Asymptotically one-dimensional diffusions on scale-irregular gaskets.

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Abstract

A new class of fractals, the scale-irregular *abb*-gaskets, is defined, and the asymptotically one-dimensional diffusion processes are constructed on them. The class contains infinitely many fractals which lack exact self-similarity, and which also lack non-degenerate fixed points of renormalization maps (hence are not in the class of nested fractals).

An essential step in the construction of diffusion is to prove the existence of appropriate time-scaling factors. For this purpose, a limit theorem for a discrete-time multi-type supercritical branching processes with singular and irregular (varying) environment, is developed.

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1 Introduction.

In this paper we define a new class of fractals which we call the scale-irregular *abb*-gaskets, and construct asymptotically one-dimensional diffusion processes [16] on the scale-irregular *abb*-gaskets. The class scale-irregular *abb*-gasket is a generalization of the Sierpiński gasket, a triangle based fractal, which we introduce as examples of finitely ramified fractals which are scale-irregular, i.e. do not have exact self-similarity, and moreover, which lack non-degenerate fixed points of renormalization maps (hence are not in the class of nested fractals). See [17, 16, 15] for the motivation on the latter point. The class scale-irregular *abb*-gasket is a scale-irregular extension of the *abc*-gasket defined in [17].

Intuitively speaking, a scale-irregular *abb*-gasket is obtained by recursively repeating a procedure of joining 'triangle graphs' to form a larger triangle, and

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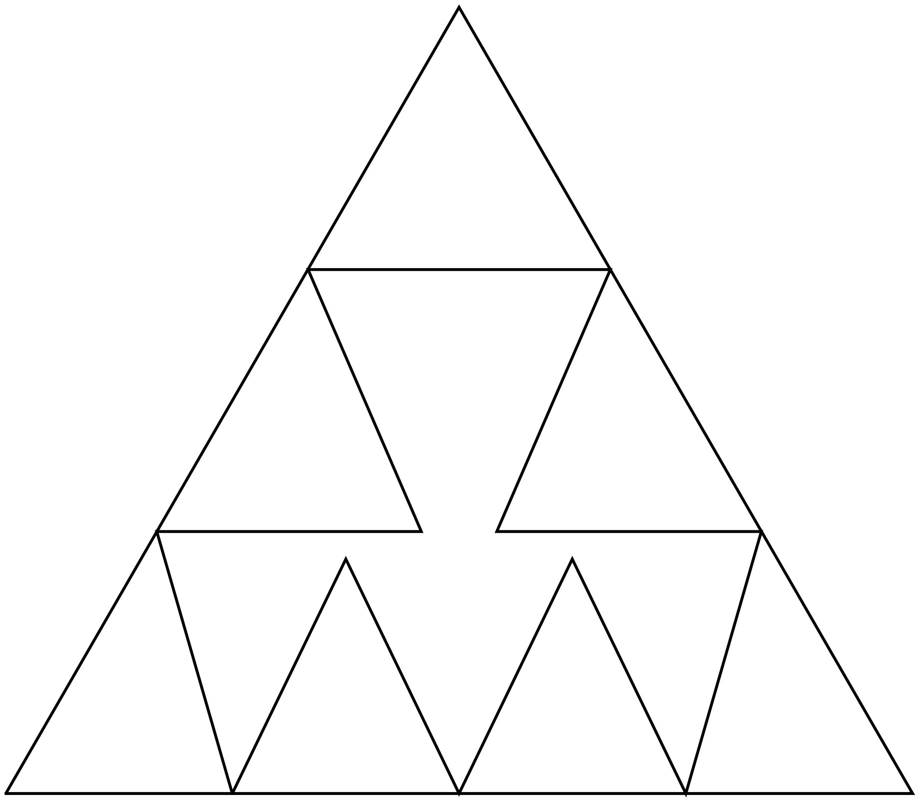

'shrinking' them by giving appropriate metrics. Namely, join (a_0+2b_0) copies of a triangle $\tilde{H}_0$ as in Figure 1 to form a triangle $\tilde{H}_{-1}$. In a similar way, for $n \in \mathbf{Z}_+$, form $\tilde{H}_{-n-1}$ from $(a_{-n}+2b_{-n})$ copies of $\tilde{H}_{-n}$. A scale-irregular pre-*abb*-gasket as a graph $\tilde{H}'_\infty$ is defined to be the inductive limit of two copies of $\tilde{H}_{-n}$ joined at origin O. $\tilde{H}'_\infty$ is specified by a sequence of pairs of positive integers $\Sigma = ((a_0,b_0),(a_{-1},b_{-1}),(a_{-2},b_{-2}),\cdots)$. Denote the vertex set of $\tilde{H}'_\infty$ by $G_0(\Sigma)$.

