Self-repelling Walk on the Sierpiński Gasket

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Abstract

We construct a one-parameter family of self-repelling processes on the Sierpiński gasket, by taking continuum limits of self-repelling walks on the pre-Sierpiński gaskets. We prove that our model interpolates between the Brownian motion and the self-avoiding process on the Sierpiński gasket. Namely, we prove that the process is continuous in the parameter in the sense of convergence in law, and that the order of Hölder continuity of the sample paths is also continuous in the parameter. We also establish a law of the iterated logarithm for the self-repelling process. Finally we show that this approach yields a new class of one-dimensional self-repelling processes.

1 Introduction.

In this paper we construct and study a one-parameter family of self-repelling processes on the Sierpiński gasket by taking continuum limits of self-repelling random walks on the pre-Sierpiński gaskets.

The continuum limits for the following two cases are known on the pre-Sierpiński gaskets:

(1) The simple random walk, whose continuum limit is the Brownian motion on the Sierpiński gasket [2, 8, 19].

(2) The self-avoiding path, whose continuum limit is a self-avoiding process on the Sierpiński gasket [12, 13, 15].

Note that the self-avoiding process on the Sierpiński gasket has a non-trivial distribution [12, 13, 15] (in contrast to the one-dimensional self-avoiding process, which is a deterministic linear motion).

Our family of processes is parametrized by $u \in [0, 1]$ such that the Brownian motion corresponds to $u = 1$, and the self-avoiding process corresponds to $u = 0$. The processes corresponding to $0 < u < 1$ interpolate between the two extreme cases in the sense that the path measure $P^u$ (the image measure of the process defined on a space of continuous functions on the Sierpiński gasket) converges weakly as $u \to u_0$ for any $0 \leq u_0 \leq 1$, and that the scaling exponent $\gamma$ for the ‘speed’ of the process is continuous in $u \in [0, 1]$.

The initial work on self-avoiding and weakly self-avoiding (self-repelling) random walk arises from defining models for polymers. The classical problem is to define a path measure on $\mathbb{R}^d$ in which self intersections are penalized by an exponential weighting factor and then study the almost sure behaviour of the paths under this measure. The one dimensional case is reasonably well understood as well as for $d > 4$ but in two and three dimensions there are still major open problems.
An important property is the behaviour of the end to end length of the polymer. This is captured in an exponent for the speed of the walk, (the reciprocal of the walk dimension) $\gamma$, which can be defined by $\gamma = \lim_{n \to \infty} \log|X_n|/\log(n)$. On the one dimensional integer lattice $[5, 9, 18]$ proved that there is ‘ballistic motion’ in that $\gamma = 1$. The models proposed in [21, 22] are consistent with $\gamma = 2/3$, $\gamma = 1/2$ respectively. In these cases the exponents $\gamma$ are independent of the self-repelling factor. A model which continuously interpolates between $1/2 \leq \gamma \leq 2/3$ is given in [20].

All these models obtain the self-repelling property by introducing weights depending on the number of returns to bonds [20, 21, 22] or sites [5, 9, 18]. For a recent review of the site case see [17]. For some models it is possible to construct continuum limit processes, and [6] constructs such a process on $\mathbb{R}$ in the diffusive phase $(\gamma = 1/2)$. In [23] a continuous self-repelling process is constructed in the case $\gamma = 2/3$ and many of its path properties are examined.

Our approach is different in that we introduce a parameter $u$ which allows us to interpolate between the simple random walk and a self-avoiding walk. Our path is weighted according to a revisiting factor, which counts visits to ‘higher level’ points, and a reversing factor, which counts back tracks. (By reversing we mean, for the Sierpiński gasket, that the path remains within the same triangle, not necessarily going back the way it came.) We will take a continuum limit of the random walks to obtain a self-repelling process. It is known that the scaling exponent $\gamma$ is different for the Brownian motion and the self-avoiding process. We prove that $\gamma$ is continuous for $0 \leq u \leq 1$. Since all the processes we consider are self-similar, this should imply that various exponents of the sample paths of the processes are also continuous in $u$ in our model, and interpolate between those of the Brownian motion and the self-avoiding process on the Sierpiński gasket.

We note that our parameter does not directly count the number of returns to bonds or sites. Therefore, our construction gives an alternative model for self-repelling walks on $\mathbb{Z}$. We can construct a one-parameter family of self-repelling processes on $\mathbb{R}$ by taking continuum limits of these self-repelling random walks, and the resulting family of processes continuously interpolates the exponent $\gamma$ between that of the one-dimensional Brownian motion and the one-dimensional self-avoiding process (deterministic linear motion).

The structure of the paper is as follows. We will begin with a sequence of random walks on graph approximations to the Sierpiński gasket and show in Section 2 that there is a continuum limit process for each value of the interpolating parameter. In Section 3 we prove that the processes are continuous in the interpolating parameter. We also note that though we construct our continuum limit processes on a finite Sierpiński gasket, the extension of our processes to the (infinite) Sierpiński gasket in a self-similar way apparently poses no difficulties. The corresponding results, with the expected self-similarity, will imply that the exponents of mean square displacement lie between the value of the Brownian motion and the self-avoiding process for our model, and that the value is a continuous function of the parameter. Thus it is natural to consider our model as a family of self-repelling processes which interpolates between the Brownian motion and the self-avoiding process on the Sierpiński gasket.

In Section 4 we will discuss the path properties of our self-repelling process. In particular we show that $\gamma$ controls the mean square displacement and prove a law of the iterated logarithm. A crucial part of the proof is that there is a supercritical branching process which describes the path. In what follows we will mainly work on the more difficult case of the Sierpiński gasket. In the final section we will summarize the basic ingredients (and the main differences from the Sierpiński gasket case) of the construction of the corresponding self-repelling processes on $\mathbb{R}$.

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2 Construction of the processes.

The pre-Sierpiński gaskets and the Sierpiński gasket are defined as follows. Let $O = (0, 0), \quad a = (1, \sqrt{3}/2), \quad b = (1, 0)$, and let $F_1^0$ be the set of all the points on the vertices and edges of $\Delta O a b$. We define a sequence of sets $F_0^1, F_1^1, F_2^1, \ldots$, inductively by

$$F_{n+1} = \frac{1}{2} F_n^1 \cup \frac{1}{2} (F_n^1 + a) \cup \frac{1}{2} (F_n^1 + b), \quad n = 0, 1, 2, \ldots$$
where $A + a = \{x + a : x \in A\}$ and $kA = \{kx : x \in A\}$. Let

$$F_n = F_n^0 \cup (F_n^1 - b).$$

We call $F_n$ the (finite) pre-Sierpiński gaskets, and $F = \bigcup_{n=0}^{\infty} F_n$ the (finite) Sierpiński gasket. We denote the set of vertices in $F_n$ by $G_n$. Let us denote by $\mathcal{T}_n$ the set of all the closed triangles in $\R^2$ that are the translations of $2^{-n}\Delta{Oab}$ (without rotation) and whose vertices are in $G_n$.

Let

$$C = \{w \in \mathcal{C}(\[0, \infty) \to F) : w(0) = O, \lim_{t \to \infty} w(t) = a\}.$$ 

Let $|x - y|, x, y \in \R^2$, denote the Euclidean distance. $C$ is a complete separable metric space with the metric

$$d(u, v) = \sup_{t \in [0, \infty)} |u(t) - v(t)|, \ u, v \in C.$$

We define the ‘hitting times,’ $T^k_i : C \to \R_+ \cup \{\infty\}$, $k, i \in \mathbb{Z}_+$, as follows. Let $T^k_0(w) = 0$, and by induction, for $i \geq 1$, let

$$T^k_i(w) = \inf \{t > T^k_{i-1}(w) : w(t) \in G_k \setminus \{w(T^k_{i-1}(w))\}\},$$

if the right hand side is finite, otherwise, $T^k_i(w) = \infty$. $T^k_i$ is the time when the path $w$ hits a vertex of $G_k$ for the $i$-th time under the condition that if $w$ hits the same element of $G_k$ more than once in a row, we consider it ‘once’. Writing $w(\infty) = a$, and noting that $w(t) \to a$ as $t \to \infty$, we obtain a finite sequence $(T^k_i)_{i=1, \ldots, M}$ such that $w(T^k_M(w)) = a$ and $w(T^k_{M-1}(w)) \neq a$. Let $S^k_i(w) = T^k_i(w) - T^k_{i-1}(w)$.

For $n \in \mathbb{Z}_+$, we define a ‘decimation’ map $Q_n : C \to C$ by setting

$$(Q_nw)(i) = w(T^n_i(w)).$$

for $i = 0, 1, 2, \ldots, M$, where $M$ is as above, and by using linear interpolation

$$Q_nw(t) = \begin{cases} (i + 1 - t) \cdot (Q_nw)(i) + (t - i) \cdot (Q_nw)(i + 1), & \text{if } i \leq t < i + 1, \ i = 0, 1, 2, \ldots, M - 1, \\ a, & \text{if } t \geq M. \end{cases}$$

Note that

$$(Q_k \circ Q_n = Q_n) \quad \text{if} \quad k \leq n. \quad \text{(2.1)}$$

Let us denote by $W_n$ the set of continuous functions $w : [0, \infty) \to F_n$ such that there exists $L(w) \in \mathbb{N}$ for which

$$w(0) = O,$$

$$w(t) = a, \quad \text{if } t \geq L(w),$$

$$w(t) \notin G_0 \setminus \{O\}, \quad \text{if } t < L(w),$$

$$|w(i) - w(i + 1)| = 1, \quad \text{if } i = 0, \ldots, L(w) - 1,$$

$$w(i)w(i + 1) \subset F_n, \quad \text{if } i = 0, \ldots, L(w) - 1,$$

$$w(t) = (i + 1 - t)w(i) + (t - i)w(i + 1), \quad \text{if } i \leq t < i + 1, \ i = 0, 1, 2, \ldots.$$ 

These are all the paths from 0 to $a$ which remain in a pair of triangles about 0 and first exit at $a$.

