Scaling limit of successive approximations for $w' = -w^2$, from analysis on single layer solutions to a non-linear non-local recursion

2005.07, Sendai

Tetsuya HATTORI (Tohoku U)

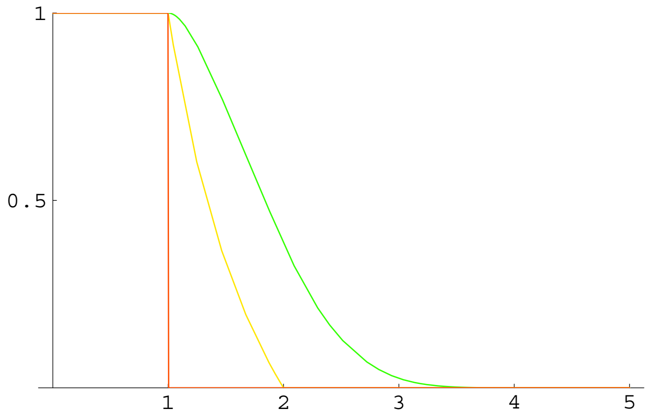
Hiroyuki OCHIAI (Nagoya U)

0. Motivation — Rod bisection.

Consider a non-linear non-local recursion

$$\boxed{\begin{array}{l} f_{n+1}(y) = \dfrac{1}{y}\displaystyle\int_0^y f_n(y')f_n(y-y')\,dy',\ y>0, \quad n=0,1,2,\cdots \\ \text{Initial function: } f_0(y)=1,\ 0\leqq y<1,\ =0, y\geqq 1 \end{array}}$$

'Propagating single layer (Tsunami) solutions'



- Random sequential bisections of a rod:

  Start from a rod of length 1.

  Break into 2 pieces randomly with uniform distribution.

  Then break the resulting pieces independently.

  Contiue the procedure recursively.

X_n: length of the longest at nth stage ($X_0 = 1$)

Then $f_n(y) = \mathrm{P}[\, 1/X_n > y \,]$ (M. Sibuya and Y. Itoh, 1987).

- Further related to binary search trees in data analysis.

○ $0 \leqq f_n(y) \leqq 1$, decreasing in y and increasing in n

$\to$ (discrete time) 'Tsunami' solutions

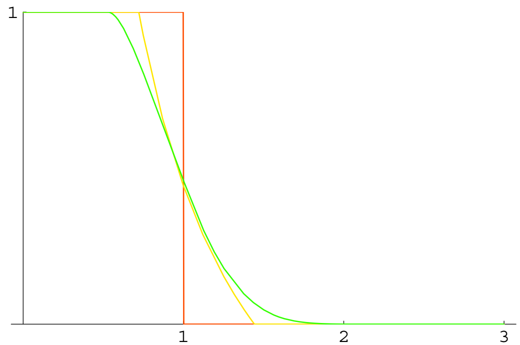
$$f_{n+1}(y) = \frac{1}{y}\int_0^y f_n(y')f_n(y-y')\,dy'$$
$$f_0(y) = 1,\ 0 \leqq y < 1, = 0, y \geqq 1$$

Problem:

(Exponential) speed of propagation (wavefront)? (**known**)

Existence of scaling limit $\lim f_n(q_n y)$? (**unknown**)

Shape of scaling limit? (**Solved** — This work)

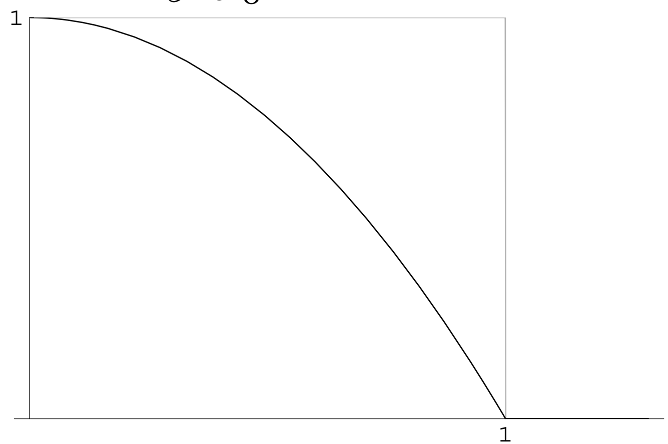


1. Scaling limit — Existence.

Existence of scaling limit for other initial functions? **YES!**

$$\boxed{f_{b,-}(y) = \max\{1 - y^{b-1},\ 0\}}$$

$$f_0 = f_{b,-},\ f_{n+1}(y) = \frac{1}{y}\int_0^y f_n(y')f_n(y-y')\,dy',\ n \in \mathbb{Z}_+$$



