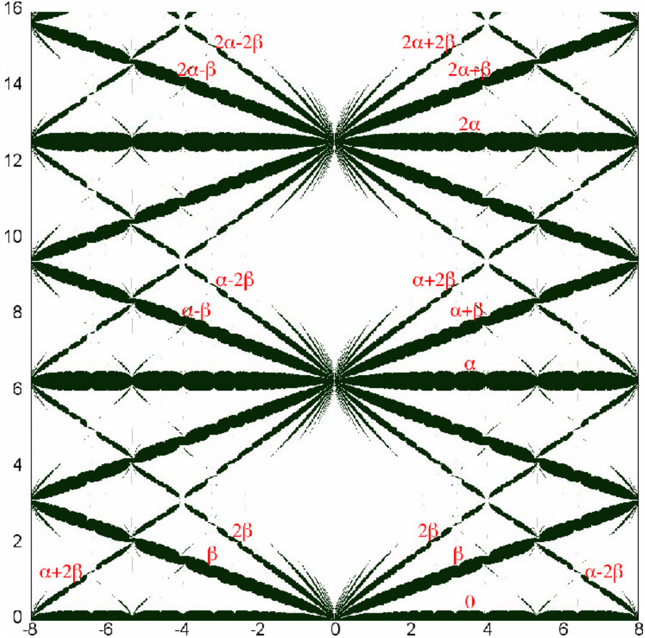


- Moreover we can use $\mu \in \mathcal{ML}(S)_{\mathbb{Z}}$ instead of β provided $i(\mu, \alpha) \neq 0$. If $\mu = p\alpha + \beta$, $Q_{p\alpha+\beta}$ is near $(-p\ell, 0)$ by the above argument.

Final Remarks



Final Remarks



Final Remarks

The intersection $\mathrm{Eq}(\ell) \cap Q_\mu$ may consist of more than one component.

Moreover, the non-local connectivity of $AH(S)$ (Bromberg) implies there may be infinitely many.