Also we denote by $W^{(1)}_n$ the set of continuous functions $w : [0, \infty) \to F^1_n$ such that there exists $L_1(w) \in \mathbb{N}$ for which

$$w(0) = O,$$

$$w(t) = O, \quad \text{if } t \geq L_1(w),$$

$$w(t) \notin G_0, \quad \text{if } 0 < t < L_1(w),$$

$$|w(i) - w(i + 1)| = 1, \quad \text{if } i = 0, \ldots, L_1(w) - 1,$$

$$w(i)w(i + 1) \subset F_n, \quad \text{if } i = 0, \ldots, L_1(w) - 1,$$

$$w(t) = (i + 1 - t)w(i) + (t - i)w(i + 1), \quad \text{if } i \leq t < i + 1, \ i = 0, 1, 2, \ldots.$$ 

These are the excursions from 0 which do not reach $a$ or $b$.

Finally we denote by $W^{(2)}_n$ the set of continuous functions $w : [0, \infty) \to F_n$ such that there exists $L_2(w) \in \mathbb{N}$ for which

$$w(0) = O,$$

$$w(t) = a, \quad \text{if } t \geq L_2(w),$$

$$w(t) \notin G_0, \quad \text{if } t < L_2(w),$$

$$|w(i) - w(i + 1)| = 1, \quad \text{if } i = 0, \ldots, L_2(w) - 1,$$

$$w(i)w(i + 1) \subset F_n, \quad \text{if } i = 0, \ldots, L_2(w) - 1,$$

$$w(t) = (i + 1 - t)w(i) + (t - i)w(i + 1), \quad \text{if } i \leq t < i + 1, \ i = 0, 1, 2, \ldots.$$ 

These are the excursions from 0 which do not reach $a$ or $b.
These paths are the excursions from 0 which exit at $a$.

We call $L(w)$, $L_1(w)$ and $L_2(w)$ the length of the path $w$.

$W_n$ and $W_n^{(2)}$ are subsets of $C$. We define $T_i^{(1)}$’s and $Q_n$’s also on $W_n^{(1)}$ analogously to the definitions on $C$.

Each $w \in W_n$ makes a polygonal curve on $F_n$. For $w \in \bigcup_{n \geq k} (W_n \cup W_n^{(1)} \cup W_n^{(2)})$, define the reversing number $N_k(w)$ and the returning number $M_k(w)$ for level $k$ by

$$N_k(\ell)(w) = \frac{1}{2} \left\{ T_{\ell-1}^{k-1} < i < T_{\ell}^{k-1} : \right.\left. (Q_kw)(i) \cdot (Q_kw)(i+1) < 0, \right.$$

$$\left. (Q_kw)(i) \neq w(T_{\ell-1}^{k-1}(w)) \right\},$$

where $\vec{a} \cdot \vec{b}$ denotes the inner product of $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^2$, and

$$M_k(\ell)(w) = \frac{1}{2} \sum_{\ell=1}^{L(Q_{k-1}w)} N_k(\ell)(w),$$

$$N_k(w) = \sum_{\ell=1}^{L(Q_{k-1}w)} M_k(\ell)(w).$$

Thus $N_k(w)$ counts the number of times the path $w$ on $G_k$ makes two steps within the same triangle and $M_k(w)$, the number of times the path on $G_k$ revisits a vertex in $G_{k-1}$. It is these types of steps that we will suppress in our self-repelling path measure.

For $x > 0$ and $0 \leq u \leq 1$, define

$$\Phi_n(x,u) = \sum_{w \in W_n} \left( \prod_{k=1}^{n} u^{N_k(w)+M_k(w)} \right) x^{L(w)},$$

$$\Theta_n(x,u) = \sum_{w \in W_n^{(1)}} \left( \prod_{k=1}^{n} u^{N_k(w)+M_k(w)} \right) x^{L_1(w)},$$

$$\Psi_n(x,u) = \sum_{w \in W_n^{(2)}} \left( \prod_{k=1}^{n} u^{N_k(w)+M_k(w)} \right) x^{L_2(w)}.$$
Proposition 2.1. For \( n > m \),
\[
\Phi_n(x, u) = \Phi_m(\Phi_{n-m}(x, u), u).
\]

Proof. Assume \( n > m \) and \( w \in W_n \). Note that
\[
(2.3) \quad L(w) = T_{L(Q_mw)}^m(w) = \sum_{i=1}^{L(Q_mw)} \{ T_i^m(w) - T_{i-1}^m(w) \} = \sum_{i=1}^{L(Q_mw)} S_i^m(w).
\]
This together with (2.1) implies that for \( k = m+1, \ldots, n \)
\[
N_k(w) = \sum_{\ell=1}^{L(Q_{k-1}w)} N_k(\ell)(w) = \sum_{i=1}^{L(Q_{k-1}w)} \sum_{\ell=1}^{T_i^{Q_{k-1}w} - 1} N_k(\ell)(w) = \sum_{i=1}^{L(Q_{k-1}w)} N_k(\ell)(w).
\]
A similar decomposition holds also for \( M_k(w) \). Using these and (2.3), we can rewrite the summands in (2.2) as
\[
\prod_{k=1}^{n} u^{N_k(w) + M_k(w)} x^{L(w)} = \prod_{k=1}^{m} u^{N_k(w) + M_k(w)} \cdot \prod_{k=m+1}^{n} \prod_{i=1}^{L(Q_{k-1}w)} \{ u^{\sum_{\ell=1}^{T_i^{Q_{k-1}w} - 1} N_k(\ell)(w) + M_k(\ell)(w)} \} \prod_{i=1}^{L(Q_{n-1}w)} x^{S_i^m(w)},
\]
where \( \sum_{\ell} \) is taken over \( \ell = \sum_{i=1}^{L(Q_{k-1}w)} (Q_{k-1}w) + 1, \ldots, T_i^{Q_{k-1}w} \).

For \( i = 1, \ldots, L(Q_{n-1}w) \), let \( \Delta_i \) and \( \Delta_i' \) be two adjacent elements of \( \mathcal{T}_m \) determined by
\[
w(T_{L_i^{Q_{n-1}w}}(w)) \in \Delta_i \cap \Delta_i',
\]
\[
w(T_{L_i^{Q_{n-1}w}}(w)) \in \Delta_i \cap (\Delta_i')^c.
\]
Consider each path segment \( w_i = \{ w(t) : T_{L_i^{Q_{n-1}w}}(w) \leq t \leq T_{L(Q_{n-1}w)}(w) \} \). Note that \( w_i \subseteq \Delta_i \cup \Delta_i' \). Since \( \Delta_i \cap F_n \) and \( \Delta_i' \cap F_n \) are both similar to \( F'_n \), there are \( \tilde{w}_i \in W_{n-m} \) such that \( \{ \tilde{w}_i(t) : 0 \leq t \leq L(\tilde{w}_i) \} \cap F_{n-m} \) and \( \{ \tilde{w}_i(t) : 0 \leq t \leq L(\tilde{w}_i) \} \cap (F_{n-m} - b) \) are similar to \( w_i \cap \Delta_i \) and \( w_i \cap \Delta_i' \) (or, to their reflections), respectively. In terms of \( \tilde{w}_i \)'s, the right-hand side of the above expression of the summand can further be rewritten as
\[
\prod_{k=1}^{m} u^{N_k(\tilde{w}_i) + M_k(\tilde{w}_i)} \prod_{i=1}^{L(Q_{n-1}w)} \{ \prod_{k=1}^{n-m} u^{\sum_{\ell=1}^{L(Q_{k-1}w)} N_k(\ell)(\tilde{w}_i) + M_k(\ell)(\tilde{w}_i)} \} x^{L(\tilde{w}_i)}
\]
Summing up over \( W_n \), we have
\[
\Phi_n(x, u) = \sum_{v \in W_m} \left\{ \prod_{k=1}^{m} u^{N_k(v) + M_k(v)} \{ \Phi_{n-m}(x, u) \}^{L(v)} \right\} \left[ \prod_{\tilde{w}_i \in W_{n-m}} \sum_{\tilde{w}_i \in W_{n-m}} \frac{1}{\Phi_{n-m}(x, u)} \prod_{k=1}^{n-m} u^{N_k(\tilde{w}_i) + M_k(\tilde{w}_i)} x^{L(\tilde{w}_i)} \right]
\]
\[
= \sum_{v \in W_m} \left\{ \prod_{k=1}^{m} u^{N_k(v) + M_k(v)} \{ \Phi_{n-m}(x, u) \}^{L(v)} \right\} \left[ \prod_{i=1}^{L(v)} \left\{ \frac{1}{\Phi_{n-m}(x, u)} \sum_{\tilde{w}_i \in W_{n-m}} \left( \prod_{k=1}^{n-m} u^{N_k(\tilde{w}_i) + M_k(\tilde{w}_i)} x^{L(\tilde{w}_i)} \right) \right\} \right]
\]
\[
= \Phi_m(\Phi_{n-m}(x, u), u).
\]
\( \square \)
We next define a family of probability measures \( \{ \hat{P}_n^u(x) \} \) on \( C \) (supported on \( W_n \)) by assigning to each \( w \in W_n \),
\[
(2.4) \quad \hat{P}_n^u(x)(w) = \left( \prod_{k=1}^{n} u^{N_k(w) + M_k(w)} x^{L(w)} / \Phi_n(x, u) \right).
\]

Let \( n > m \) and \( w \in W_n \). Using the decomposition in the proof of Proposition 2.1 we rewrite (2.4) as
\[
(2.5) \quad \hat{P}_n^m(x)(w) = \hat{P}_m^m(\Phi_{n-m}(x, u))(Q_m w) \cdot \prod_{i=1}^{L(Q_m w)} \hat{P}_n^u(\Phi_{n-m}(x, u))(\tilde{w}_i),
\]
where \( \tilde{w}_i \)'s are as in the proof of Proposition 2.1.