Theorem 1. Let $b > 2$ and $r = r(b) := (b/2)^{1/(b-1)}$.
(Then $f_n(r^n y) \uparrow$ in n ($\forall y > 0$) hence) $\exists \tilde{f}(y) = \lim_{n\to\infty} f_n(r^n y)$.
Following dichotomy, depending on b holds: Either
(i) $\tilde{f}(y) = 1$, $y \geqq 0$, or, (ii) $Q := \int_0^\infty \tilde{f}(y)\,dy < \infty$.
If in addition $b < (\log\rho)^{-1}$, then (ii) holds,
where, $0 < 2e\log\rho = \rho < e$ ($\rho = 1.26\cdots$). $\diamond$

• $r = r(b)$ is the correct scaling factor for $2 < b < (\log\rho)^{-1} = 4.311\cdots$. (Case (i) means that r^n is slower than the wavefront.)

Essence of Proof. — Monotonicity argument.

• $f_n(r^n y) \uparrow$ in n means r^n is no faster than the correct scaling sequence.

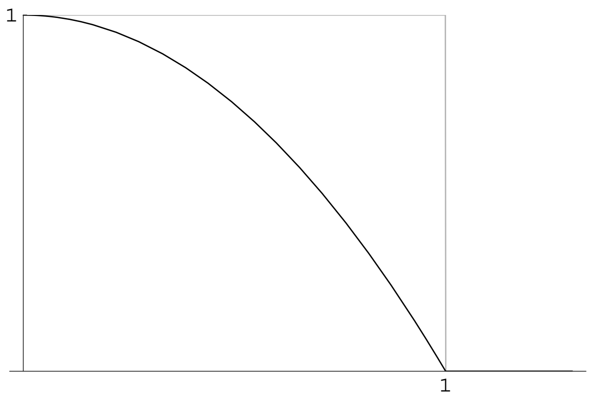
• A bound in other direction is possible for $\boxed{2 < b < (\log\rho)^{-1}}$ $(1 < r(b) < \rho)$ by $\boxed{f_{b,b',+}(y) = \min\{1 - y^{b-1} + C y^{b'-1},\ 1\}}$; $b < b' \leqq \min\{(\log\rho)^{-1},\ 2b-1\}$ and large C.

$$f_0 = f_{b,b',+},\ f_{n+1}(y) = \frac{1}{y}\int_0^y f_n(y') f_n(y-y')\, dy'.$$

Lemma. $f_n(r(b)^n y) \downarrow$ in n. $\diamond$

This is insufficient to prove $Q < \infty$, but the non-linearlity of the recursion (with r $(< \rho) < 2$) implies integrability.

- $b = (\log \rho)^{-1}$: $f_0 = f_{b,-}$ gives monotone sequence but $f_{b,b',+}$ does not exist!
- $b > (\log \rho)^{-1}$: Monotonicity arguments insufficient.
- Our next results suggest $r = \rho$ for $b \geqq (\log \rho)^{-1}$ (including '$b = \infty$', the rod bisection case), possibly with 'corrections'.



2. A sufficient condition for existence.

To state a sufficient condition for existence of scaling limit, we will work with the **Laplace transforms**:

$$\boxed{w_n(x) = \int_0^\infty e^{-xy} f_n(y)\, dy}.$$

The recursion $f_{n+1}(y) = \dfrac{1}{y}\displaystyle\int_0^y f_n(y') f_n(y-y')\, dy'$

corresponds to

$$\boxed{w_{n+1}(x) = \int_x^\infty w_n(x')^2\, dx'}.$$

• **Successive approximation** (approximation by integration) of a differential equation $w'(x) = -w(x)^2$

- $r > 1$ (un-scaled $f_n(y) \uparrow 1$) corresponds to successive approximation to a solution $w(x) = x^{-1}$
$\Leftrightarrow \boxed{w_0(x) = x^{-1} + o(x^{-2}),\ x \to \infty}$.
- Starting from a bounded function w_0, the sequence of approximate functions $\{w_n\}$ should increase near $x = 0$. Our problem is to find whether this 'blow-up' has a scaling limit, namely, to find whether w_n approach the exact solution in an asymptotically conformal way, $w_n(x) \asymp q_n \bar{w}(q_n x)$, for some (regular) function $\bar{w}$ and a sequence of numbers $\{q_n\} \uparrow \infty$.