The following proposition follows immediately from (2.5).

**Proposition 2.2.** If \( w \in W_n \) and \( m \leq n \), then \( Q_m w \in W_m \). The probability law of \( Q_m w \) under \( \hat{P}_n^u(x) \) is \( \hat{P}_m^m(\Phi_{n-m}(x, u)) \).

Let \( r_u \) be the radius of convergence for \( \Phi(x, u) \) as a power series in \( x \).

**Proposition 2.3.**

1. For each \( u \in [0, 1] \), there is a unique fixed point \( x_u \) of the mapping \( \Phi(\cdot, u) : (0, r_u) \to (0, \infty) \), that is,
\[
\Phi(x_u, u) = x_u, \quad x_u > 0.
\]

As a function in \( u \), \( x_u \) is continuous and strictly decreasing on \( [0, 1] \).

2. Let \( \lambda_u = \frac{\partial \Phi}{\partial x}(x_u, u) \). Then \( \lambda_u \) is continuous in \( u \) and \( \lambda_u > 2 \).

**Proof.**

1. \( \Phi_u(x) = \Phi(x, u) \) is expressed as a power series in \( x \) with non-negative coefficients, starting from a quadratic term. It follows that \( \Phi_u(0) = \Phi_u'(0) = 0 \), \( \inf_{x \geq 0} \Phi_u'(x) = \Phi_u'(0) > 0 \). Therefore \( \Phi_u'(x) - 1 \) is increasing in \( x \geq 0 \), negative at \( x = 0 \) and diverges to \( +\infty \) as \( x \to r_u \). Existence and uniqueness of the fixed point follow. The rest of the statement follows from the application of the implicit function theorem to \( F(x, u) = \Phi(x, u) - x \).

2. Since the terms in \( \Phi_u(x) \) are degree 2 or higher, and since there are terms of degree strictly higher than 2, we have
\[
\Phi_u'(x) > \frac{2}{x} \Phi_u(x), \quad 0 < x < r_u.
\]

Therefore \( \lambda_u > 2 \frac{\Phi_u(x_u)}{x_u} = 2 \). The continuity of \( x_u \) in \( u \) and the continuity of \( \frac{\partial \Phi}{\partial x}(x, u) \) in \( (x, u) \) imply the continuity of \( \lambda_u \).

\[ \square \]

In the two extreme cases, we know that \( x_0 = \frac{\sqrt{5} - 1}{2} \), \( \lambda_0 = \frac{7 - \sqrt{5}}{2} \) (see [14], [12]), and \( x_1 = \frac{1}{4} \), \( \lambda_1 = 5 \) (see [2], [19]). For \( m \leq n \), let \( Q_m \hat{P}_n^u \) be the image measure of \( \hat{P}_n^u \) induced by \( Q_m \). Combining Proposition 2.2 and Proposition 2.3, we have

**Proposition 2.4.** If \( w \in W_n \) and \( m \leq n \), then \( Q_m w \in W_m \) and
\[
Q_m \hat{P}_n^u(x_u) = \hat{P}_m^m(x_u).
\]

By virtue of Proposition 2.4 and Kolmogorov’s extension theorem, for each \( u \in [0, 1] \), there is a probability measure \( P_u^w \) on \( \Omega = C^\infty = C \times C \times \cdots \) such that
\[
P_u^w[ \omega = (w_1, w_2, \cdots) : Q_m w_n = w_m, \ n \geq m ] = 1,
\]
and
\[
Y_n P_u^w = \hat{P}_n^u(x_u),
\]

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where $Y_n^{P^m}$ denotes the image measure of $P^m$ induced by the natural projection $Y_n$ from $\Omega$ to the $n$-th $C$ in the product. We regard each $Y_n(\omega, t)$ as an $F$-valued process on $(\Omega, \mathcal{B}, P^m)$, where $\mathcal{B}$ is the Borel algebra on $\Omega$. The following properties are used later.

$$Y_n \in W_m, \quad \text{a.s.}$$

$$Q_m Y_n = Y_m, \quad m < n, \quad \text{a.s.}$$

In particular,

(2.6) \hspace{1cm} Y_n(T^m_i(Y_n)) = Y_m(i), \quad m < n, \quad \text{a.s.}

(2.7) \hspace{1cm} T^m_i(Y_k) = T^m_i(Y_{m,n}(k)), \quad m \leq n \leq k, \quad \text{a.s.}

(2.5) and Proposition 2.4 imply the following.

**Proposition 2.5.** Assume $n \geq m$ and $N \geq 2^m$. Under the conditional probability $P^m[ \cdot \mid Y_m \in W_m, L(Y_m) \geq N]$, \hspace{1cm} \{S^m_1(Y_n), S^m_2(Y_n), \ldots, S^m_N(Y_n)\}

are i.i.d. random variables. They are jointly independent of $Y_m$. The law of $S^m_1(Y_n)$ is equal to that of $S^m_1(Y_{m,n})$ under $P^m$.

**Proposition 2.6.** Fix $m \in \{0, 1, 2, \cdots\}$, $N \geq 2^m$ and $i \in \{0, 1, 2, \cdots, N\}$. Under $P^m[ \cdot \mid Y_m \in W_m, L(Y_m) \geq N]$, \hspace{1cm} \{S^m_i(Y_{m+n})\}, n = 0, 1, 2, \cdots is a supercritical branching process starting at $S^m_1(Y_m) = 1$ and with offspring distribution equal to the law of $S^m_1(Y_1)$ under $P^m$ with the properties:

(1) The characteristic function of $S^m_1(Y_1)$ is given by

$$E^m[\exp(itS^m_1(Y_1))] = \frac{1}{x_u} \Phi(x_u e^{it}, u), \quad t \in \mathbb{R}.$$  

(2) $E^m[S^m_1(Y_1)] = \lambda_u$.

(3) $0 < E^m[(S^m_1(Y_1) - \lambda_u)^2] < +\infty$.

(4) $P^m[S^m_1(Y_1) \geq 2] = 1$.

**Proof.** (2.7) implies that

$$T^m_i(Y_{m+n+1}) = T^m_{i+n}(Y_{m+n+1}) = T^m_{i+n}(Y_{m+n+1}) = \sum_{j=1}^{T^m_{i+n}(Y_{m+n+1})} S^m_{i+n}(Y_{m+n+1}).$$

Hence,

$$S^m_{i+n}(Y_{m+n+1}) = \sum_{j=1}^{T^m_{i+n}(Y_{m+n+1})} S^m_{i+n}(Y_{m+n+1}).$$

This combined with Proposition 2.5 implies that \{S^m_i(Y_{m+n})\}, $n = 0, 1, 2, \cdots$ is a branching process. 
(1) is immediate from (2.4), and (2) through (4) are the immediate consequences of (1). Since $\lambda_u > 2$, \{S^m_i(Y_{m+n})\} is a supercritical branching process.
Proposition 2.6 suggests that we consider $F$-valued processes with time appropriately scaled. We introduce a time-scale transformation $U_n(\alpha) : C \to C$, $\alpha \in (0, \infty]$, $n \in \mathbb{N}$. For $w \in C$, define

$$(U_n(\alpha)w)(t) \overset{\text{def}}{=} w(\alpha^n t).$$

Let us denote by $P^w_n$ the image measure of $\tilde{P}^w_n(x_u)$ induced by $U_n(\lambda_u)$. Define

$$X_n = U_n(\lambda_u)Y_n, \ n = 1, 2, \ldots.$$

The convergence theorem for supercritical branching processes (See [1]) leads to the following proposition.

**Proposition 2.7.** Assume $N \geq 2^m$. Under $P^w[u \cdot | Y_m \in W_m, \ L(Y_m) \geq N]$, we have

1. For each $i \in \{1, \ldots, N\}$, $S_i^m(X_n)$ converges a.s. and in $L^2$ as $n \to \infty$ to a random variable $S_i^m$.
2. $S_i^m$, $i = 1, \ldots, N$ are i.i.d. random variables and are jointly independent of $Y_m$.
3. $S_i^m$ is equal in law to $\lambda_i^m S_i^0$.
4. $P^w[S_i^0 > 0] = 1$, $E^w[S_i^0] = 1$.

The characteristic function of $S_i^0$, $\phi_u(t) = E^w[\exp(it S_i^0)]$ is the unique solution to

$$\phi_u(\lambda_0 t) = \frac{1}{x_u} \Phi(x_u \phi_u(t), u), \ \phi_u'(0) = 1.$$

We denote by $p^w$ and $p_i^w$ the law of $S_i^0$ and $S_i^0(X_n)$, respectively.

**Theorem 2.8.** $p^w$ has a $C^\infty$ density $\rho$, which satisfies $\rho(x) = 0$ for $x \leq 0$, and $\rho(x) > 0$ for $x > 0$.

In general, a $C^\infty$ density exists for the limit distribution of supercritical branching processes such that the number of offspring is almost surely greater than one. In our case the condition is ensured by Proposition 2.6 (2). For a proof see [2], [16].

Let $T_i^m = \sum_{j=1}^i S_j^m$.

**Theorem 2.9.** For each $u \in [0, 1]$, $X_n$ converges uniformly in $t$ a.s. as $n \to \infty$ to a continuous process $X$.