Interesting things happen because $w'(x) = -w(x)^2$ has no singluarities while its solutions do (**moving singularities**). To be specific, we define the **scaling limit** of $\{w_n\}$ by

$$\boxed{\bar{w}(x) = \lim_{n\to\infty} q_n^{-1} w_n(q_n^{-1} x), \text{ where } q_n = w_n(0)}.$$

- $\bar{w}(0) = 1$ for this choice of $\{q_n\}$.

Our previous results on existence of scaling limits for $2 < b < (\log \rho)^{-1}$ are restated in terms of successive approximations, as follows. For $b > 2$, consider

$$\boxed{w_0(x) = \frac{1}{x}(1 - e^{-x}) - \frac{1}{x^b}\gamma(b, x),\ \gamma(b, x) = \int_0^x y^{b-1} e^{-y} dy}.$$

Note that $w_0(x) = x^{-1} + O(x^{-b})$, $x \to \infty$.

Theorem 2. Let $2 < b < (\log \rho)^{-1}$, and
$$w_{n+1}(x) = \int_x^\infty w_n(x')^2\, dx',\ n \in \mathbb{Z}_+,$$
with $w_0(x) = \frac{1}{x}(1 - e^{-x}) - \frac{1}{x^b}\gamma(b, x)$. Then the scaling limit exists and satisfies $\bar{w}(x) = \sum_{k=0}^{\infty}(-1)^k \alpha_k x^k$ with
$$\alpha_0 = 1,\ \alpha_k = \frac{1}{kr^{k+1}} \sum_{j=1}^{k} \alpha_{k-j}\alpha_{j-1},\ k \in \mathbb{N}, \text{ and}$$
$r = \lim_{n\to\infty} q_{n+1}/q_n > 1$ given by $r = r(b) := (b/2)^{1/(b-1)}$. $\diamond$

We thus have existence of scaling limits and precise form!

Outline of proof.

The correspondence $w_n(x) = \displaystyle\int_0^\infty e^{-xy} f_n(y)\,dy$ and Theorem 1, combined with the next Theorem 3, which gives a sufficient condition for a sequence of successive approximations $\{w_n\}$ to have a scaling limit, prove Theorem 2.

The integrability condition $Q = \displaystyle\int_0^\infty \tilde{f}(y)dy < \infty$ in Theorem 1 is crucial in proving that the scaling sequence $r(b)^n$ in Theorem 1 is asymptotically equivalent to the scaling sequence $q_n = w_n(0) = \displaystyle\int_0^\infty f_n(y)\,dy$ in Theorem 2.

$\circ$ **Sufficient condition for existence of scaling limits.**

$\mathcal{C}$: a set of entire functions $\bar{w}: \mathbb{C} \to \mathbb{C}$, with $\bar{w}(0) = 1$ and

$$\bar{w}(z) = \sum_{k=0}^{\infty} (-1)^k a_k z^k, \ a_k \geqq 0, \ k \in \mathbb{Z}_+, \ \bar{w}(x) > 0, \ x > 0,$$

and $\bar{w}(x) = x^{-1} + o(x^{-2})$, $x \to \infty$.

Theorem 3. Let $\bar{w}_0 \in \mathcal{C}$ and $\bar{w}_n$, $n \in \mathbb{Z}_+$, defined by

$$\bar{w}_{n+1}(x) = \frac{1}{r_n} \int_{x/r_n}^{\infty} \bar{w}_n(x')^2 \, dx', \ r_n = \int_0^{\infty} \bar{w}_n(x')^2 \, dx'.$$

If $\exists r = \lim_{n\to\infty} r_n > 1$, then $\bar{w}_n$ converges uniformly on $\forall K \subset\subset \mathbb{C}$ to

$$\bar{w}(z) = \sum_{k=0}^{\infty} (-1)^k \alpha_k z^k,$$

where $\{\alpha_k\}$ as in Theorem 2. $\diamond$

3. Random sequential bisection revisited — Suggestions from numerical results.

The sufficient condition for existence of scaling limits (Theorem 3) holds for any $r > 1$, in particular, for the rod bisection case: $f_0(y) = \begin{cases} 1, & 0 \leqq y < 1, \\ 0, & y \geqq 1\,, \end{cases}$ or, in terms of Laplace transform: $w_0(x) = \dfrac{1}{x}\,(1 - e^{-x}) = \displaystyle\int_0^\infty e^{-xy} f_0(y)\,dy.$

$w_{n+1}(x) = \displaystyle\int_x^\infty w_n(x')^2\,dx'$, $n \in \mathbb{Z}_+$.