**Proof.** Choose $\omega \in \Omega$ such that $Y_m \in W_m$, $\lim_{n \to \infty} S_i^m(X_n) = S_i^m$ exists and $S_i^m > 0$ for all $m \in \mathbb{N}$ and $i \in \{1, \ldots, T_i^0(Y_m)\}$. Let $M = T_i^0 + \varepsilon$, where $\varepsilon > 0$ is an arbitrary constant. It suffices to show that $X_n(\omega)$ converges uniformly in $[0, M]$. In fact, if $t > M$, $X_n(t) = \alpha$ for large enough $n$.

Fix $m \geq 0$. Let $L = T_i^0(Y_m)$. Note that $T_i^m(X_n) = T_i^0(X_n)$ a.s. Letting $n \to \infty$, we have $T_i^m = T_i^0$ a.s.

By the choice of $\omega$, there exists $n_1 = n_1(\omega) \in \mathbb{N}$ such that

$$\max_{0 \leq i \leq L} |T_i^m(X_n) - T_i^m| \leq \min_{0 \leq i \leq L} S_i^m,$$

and

$$|T_i^m(X_n) - T_i^m| < \varepsilon,$$

for $n \geq n_1$.

For each $t \in [0, M]$, either of the following holds.

(i) $0 \leq t < T_i^m$.

(ii) $T_i^m \leq t \leq T_i^m + \varepsilon$.
In case (i), choose \( j \in \{1, \cdots, L \} \), such that \( T_{j+1} \leq t < T_j \). Then we have \( T_{j+1}^m(X_n) \leq t \leq T_j^m(X_n) \), for \( n \geq n_1 \). Thus
\[
|X_n(T_j^m(X_n)) - X_n(t)| \leq 3 \cdot 2^{-m}.
\]
In case (ii), let \( j = L \). Since \( T_{L+1}^m(X_{n-1}) \leq t \),
\[
|X_n(T_j^m(X_n)) - X_n(t)| \leq 2 \cdot 2^{-m}.
\]
(2.6) implies that \( X_n(T_j^m(X_n)) = Y_m(j) \) for any \( n \geq m \) and \( j \in \{0, 1, \cdots, L(Y_m) \} \). Therefore, if \( n, n' \geq n_1 \), then for any \( t \in [0, M] \),
\[
|X_n(t) - X_{n'}(t)| \\
\quad \leq |X_n(T_j^m(X_n)) - X_n(t)| + |X_n'(T_j^m(X_n)) - X_{n'}(t)| + |X_n(T_j^m(X_n)) - X_{n'}(T_j^m(X_n))| \\
\quad \leq 6 \cdot 2^{-m}.
\]
Since \( m \) is arbitrary, we have the uniform convergence. \( \Box \)

**Corollary 2.10.** Assume \( n \geq m \).

1. \( X(T_j^m) = X_n(T_j^m(X_n)) = Y_m(j) \in G_m \), for all \( j \in \{0, 1, \cdots, T_1^m(Y_m) \} \), a.s.

2. With probability 1, there exist adjacent closed triangles \( \Delta, \Delta' \in T_m \) such that
   \[
   X(T_j^m) \in \Delta \cap \Delta',
   \]
   \[
   X(T_j^m) \in \Delta \cap (\Delta')^c,
   \]
   and
   \[
   X(t) \in \Delta \cup \Delta' \quad \text{for} \quad T_{j-1}^m \leq t \leq T_j^m.
   \]

3. \( X(t) = a, \quad t \geq T_1^0 \), a.s.

**Proof.** \( X_n(T_j^m(X_n)) = Y_m(j) \) for \( n \geq m \) and the a.s. uniform convergence of \( X_n \) to \( X \) imply the statement of (1). The definition of hitting times and (1) imply that \( X_n(t) \in \Delta \cup \Delta' \) for \( T_{j-1}^m(X_n) \leq t \leq T_j^m(X_n) \) a.s.

Letting \( n \to \infty \), we have (2). (3) follows from \( X_n(t) = a, \ t \geq T_1^0(X_n) \) a.s. \( \Box \)

This result implies that the limit process for \( 0 < u < 1 \) maintains the self-repelling property of the original walk when observed at the \( 2^{-m} \)-scale for each \( n \in \mathbb{N} \).

### 3 Continuity of the limit processes in the self-repelling parameter.

In this section, we show that the process is ‘continuous’ in \( u \). Denote by \( P^u \) and \( P^n \) the laws of \( X \) and \( X_n \) under \( P^{**} \), respectively. \( P^n \) and \( P^u \) are measures on \( C \). We will show \( P^u \rightarrow P^u \) weakly as \( u \to u_0 \).

We start with the ‘continuity’ of \( P^u \) in \( u \). Since \( p^n \) and \( p^u \) have supports on \( [0, \infty) \) as stated in Theorem 2.8, their Laplace transforms,
\[
G_n^u(s) \overset{def}{=} \int_0^\infty \exp(-s \eta) \ p^n_d(\eta) = \frac{1}{x_u} \Phi_n(x_u \exp(-\lambda^{-n} s), u),
\]
and
\[
G^u(s) \overset{def}{=} \int_0^\infty \exp(-s \eta) \ p^u(\eta),
\]
are holomorphic in \( \{ s \in \mathbb{C} : \Re(s) > 0 \} \).

Let
\[
g_n^u(s) \overset{def}{=} -\log \left\{ \frac{1}{x_u} \Phi_n(x_u \exp(-\lambda^{-n} s), u) \right\}, \quad s \in \mathbb{C},
\]

\[
9
\]
\[ H^n(q) \overset{\text{def}}{=} \frac{1}{x_u} \Phi(x_u q, u). \]

Proposition 2.1 with \( m = n - 1 \) and \( x = x_u \) implies
\[ \exp(-g^n_u(s)) = H^n(\exp(-g^n_{n-1}(\lambda_u^{-1}s))) \]
and
\[ (3.2) \quad g^n_u(s) = g^n_1(\lambda_u g^{n-1}_{n-1}(\lambda_u^{-1}s)). \]

**Proposition 3.1.** There exist positive constants \( C_\infty \) and \( M_\infty \) such that for any \( u \in [0, 1] \) and any \( n \in \mathbb{N} \), \( g^n_u(s) \) is holomorphic on \( |s| \leq C_\infty \) and satisfies
\[ (3.3) \quad |g^n_u(s) - s| \leq M_\infty |s|^2, \quad |s| \leq C_\infty \]

**Proof.** The implicit function theorem implies that for each \( u \in [0, 1] \), there exists a positive solution \( x = a_u \) to
\[ \Phi(x, u) = 1 \]
and that \( a_u \) is a continuous function of \( u \). Note that \( \Phi(x_u, u) = x_u \leq x_0 < 1 \). Hence \( C_1 \overset{\text{def}}{=} \min_{u \in [0,1]} \frac{a_u}{x_u} > 1. \)

Since \( \left| \frac{1}{x_u} \Phi(x_u e^{-t}, u) - 1 \right| \) is finite and continuous on the compact set \( \{(u, s) \in [0, 1] \times \mathbb{C} : |s| \leq \log C_1\} \), it is uniformly continuous. It follows from this and \( \Phi(x_u, u) = x_u \) that there exists \( C_2 > 0 \) such that
\[ \left| \frac{1}{x_u} \Phi(x_u e^{-t}, u) - 1 \right| < \frac{1}{2} \text{ for } \{u, s\} \in [0, 1] \times \mathbb{C} : |s| \leq C_2 \}. \]
Thus (3.1) implies that \( g^n_i(s) \) is holomorphic in \( \{s \in \mathbb{C} : |s| \leq C_2/\lambda_u\} \). This combined with \( g^n_i(0) = 0 \) and \( (g^n_i)'(0) = 0 \), implies that there exist positive constants \( C \) and \( M \) such that for any \( u \in [0, 1] \), \( g^n_i(s) \) is holomorphic on \( \{s \in \mathbb{C} : |s| < C\} \) and satisfies
\[ (3.4) \quad |g^n_i(s) - s| \leq M|s|^2, \quad |s| \leq C \]

Fix \( \varepsilon > 0. \) Let \( \lambda_- = \inf_{u \in [0,1]} \lambda_u \), and
\[ (3.5) \quad M_\infty = \frac{M(1 + \varepsilon)^2}{1 - \frac{1}{\lambda_-}}, \quad C_\infty = \frac{\varepsilon \lambda_-}{M_\infty} \wedge \tilde{C}, \]
where \( \tilde{C} \) is a positive constant defined by
\[ (3.6) \quad \tilde{C} \left( 1 + \frac{\tilde{C}M_\infty}{\lambda_-} \right) = C. \]

Note that \( C_\infty \leq \tilde{C} < C. \)

Define a sequence \( M_n, n = 1, 2, 3, \ldots \), by
\[ M_n = M_\infty \left( 1 - \frac{1}{\lambda_-} \right) - \frac{M \varepsilon (1 + \varepsilon)}{\lambda_-^{n-1}}. \]

It is straightforward to see that
\[ (3.7) \quad M_n < M_\infty, \quad n = 1, 2, 3, \ldots, \]
\[ (3.8) \quad M_1 = M, \]
\[ M_{n+1} = M_\infty \left( 1 - \frac{1}{\lambda_-} \right) + \frac{1}{\lambda_-} M_n \]
\[ = M(1 + \varepsilon)^2 + \frac{1}{\lambda_-} M_n, \quad n = 1, 2, 3, \ldots. \]

We prove by induction in \( n \) that \( g^n_u(s) \) is holomorphic on \( |s| \leq C_\infty \) and satisfies
\[ (3.10) \quad |g^n_u(s) - s| \leq M_n |s|^2, \quad |s| \leq C_\infty. \]
Then (3.7) and (3.10) imply Proposition 3.1.

$C_\infty < C$, (3.4) and (3.8) imply that $g^u_t(s)$ is holomorphic on $|s| \leq C_\infty$ and (3.10) holds for $n = 1$.

Assume that for $n = k$, $g^u_t(s)$ is holomorphic on $|s| \leq C_\infty$ and (3.10) holds. Note that $\lambda_- > 1$ implies that if $|s| \leq C_\infty$ then $|s/\lambda_u| < C_\infty$. Hence by induction hypothesis, $g^u_t(s/\lambda_u)$ is holomorphic and satisfies

\begin{equation}
\lambda_u |g^u_t(s/\lambda_u) - s/\lambda_u| \leq \frac{M_k}{\lambda_u} |s|^2.
\end{equation}

This and (3.7) and (3.6) further imply, for $|s| \leq C_\infty$,

$$
\lambda_u |g^u_t(s/\lambda_u)| \leq |s| \left(1 + \frac{M_k}{\lambda_u} |s| \right) \leq C_\infty \left(1 + \frac{M_k C_\infty}{\lambda_-} \right) \leq C.
$$

Therefore $g^u_{k+1}(s) = g^u_t(\lambda_u g^u_t(s/\lambda_u))$ is holomorphic on $|s| \leq C_\infty$ and

\begin{align}
|g^u_t(\lambda_u g^u_t(s/\lambda_u)) - \lambda_u g^u_t(s/\lambda_u)| &\leq M|\lambda_u g^u_t(s/\lambda_u) - s/\lambda_u| \\
&\leq M|\lambda_u g^u_t(s/\lambda_u)|^2 \leq M \left(|s| + \frac{M_k}{\lambda_u} |s|^2\right)^2 \leq M \left(1 + \frac{M_k C_\infty}{\lambda_-} \right)^2 |s|^2
\end{align}

where we also used (3.7) and $C_\infty \leq \frac{\varepsilon \lambda_-}{M_\infty}$ in the last line.

Using (3.11) and (3.12) we finally have

\begin{align}
|g^u_{k+1}(s) - s| &\leq |g^u_t(\lambda_u g^u_t(s/\lambda_u)) - \lambda_u g^u_t(s/\lambda_u)| + \lambda_u |g^u_t(s/\lambda_u) - s/\lambda_u| \\
&\leq \left(\frac{M_k}{\lambda_u} + M(1 + \varepsilon)^2 \right) |s|^2 \leq \left(\frac{M_k}{\lambda_-} + M(1 + \varepsilon)^2 \right) |s|^2 = M_{k+1} |s|^2, \quad |s| \leq C_\infty,
\end{align}

which implies (3.10) for $n = k + 1$. By induction, we have (3.10) for all $n$. \hfill $\square$

**Proposition 3.2.** There is a positive constant $C_\infty$ such that for each $u \in [0, 1]$, $g^u_n(s)$ converges uniformly on any compact subset of $\{s \in \mathbb{C} : |s| < C_\infty\}$ to a holomorphic function $g^u(s)$ as $n \to \infty$. $g^u(s)$ satisfies the following functional relation:

\begin{equation}
g^u(s) = g^u_1(\lambda_u g^u(\lambda_u^{-1} s)),
\end{equation}

and

\begin{equation}
\exp(-g^u(s)) = H^u(\exp(-g^u(\lambda_u^{-1} s))).
\end{equation}

**Proof.** Fix $u \in [0, 1]$. (3.3) implies that the family of functions, $\{g^u_n(s) : n = 1, 2, \ldots\}$ is uniformly bounded. This combined with $g^u_0(0) = 0$ implies that $\{g^u_n(s) : n = 1, 2, \ldots\}$ forms a normal family on $\{s \in \mathbb{C} : |s| < C_\infty\}$. Therefore for any subsequence of $\{g^u_n(s)\}$, there exists a sub-subsequence that converges to a holomorphic function $g^u$ uniformly on any compact subset of $\{s \in \mathbb{C} : |s| < C_\infty\}$. Note that

$$
\exp(-g^u_n(s)) = G^u_n(s)
$$
on $\{s \in \mathbb{C} : |s| < C_\infty, \Re(s) > 0\}$. Proposition 2.7 implies that $p^u_n$ converges weakly to $p^u$ as $n \to \infty$, which further implies $G^u_n(s) \to G^u(s)$ as $n \to \infty$. Thus

$$
g^u(s) = -\log G^u(s)
on\quad\text{on } \{s \in \mathbb{C} : |s| < C_\infty, \Re(s) > 0\}.\non$$

This combined with the principle of analytic continuation implies that the limit $g^u$ is independent of the subsequences. Therefore $\{g^u_n\}$ converges uniformly on any compact subset of $\{s \in \mathbb{C} : |s| < C_\infty\}$ to a holomorphic function $g^u$ satisfying

\begin{equation}
\exp(-g^u(s)) = G^u(s),
\end{equation}
on $\{s \in \mathbb{C} : |s| < C_\infty, \Re(s) > 0\}$. Letting $n \to \infty$ in (3.2), we have (3.13). Rewriting (3.13), we also have (3.14).
Let

\[ F_0(s) = \exp(-g^u(s)), \]
\[ F_n(s) = (H^u)^n(\exp(-g^u(\lambda_u^{-n}s))), \quad n = 1, 2, \ldots, \]

and \( A_0 = \{ s \in \mathbb{C} : |s| < C_\infty \}, \quad A_n = \{ s \in \mathbb{C} : \lambda_u^{-n}C_\infty \leq |s| < \lambda_u^nC_\infty \}, \) for \( n = 1, 2, \ldots. \) Define \( \tilde{G}^u : \mathbb{C} \rightarrow \mathbb{C} \) by

\[ \tilde{G}^u(s) = F_n(s), \quad \text{on} \quad A_n, \quad n = 0, 1, 2, \ldots. \]

Since \( H(g) \) is a rational function and (3.14) implies \( F_n(s) = F_{n-1}(s) \) on \( \bigcup_{k=0}^{n-1} A_k, \) \( \tilde{G}^u(s) \) is meromorphic on \( \mathbb{C}. \) This combined with (3.15) implies

\[ \tilde{G}^u(s) = G^u(s), \quad \text{on} \quad \{ s \in \mathbb{C} : \Re(s) > 0 \}. \]

**Proposition 3.3.** For any \( u_0 \in [0, 1], \) \( p^{u_n} \) converges weakly to \( p^{u_0} \) as \( u \rightarrow u_0. \)

**Proof.** Letting \( n \rightarrow \infty \) in (3.3), we see that

\[ |g^u(s) - s| \leq M_\infty|s|^2, \quad |s| < C_\infty. \]

This implies \( \{ g^u : u \in [0, 1] \} \) forms a normal family on \( \{ s \in \mathbb{C} : |s| < C_\infty \} \). Let \( \{ u_n \} \) be a sequence in \( [0, 1] \) that converges to \( u_0. \) Then \( \{ g^{u_n} : n = 1, 2, \ldots \} \) has a subsequence that converges to a holomorphic function \( h \) uniformly on any compact subset of \( \{ s \in \mathbb{C} : |s| < C_\infty \}. \) \( h \) satisfies the functional relation,

\[ h(s) = g^{u_0}(\lambda_u h(\lambda_u^{-1}s)). \]
\[ h(0) = 0, \quad h'(0) = 1. \]

This equation has a unique solution \( g^{u_0}. \) Thus \( g^{u_n} \rightarrow g^{u_0} \) uniformly on any compact subset of \( \{ s \in \mathbb{C} : |s| < C_\infty \} \) as \( n \rightarrow \infty. \) This combined with (3.16) implies that \( \tilde{G}^{u_n}(s) \rightarrow \tilde{G}^{u_0}(s) \) for every \( s \in \mathbb{C} \) with \( \Re(s) > 0. \) We have, from (3.17),

\[ \lim_{n \rightarrow \infty} G^{u_n}(s) = G^{u_0}(s), \quad \text{on} \quad \{ s \in \mathbb{C} : \Re(s) > 0 \}. \]

Since a Laplace transform determines a measure, we see that \( p^{u_n} \) converges weakly to \( p^{u_0} \) as \( n \rightarrow \infty. \)

Now we go on to the 'continuity' of \( P^u \) in \( u, \) where \( P^u \) is defined at the beginning of this section.

**Proposition 3.4.** The family of measures \( \{ P^u \} \), \( 0 \leq u \leq 1 \) is tight.

**Proof.** Since \( \{ w \in C : w(0) = 0 \} \) is a closed subset of \( C, \) we have

\[ P^u[ w(0) = 0 ] \geq \limsup_{n \rightarrow \infty} P^u_n[ w(0) = 0 ] = 1. \]

We will show that for any \( \varepsilon > 0 \) and \( u_0 \in [0, 1], \) there exist positive numbers \( \alpha \) and \( \delta \) such that

\[ P^u[ \sup_{|s|=\varepsilon, 0 < t < s} |w(s) - w(t)| \geq \varepsilon ] \leq \frac{\varepsilon}{2}, \quad \text{for any} \quad u \quad \text{with} \quad |u - u_0| < \alpha. \]

For an arbitrarily given \( \varepsilon, \) choose \( k \in \mathbb{Z} \) satisfying

\[ 2 \cdot 2^{-k} < \varepsilon. \]

Choose \( N \) large enough so that

\[ \tilde{P}^u_k[L(w) > N] < \frac{\varepsilon}{2}, \quad \text{for all} \quad u \in [0, 1]. \]

Let

\[ V = \{ w \in W_k : L(w) \leq N \}, \]

where
and 

\[ D = \{ w \in C : \text{There exist } s, t \geq 0 \text{ with } |s - t| < \delta \text{ and } \Delta_1, \Delta_2 \in \mathcal{T}_k \text{ with } \Delta_1 \cap \Delta_2 = \emptyset \text{ such that } w(s) \in \text{int}(\Delta_1), w(t) \in \text{int}(\Delta_2) \}, \]

where \( \text{int}(\Delta_i) = \Delta_i \cap \mathcal{G}_h \), \( i = 1, 2 \). \( D \) is an open subset of \( C \). Note that Theorem 2.9 implies that \( P_n^u \rightarrow P^u \) weakly as \( n \to \infty \). The choice of \( k \) and the continuity of \( w \) imply that if \( \sup_{|t-t'|<\delta} |w(s) - w(t)| > \varepsilon \), then \( w \in D \).