Then $w_n(x) = \int_0^\infty e^{-xy} \mathrm{P}[\,1/X_n > y\,]\,dy$;

X_n: length of the longest piece among 2^n pieces at nth stage of random sequential bisection of a rod, starting from $X_0 = 1$.

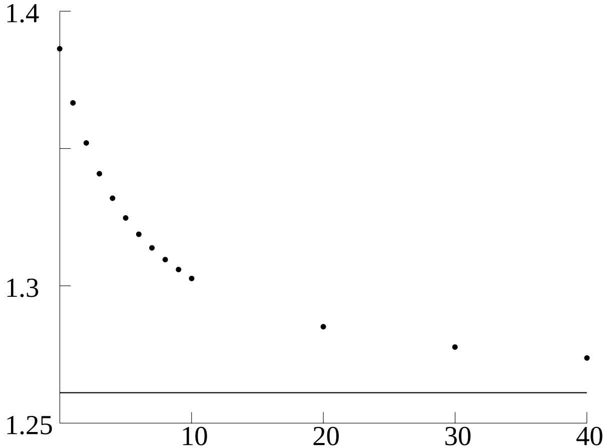
Theorem 4. If a limit $r = \lim_{n\to\infty} q_{n+1}/q_n > 1$ exists, then the scaling limit $\bar{w}(x) = \sum_{k=0}^{\infty} (-1)^k \alpha_k x^k$ exists with $r = \rho$ and $\alpha_0 = 1$, $\alpha_k = \dfrac{1}{kr^{k+1}} \sum_{j=1}^{k} \alpha_{k-j}\alpha_{j-1}$, $k \in \mathbb{N}$. $\diamond$

○ **Note.** Theorem 4 in particular implies $1/(q_n X_n)$ converges weakly to a distribution (*scaling limit*) whose generating function is $\boxed{\lim_{n\to\infty} \mathrm{E}[\, e^{-z/(q_n X_n)}\,] = 1 - z\bar{w}(z)}$.

○ **Proof of $r = \rho$.** The assumed limit $r = \lim_{n\to\infty} q_{n+1}/q_n > 1$ is equal to a weaker limit $\lim_{n\to\infty} q_n^{1/n}$, which can be derived from $\lim_{n\to\infty} X_n^{-1/n} = \rho$, a.s., a result of J. D. Biggins (1977) applied to the problem along the lines of L. Devroye (1986).

Implications of numerical results.

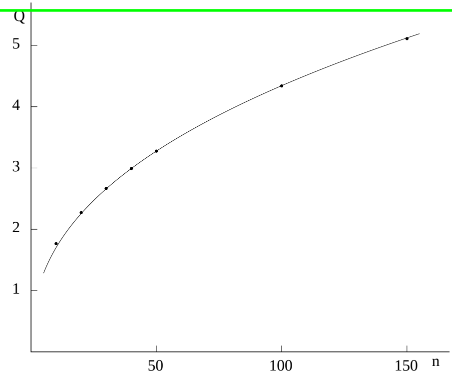
- ○ **Assumption** $r = \lim_{n\to\infty} q_{n+1}/q_n > 1$ **in Theorem 4.**



Numerical results for q_{n+1}/q_n vs n.

- The results suggest $\exists r = \lim_{n\to\infty} q_{n+1}/q_n > 1$.

○ **Integrability of** $Q = \int_0^\infty \tilde{f}(y)dy$.



Numerical results for $Q_n = q_n/\rho^n$ vs n.

• The results suggest $Q = \lim_{n\to\infty} Q_n = \infty$. (The curve is a fit to the data: $Q_n = 0.666\, n^{0.407}$.) In particular, ρ^n isn't a correct scaling sequence; the choice $q_n = w_n(0)$ is essential.

Height of binary search trees.

Maximal length X_n of random sequential bisections of a rod is closely related to the height H_N of binary search trees with data size N (L. Devroye (1986)):

$\mathrm{P}[\ X_n \geqq (1+n)/N\ ] \leqq \mathrm{P}[\ H_N \geqq n\ ] \leqq \mathrm{P}[\ X_n \geqq 1/N\ ]$.

With some extra assumptions, we could conjecture (in connection with our results) 'sum rules':

$$\lim_{n\to\infty} \frac{\sum_{N=1}^{\infty} N^k \mathrm{P}[\ H_N \leqq n\ ]}{(\sum_{N=1}^{\infty} \mathrm{P}[\ H_N \leqq n\ ])^{k+1}} = \boxed{\alpha_k} k!$$