We have,

\[
P^u \left[ \sup_{|t-t'|<\delta} |w(s) - w(t)| > \varepsilon \right] \leq P^u[D] \leq \liminf_{n \to \infty} P_n^u[D] \leq \liminf_{n \to \infty} P_n^u[ S_k^h < \delta \text{ for some } i = 1, \ldots, L(Q_kw) ] \leq \liminf_{n \to \infty} \left\{ \sum \limits_{v \in V} \sum \limits_{i=1}^{L(v)} P_n^u[ S_k^h < \delta | Q_kw = v ] \cdot P_n^u[ Q_kw = v ] + P_n^u[ Q_kw \in W_k \setminus V ] \right\} < N \sum \limits_{v \in V} \lim_{n \to \infty} P_n^u[ s : s < \lambda_k^h \delta ] P^u_{}[ v ] + \frac{\varepsilon}{2} = N \sum \limits_{v \in V} P^u_{}[ s : s < \lambda_k^h \delta ] P^u_{}[ v ] + \frac{\varepsilon}{2} \leq N P^u_{}[ s : s < \lambda_k^h \delta ] + \frac{\varepsilon}{2}
\]

where we used Proposition 2.4, Proposition 2.5 and Theorem 2.8.

The continuity of \( p^u \) in \( u \) proceed in Proposition 3.3 combined with \( p^u[(0)] = 0 \) implied by Theorem 2.8 shows that we can choose a \( \delta > 0 \) such that

\[ p^u_{}[ s : s < \lambda_k^h \delta ] \leq p^u_{}[ s : s < ( \max_{u \in [0,1]} \lambda_u )^k \delta ] \leq \frac{\varepsilon}{2N} \text{ for all } u \in [0,1]. \]

Then we have

\[ P^u_{} \left[ \sup_{|t-t'|<\delta} |w(s) - w(t)| > \varepsilon \right] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ for all } u \in [0,1]. \]

This completes the proof.

Now we will show the convergence of the finite dimensional distributions. Fix an arbitrary \( m \in \mathbb{Z} \) and \( 0 < t_1 < \ldots, \leq t_m \). For \( w \in C \), let us define

\[ h(w, x) = e^{ix_1 w(t_1) + \ldots + ix_m w(t_m)}, x = (x_1, \ldots, x_m) \in (\mathbb{R}^2)^m, \]

where \( x \cdot y \) denotes the inner product in \( \mathbb{R}^2 \). For a probability measure \( Q \) on \( C \), define \( F(Q) : (\mathbb{R}^2)^m \to \mathbb{C} \) by,

\[ F(Q)_{}[x] \overset{\text{def}}{=} E^Q[h(., x)]. \]

**Proposition 3.5.** For each \( x \in (\mathbb{R}^2)^m \), \( F(P^u) = F(P^u_{})(x) \) is continuous in \( u \in [0,1] \).

**Proof.** Fix \( x \in (\mathbb{R}^2)^m \). Let \( \varepsilon \) be an arbitrary positive number. Define \( f : (\mathbb{R}^2)^m \to \mathbb{R}, \) by \( f(y) = \exp(i \sum_{j=1}^{m} x_j y_j), y = (y_1, \ldots, y_m) \in (\mathbb{R}^2)^m \). Since \( f \) is uniformly continuous, we can choose a positive integer \( k \) such that

\[ |f(y) - f(z)| < \varepsilon, \text{ for any } y, z \in (\mathbb{R}^2)^m, \text{ with } |y_j - z_j| < 2^{-k}, j = 1, \ldots, m, \]

where \( y = (y_1, \ldots, y_m), z = (z_1, \ldots, z_m) \). Furthermore, we can choose a positive integer \( N \) such that

\[ P_k^u[L(w) > N] < \varepsilon, \text{ for all } u \in [0,1]. \]
Let $V = \{ w \in W_k : L(w) \leq N \}$.
For $n \geq k$,

$$\begin{align*}
F(P^n) &= E^{P^{w}}[ h(X, x) ] \\
&= \sum_{v \in V} E^{P^{v+w}}[ h(X, x) \mid X_k = v ] P^{v}[ X_k = v ] \\
&\quad + E^{P^{w}}[ h(X, x) \mid X_k \in W_k \setminus V ] P^{w}[ X_k \in W_k \setminus V ].
\end{align*}$$

The first term on the right-hand side is further decomposed as

$$\begin{align*}
\sum_{v \in V} E^{P^{v+w}}[ h(X, x) \mid X_k = v ] P^{v}[ X_k = v ] \\
&= \sum_{v \in V} \sum_{r_i} E^{P^{v+w}}[ h(X, x) \mid X_k = v, T_{r_i}^{v+k} \leq t_i < T_{r_i+1}^{v+k}, i = 1, \ldots, m ] \\
&\quad \times P^{v}[ T_{r_i}^{v+k} \leq t_i < T_{r_i+1}^{v+k}, i = 1, \ldots, m \mid X_k = v ] P^{w}[ X_k = v ],
\end{align*}$$

where $\sum_{r_i}$ is taken over $(r_1, \ldots, r_m) \in \{ 1, 2, \ldots, L(v) \}^m$ with $r_1 \leq r_2 \leq \cdots \leq r_m$. Using $|h(w, x)| \leq 1$ and the definition of $N$ and $V$,

$$\begin{align*}
|E^{P^{v+w}}[ h(X, x) \mid X_k \in W_k \setminus V ] P^{w}[ Q_k w \in W_k \setminus V ] &
\leq P^{w}[ Q_k w \in W_k \setminus V ] \leq \varepsilon.
\end{align*}$$

Denote for simplicity,

$$\begin{align*}
E(u) &= E^{P^{u}}[ h(X, x) \mid X_k = v, T_{r_i}^{v+k} \leq t_i < T_{r_i+1}^{v+k}, i = 1, \ldots, m ], \\
\overline{P}^u_1 &= P^{u}[ T_{r_i}^{v+k} \leq t_i < T_{r_i+1}^{v+k}, i = 1, \ldots, m \mid X_k = v ], \\
\overline{P}^u_2 &= P^{u}[ X_k = v ].
\end{align*}$$

Fix $u_0 \in [0, 1]$ arbitrary. For any $u \in [0, 1]$,

$$\begin{align*}
|F(P^n) - F(P^{u_0})| &< \sum_{v \in V} \sum_{r_i} |E(u) - E(u_0)| \mid \overline{P}^u_1 - \overline{P}^{u_0}_1 \mid \overline{P}^u_2 + \sum_{v \in V} \sum_{r_i} |E(u_0)| \mid \overline{P}^{u_0}_1 - \overline{P}^u_1 \mid \overline{P}^u_2 \\
&\quad + \sum_{v \in V} \sum_{r_i} |E(u_0)| \mid \overline{P}^{u_0}_1 - \overline{P}^{u_0}_2 \mid + 2\varepsilon.
\end{align*}$$

Corollary 2.10 implies that if $X_k = v$ then $X(T_{r_i}^{v+k}) = v(r_i)$ a.s. and that if $T_{r_i}^{v+k} \leq t_i < T_{r_i+1}^{v+k}$, then $X(t_i)$ is almost surely either in $\Delta \in T_K$ such that $v(r_i), v(r_i + 1) \in \Delta$ or in its neighboring elements of $T_K$ adjacent at $v(r_i)$. This means $|X(t_i) - v(r_i)| \leq 2^{-k}$ a.s.

Therefore,

$$\begin{align*}
|E(u) - E(u_0)| &< |E^{P^{u}}[ f(w(t_i), \ldots, w(t_m)) - f(v(r_1), \cdots, v(r_m)) ] X_k = v, T_{r_i}^{v+k} \leq t_i < T_{r_i+1}^{v+k}, i = 1, \cdots, m | \\
&\quad + f(v(r_1), \cdots, v(r_m)) \mid - E^{P^{u_0}}[ f(w(t_i), \ldots, w(t_m)) - f(v(r_1), \cdots, v(r_m)) ] X_k = v, T_{r_i}^{v+k} \leq t_i < T_{r_i+1}^{v+k}, i = 1, \cdots, m | \\
&\quad - f(v(r_1), \cdots, v(r_m)) | \\
&< 2\varepsilon,
\end{align*}$$

where we have used (3.18) for the last inequality.

Theorem 2.8 and Proposition 3.3 imply that $\overline{P}^u_1$ is continuous in $u$, thus there exists $\delta_1 > 0$ such that

$$|\overline{P}^u_1 - \overline{P}^{u_0}_1| < \varepsilon, \quad \text{for any } u \text{ with } |u - u_0| < \delta_1,
If we note that \( P^w[X_k = v] = \tilde{P}'_k[w = v] \) for \( v \in W_k \), we see that \( P^w[X_k = v] \) is continuous in \( u \). Thus there exists \( \delta_2 > 0 \) such that

\[
|\tilde{P}'_k - \tilde{P}'_{k_0}| < \varepsilon, \quad \text{for any } u \text{ with } |u - u_0| < \delta_2.
\]

Let \( \delta = \min \{ \delta_1, \delta_2 \} \). Then

\[
|F(P^u) - F(P^{u_0})| < 6\varepsilon \quad \text{for any } u \text{ with } |u - u_0| < \delta.
\]

This completes the proof. \( \square \)

Proposition 3.4 combined with Proposition 3.5 leads to the following theorem.

**Theorem 3.6.** For any \( u_0 \in [0, 1] \), \( P^u \) converges to \( P^{u_0} \) weakly as \( u \to u_0 \).

4 Path properties.

Let

\[
\gamma = \gamma_u = \frac{\log 2}{\log \lambda_u}, \quad \beta = \frac{1 - \gamma}{\gamma}.
\]

Large deviation estimates for the supercritical branching process allow us to state the following Lemma.

**Lemma 4.1 ([4]).** There exists a multiplicatively periodic function \( H \) such that

\[
-\log P^{*u}(S_1^0 < x) = x^{-1/\beta}H(x) + o(x^{-1/\beta}).
\]

Using this result we have

**Proposition 4.2.** There exist positive constants \( C_{2,1} - C_{2,4} \) and \( K \) such that

\[
P^u \left[ |X(t)| \geq \delta \right] \leq P^u \left[ \sup_{0 \leq s \leq t} |X(s)| \geq \delta \right] \leq C_{2,3} \exp\left(-C_{2,4}(\delta^{-\gamma})^{-\frac{1}{\beta}}\right),
\]

for all \( t \geq 0 \) and \( 0 < \delta < \frac{1}{4} \) with \( \delta^{-\gamma} \geq K \), and

\[
P^u \left[ |X(t)| \geq \delta \right] \leq P^u \left[ \sup_{0 \leq s \leq t} |X(s)| \geq \delta \right] \leq C_{2,3} \exp\left(-C_{2,4}(\delta^{-\gamma})^{-\frac{1}{\beta}}\right),
\]

for all \( t \geq 0 \) and \( 0 < \delta < 1 \).

**Proof.** Consider \( \omega \in \Omega \) such that \( \lim_{n \to \infty} S_i^n(X_n)(\omega) = S_i^m(\omega) \) exists and \( S_i^m(\omega) > 0 \) for all \( m \) and \( i \).

Lemma 4.1 implies that there exist positive constants \( C_{1,1} - C_{1,4} \) such that

\[
P^u \left[ |X(t)| \geq \delta \right] \leq P^u \left[ S_1^0 < \delta \right] \leq C_{1,3} \exp\left(-C_{1,4}(\delta^{-\gamma})^{-\frac{1}{\beta}}\right), \quad x \geq 0.
\]

For the lower bound, choose \( n \in \{ 2, 3, \ldots \} \) such that \( 2^{-n-1} < \delta \leq 2^{-n} \). Note that Corollary 2.10 implies that if \( T_1^{n-1} < t \) and \( S_j^m > t \) for \( j = S_i^{n-1}(Y_n) + 1 \) then \( |X(t)| \geq \delta \) a.s. In terms of branching processes, since \( T_1^{n-1} \) and \( S_j^m \) with \( j = S_i^{n-1}(Y_n) + 1 \) are related to the limit of the numbers of offsprings coming from different children, they are independent. (In other words, \( S_j^m \) is independent of \( \sigma[S_i^{n-1}(Y_{n-1+r}) : r = 0, 1, \ldots] \).) These combined with (4.3) imply

\[
P^u \left[ |X(t)| \geq \delta \right]
\]

\[
\geq P^u \left[ T_i^{n-1} < t, \ S_j^m > t \right]
\]

\[
= P^u \left[ T_i^{n-1} < t \right] \cdot P^u \left[ T_i^n > t \right]
\]

\[
\geq C_{1,1} \exp\left(-C_{1,2}(\lambda_i^{n-1})^{-\frac{1}{\beta}}\right) \cdot \left( 1 - C_{1,3} \exp\left(-C_{1,4}(\delta^{-\gamma})^{-\frac{1}{\beta}}\right) \right)
\]

\[
\geq C_{1,1} \exp\left(-4C_{1,2}(\delta^{-\gamma})^{-\frac{1}{\beta}}\right) \cdot \left( 1 - C_{1,3} \exp\left(-C_{1,4}(\delta^{-\gamma})^{-\frac{1}{\beta}}\right) \right).
\]

Choose \( K > 0 \) large enough so that the last factor exceeds \( \frac{1}{2} \) for \( \delta^{-\gamma} \geq K \).
For the upper bound, choose \( n \in \mathbb{N} \) such that \( 2^{-n} < \delta \leq 2^{-n+1} \). Let \( \Delta, \Delta' \in \mathcal{T}_n \) be the adjacent triangles such that \( O \in \Delta \cap \Delta' \) and \( X(T_1^n) \in \Delta \cap (\Delta')^c \).

Corollary 2.10 implies that for \( 0 \leq s \leq T_1^n \), \( X(s) \in \Delta \cup \Delta' \), thus \( |X(s)| \leq 2^{-n} < \delta \). It follows that if \( \sup_{0 < \xi \leq t} |X(\xi)| \geq \delta \), then \( T_1^n < t \). Combining this with Lemma 4.1, we have

\[
P^{su}[|X(t)| \geq \delta] \leq P^{su}[T_1^n < t] = P^{su}[S_1^{00} < \lambda_0^n t] \leq C_{1,3} \exp(-C_{1,4}(\lambda_0^n t)^{-\frac{\gamma}{\delta}}) \leq C_{1,3} \exp(-C_{1,4}(\frac{\delta t^{-\gamma}}{2})^{-\frac{1}{\delta}}).
\]

This completes the proof. \( \square \)

For a sharper result we could obtain large deviation estimates using the approach of [3].

Integrating (4.1) and (4.2), we see that for each \( p > 0 \) there exist positive constants \( C_{3,1}(p), C_{3,2}(p) \) and \( \tau(p) \) such that

\[
C_{3,1}(p) t^p \leq E^u[X(t)^p] \leq C_{3,2}(p) t^{\tau(p)},
\]

for any \( t \) with \( t < \tau(p) \). Thus we have

**Theorem 4.3.** For each \( p > 0 \), there are positive constants \( C_{3,1}(p) \) and \( C_{3,2}(p) \) such that

\[
C_{3,1}(p) \leq \liminf_{t \downarrow 0} \frac{E^u[X(t)^p]}{t^p} \leq \limsup_{t \downarrow 0} \frac{E^u[X(t)^p]}{t^p} \leq C_{3,2}(p).
\]

We can conclude with a law of the iterated logarithm for our self-repelling process.

**Theorem 4.4.** There exists a positive constant \( c \) such that

\[
c \leq \limsup_{t \downarrow 0} \frac{|X(t)|}{\psi(t)} \leq 1, \quad P^{su} \text{- a.s.,}
\]

where \( \psi(t) = C_{3,4}^{-1} t^\gamma (\log \log \frac{1}{t})^{1-\gamma} \).

*Proof.* The upper bound is straightforward to prove using (4.2) with a standard Borel-Cantelli argument.

The lower bound is more difficult and the standard approach applied to Brownian motion cannot be used as it relies on the Markov property of the process. In our setting we do have a distributional self-similarity property for our path which we can exploit.

In order to prove this result we consider the sequence of stopping times \( \{T_1^n \mid n \geq 0\} \). Thus \( |X(T_1^n)| = 2^{-n} \) under \( P^{su} \). We can describe the sequence of times via the limiting random variable in the supercritical branching process defined in Proposition 2.7. Note that for \( k < m \),

\[
S_{1}^{k} = \sum_{j=1}^{k} S_{j}^{(m)}, \quad a.s.,
\]

where the summands are i.i.d. and equal in law to \( \lambda^{-m} S_1^{0} \).

The behaviour of the asymptotics of a sequence of random variables satisfying this type of equation is discussed in [10, 11] in the context of random recursive fractals. Here we have the somewhat easier task of proving an almost sure lower bound on the oscillation in \( S_1^{0} \).

Our result will follow from the following Lemma.

**Lemma 4.5.** There exist positive constants \( c, N_0 \) such that if \( m_k = kN \) for \( N > N_0 \), then

\[
P^{su}(\lambda^{-m_k} S_1^{m_k} \leq c(\log(m_k))^{-\beta} \ i.o.) = 1.
\]
Proof. Let $A_m = \{ \lambda_m S^m_t \leq \delta (\log m)^{-\beta} \}$ and write $A^c_m$ for the complementary event. Using the fact that there is a Markov structure (4.4) inherited from the branching process in the sequence of random variables $\lambda_m S^m_t$, we have

$$P^u(A_m \mid A^c_{m-1}, \ldots, A^c_2) = P^u(A_m \mid A^c_{m-1}).$$

Then a straightforward extension of the second Borel–Cantelli Lemma shows that we will have the claim of the Lemma, $P^u(\limsup_{k \to \infty} A_k) = 1$, if

$$\sum_{m=1}^{\infty} P^u(A_m \mid A^c_{m-1}) = \infty.$$  

Note that as

$$\sum_{m=1}^{\infty} P^u(A_m \mid A^c_{m-1}) = \sum_{m=1}^{\infty} \frac{P^u(A_m \cap A^c_{m-1})}{P^u(A^c_{m-1})} \geq \sum_{m=1}^{\infty} P^u(A_m \cap A^c_{m-1})$$

it is enough to establish

$$(4.7) \quad \sum_{m=1}^{\infty} P^u(A_m \cap A^c_{m-1}) = \infty$$

to obtain (4.6) and prove the claim.

At this stage we consider $A_{m_k}$, where $m_k = kN$ for some integer $N$. Using [10] Lemma 4.2 we let $x_k = b^{-\beta} \lambda^{-m_k} (\log(m_k))^{-\beta}$ $(b > 0)$ and hence

$$P^u(S^{*m_k} \leq x_k, S^{*m_{k-1}} > x_{k-1})$$

$$= P^u(S^{*m_k} \leq x_k, \sum_{i=1}^{m_k} S_i^{*m_k} > x_{k-1})$$

$$= \int_0^{x_k} P^u \left( \sum_{i=2}^{m_k} S_i^{*m_{i-1}} + y > x_{k-1} \right) P^u(S^{*m_k} \in dy)$$

$$\geq P^u(S_1^{*m_k} \in [c_1 x_{k-1}, x_k]) P^u(\sum_{i=2}^{m_k} S_i^{*m_{i-1}} > (1-c_1|x_{k-1}),$$

for some constant $0 < c_1 < 1$. Observing that, as $x_k$ is decreasing in $k$, the second term in the product will be bounded below by a constant $c_2$. If we now set $c_3 = (c_1 \lambda^{-m_k} m^{-1/\beta})$ and apply the tail estimates in Lemma 4.1, then

$$P^u(S^{*m_k} \leq x_k, S^{*m_{k-1}} > x_{k-1})$$

$$\geq c_2 \exp(-c_3 \log(m_k) H((\log(m_k))^{-\beta}))$$

$$\times (1 - \exp(-(c_2 \log(k-1) - c_3 \log k) + o(\log(k)))).$$

for small enough $b > 0$. By choosing $N$ large enough, we can make $c_2 = b \log(m_k) H(x)$ sufficiently large to ensure that the term $\exp(-c_2 \log(k-1) - c_3 \log k) + o(\log(k)) \leq \frac{1}{2}$ for large $k$, and hence we have the divergence of the sum in (4.7), giving the result. $\square$

Finally to complete the proof of Theorem 4.4 we apply Lemma 4.5 to a suitable subsequence of the stopping times to show that for $N$ sufficiently large, where $m_k = kN$ we have $T^{*m_k} \leq \lambda^{-m_k} (\log(m_k))^{-\beta}$ almost surely. Letting $t_k = T^{*m_k}$ we have

$$\log(t_k) \leq -m_k \log(\lambda) - \beta \log(\log(m_k)),$$

and, by taking the inverse, this implies the existence of a constant $c_0$ such that

$$-m_k \geq \frac{\log(t_k)}{\log(\lambda)} + \frac{\beta \log(\log(t_k))}{\log(\lambda)} + \frac{\log(c_0 \log 2)}{\log 2}.$$
Hence
\[ |X_{T_1^{m_i}}| = 2^{-m_i} \geq c_0 \lambda_k^{\log 2 / \log \lambda_k} (\log \log (\frac{1}{T_k}))^{-1} \log 2 / \log \lambda_k, \]
and hence we have a subsequence which is exceeded infinitely often with probability one. \hfill \Box

For the behavior of the path at arbitrary times, we have

**Proposition 4.6.** For any \( t, t + h > 0 \) and \( 0 < \delta < 1 \), it holds that
\[ P^{*u}[ |X(t + h) - X(t)| \geq \delta, \ t > T_{i_k}^{m_k} ] \leq C_{2,3} \exp(- C_{2,4} (\frac{\delta h^{\gamma - \gamma'} - 1}{4} + \frac{\delta h^{\gamma'}}{4}), \]
where \( C_{2,3} \) and \( C_{2,4} \) are as in Proposition 4.2.

**Proof.** It is sufficient to prove the statement for \( h > 0 \). Choose \( \omega \) as in the proof of Proposition 4.2. For any given \( \delta, 0 < \delta < 1 \), choose \( n \in \mathbb{N} \) such that
\[ 2^{-n+2} \leq \delta < 2^{-n+3}. \]
Corollary 2.10 (3) implies
\[ P^{*u}[ |X(t + h) - X(t)| \geq \delta, \ t > T_{i_k}^{m_k} ] = 0, \]
and
\[ P^{*u}[ |X(t + h) - X(t)| \geq \delta, \ T_{i_k}^{m_k} \leq t < T_{i_k}^{m_k} = T_1^{m_0}, \text{ for some } i ] = 0. \]
In the case that \( T_{i_k}^{m_k} \leq t < T_{i_k}^{m_k} < T_1^{m_0} \), if \( S_{i+1}^{m+1} > h \), then \( |X(t + h) - X(t)| < 3 \cdot 2^{-n} < \delta. \)
These, combined together, imply
\[ P^{*u}[ |X(t + h) - X(t)| \geq \delta ] = P^{*u}[ |X(t + h) - X(t)| \geq \delta, \ T_{i_k}^{m_k} \leq t < T_{i_k}^{m_k} < T_1^{m_0} \text{ for some } i ] \leq P^{*u}[ T_{i_k}^{m_k} \leq t < T_{i_k}^{m_k} < T_1^{m_0}, \ S_{i+1}^{m+1} < h, \text{ for some } i ] \leq P^{*u}[ S_{i+1}^{m+1} < h ], \]
and we have the statement from (4.3). \hfill \Box

**Theorem 4.7.** For any \( M > 0 \) and any \( \gamma' \) with \( 0 < \gamma' < \gamma \), the following holds \( P^{*u} \)-almost surely. There are positive constants \( b = b(\gamma', \omega) \) and \( H = H(\gamma', \omega) \) such that
\[ |X(t + h) - X(t)| \leq b |h|^{\gamma'}, \text{ for any } t \in [0, M] \text{ and any } h \in [-H, H]. \]

## 5 Self-repelling processes on \( \mathbb{R} \)

Here we summarize the basic ingredients of the construction of the corresponding self-repelling processes on \( \mathbb{R} \). We start with a sequence of random walks on \( \mathbb{Z} \) (instead of the pre-Sierpiński gasket in Section 2). The vertex set that we will use for our walks is \( G_n = \{ k2^{-n} | k = -2^n, -2^n + 1, \ldots, 0, 1, 2, \ldots, 2^n \}. \)

**Remark.** We could alternatively consider, for example, \( G_n = \{ k3^{-n} | k = -3^n, -3^n + 1, \ldots, 0, 1, 2, \ldots, 3^n \}. \)
That is, we have the choice of how we divide up the unit interval and any geometric dissection into halves, thirds, quarters, etc could be used. The resulting self-repelling processes will be different (even when they are constructed to have the same value of \( \gamma \)). Thus our method produces more than one family of self-repelling processes that continuously interpolate Brownian motion and straight motion on a line. On the Sierpiński gasket, there is an obvious natural unit scale, so our method naturally points at one family of processes. Here we will take the dyadic partition as it is the simplest to work with.
As in Section 2, \( W_n \) is the set of continuous functions such that at integer times it takes values in \( G_n \) with nearest neighbor jumps from 0 to 1. A sequence of decimation maps \( Q_k \) can be defined in a similar way as in the Sierpiński gasket case. The ‘reversing number’ \( N_k(w) \) is now, verbally, the number of points in \( G_k \setminus G_{k-1} \) where the decimated walk reverses its jump direction. The ‘returning number’ retains the same interpretation as before. The generating function \( \Phi_n(x, u) \) in (2.2) is much simpler than for the Sierpiński gasket and is given by Proposition 2.1 with

\[
\Phi_1(x, u) = \frac{\Psi(x, u)}{1 - 2u\Theta(x, u)}, \quad \Psi(x, u) = x^2, \quad \Theta(x, u) = ux^2.
\]

In particular, we have \( \Phi_n(x, 0) = x^{2^n} \), which implies that when \( u = 0 \) we have a single path which connects 0 and 2\(^n\) by a straight line (i.e., the self-avoiding path on \( \mathbb{Z} \)), and for \( u = 1 \) we reproduce the generating function for the simple random walk. In general, \( \Phi_n(x, u) \) has non-trivial \( (x, u) \) dependence.

We can give explicit formulas for the key quantities \( x_u > 0 \) and \( \lambda_u > 0 \) in Proposition 2.3. They are defined by \( \Phi(x_u, u) = x_u \) and \( \lambda_u = \frac{\partial \Phi}{\partial x}(x_u, u) \) and are given by

\[
x_u = \frac{1}{4u^2}(\sqrt{1 + 8u^2} - 1), \quad \lambda_u = \frac{2}{x_u} = \sqrt{1 + 8u^2} + 1.
\]

Note that \( x_u > 0 \) exists for all \( u \geq 0 \), and that \( \Phi(x, u) \) is regular at \( x_u \). Since for \( 0 \leq u < 1 \), the paths with a large number of steps are suppressed, we expect that the corresponding walk is self-repelling. For \( u > 1 \), we expect that the corresponding walk is self-attracting. Also, \( \lambda_0, 0 \leq u \leq 1 \), continuously interpolates \( \lambda_0 = 2 \) (linear motion) and \( \lambda_1 = 4 \) (simple random walk). The basic quantities are ‘smooth’ in the parameter \( u \) for all \( u \geq 0 \). Hence we expect that everything is smooth also for \( u > 1 \) and as \( u \to \infty \) we see that \( x_u \to 0 \) and \( \lambda_u \to \infty \) and the model eventually approaches a completely localized model.

Once we have established these properties of the generating function the subsequent analysis follows quite similar lines to the Sierpiński gasket case in Section 3 and Section 4. For example, the probability measures on the paths are defined by (2.4), and the existence of a continuum limit (Theorem 2.9) and the weak continuity of the path measure \( P^u \) in \( u \in [0, 1] \) (Theorem 3.6) hold. The sample path properties such as Theorem 4.3, Theorem 4.4, and Theorem 4.7 also hold with \( \gamma = \frac{\log 2}{\log \lambda_u} \).

References