Given a sequence of pairs of positive integers $\{(a_n,b_n),\ n\in\mathbf{Z}\}$, put

$$S_N = ((a_N,b_N),(a_{N-1},b_{N-1}),(a_{N-2},b_{N-2}),\cdots),$$

and define $G_N = G_0(S_N)$ for $N=1,2,3,\cdots$. G_N has a graph structure inherited from $\tilde{H}'_\infty$, which we denote by H_N and call the scale-irregular pre-*abb*-gasket of scale N. For $x \in G_N$ we call $y \in G_N$ an N-neighbor of x if $\{x,y\}$ is an edge of the graph. If $N \le N'$, there is an injection from G_N to $G_{N'}$ (see (A.9)), with which we identify G_N to a subset of $G_{N'}$. An intuitive meaning of this injection is that G_N is obtained from G_{N-1} by adding a substructure specified by (a_N,b_N). We can define a metric d on $G_\infty \stackrel{\text{def}}{=} \bigcup_{N=1}^{\infty} G_N$ such that $d(x,y) = \prod_{k=1}^{N} \min\{a_k+1,b_k+1\}^{-1}$ if x and y are N-neighbors (see (A.8)). We define the scale-irregular *abb*-gasket, which we denote by G, as the completion of G_∞ by the metric d.

If a_N and b_N are independent of N, the corresponding fractal is an *abc*-gasket (with $b=c$) of [17]. The Sierpiński gasket is the scale-irregular *abb*-gasket with $a_N=b_N=1$, $N\in\mathbf{Z}$. Inspired by the case of the Sierpiński gasket, we use the terminology 'horizontal' (edges) and a 'unit triangle' of scale-irregular pre-*abb*-gaskets, in the following. We give a precise definition of the scale-irregular *abb*-gasket, together with the definition of these terms, in Appendix A.

For a process X taking values in G we define $T_{n,i}(X)$, $n\in\mathbf{Z}$, by $T_{n,0}(X) = \inf\{t\ge 0 \mid X(t)\in G_n\}$, and

$$T_{n,i+1}(X) = \inf\{t > T_{n,i}(X) \mid X(t) \in G_n \setminus \{X(T_{n,i}(X))\}\}\ ,\ i=0,1,2,\cdots.$$

For an integer n and a Markov process X on G or on G_N for some $N\ge n$, we call a random walk $X^{(n)}$ on G_n defined by $X^{(n)}(i) = X(T_{n,i}(X))$ the n-decimated walk of X. By definition,

Proposition 1.1. *If $n<N$ and $X^{(n)}$ and $X^{(N)}$ are the n and N-decimated walks of X respectively, then $X^{(n)}$ is the n-decimated walk of $X^{(N)}$.*

For $N\in\mathbf{Z}$ and $w>0$, we define a simple random walk $X_{N,w}$ on G_N as follows. At each integer time, the random walker jumps to one of the four N-neighbors, and the relative rates of the jumps are 1 for a jump in horizontal direction and w in the other directions. We prove in this paper the following.

Theorem 1.2. *Assume that* $\{(a_N, b_N),\ N \in \mathbf{Z}\}$ *is a bounded sequence of pairs of integers satisfying*

$$a_N \geq 2,\ b_N \geq 2,\ b_N < 2a_N, \quad N \in \mathbf{Z}, \tag{1.1}$$

and let G *be the scale-irregular abb-gasket defined by this sequence. Then there exist a sequence of positive numbers* w_N, $N \in \mathbf{Z}$ *satisfying*

$$\lim_{N\to\infty} w_N = 0, \tag{1.2}$$

and a symmetric Feller diffusion process X *with a measure* μ *on* G *defined by* $\int f\, d\mu = \lim_{N\to\infty} \left(\prod_{k=0}^N (a_k + 2b_k)^{-1}\right) \sum_{x\in G_N} f(x)$, *such that for* $N \in \mathbf{Z}$, *the* N*-decimated walk of* X *is equal in law to the random walk* X_{N,w_N}.

The assumptions (1.1) are to avoid complications. We will prove Theorem 1.2 for any w_0 satisfying

$$w_0 \in I \stackrel{\text{def}}{=} (0, \inf_N \{2a_N/b_N\} - 1). \tag{1.3}$$

Proposition 1.1 and Theorem 1.2 imply that the $(N-1)$-decimated walk of X_{N,w_N} is equal in law to $X_{N-1,w_{N-1}}$, from which it follows that the sequence $\{w_N\}$ satisfies a recursion relation

$$w_{N-1} = f_{(a_N,b_N)}(w_N)\,,\ N \in \mathbf{Z}, \tag{1.4}$$

where

$$f_{(a,b)}(w) = \frac{2w\{(1+a)b + (a\,b + a + b)w\}}{b(b+2) + 2(b^2 + a + b)w + b^2 w^2}\,. \tag{1.5}$$

Proof of (1.4) is elementary (but lengthy), and is similar to that of [16, Proposition 1.1]. It is elementary to see that

Proposition 1.3. *If (1.1) and (1.3) are satisfied, there exists one and only one sequence* $\{w_N\}$ *which satisfies (1.4) and which is in the open interval* I. *Moreover,* $\{w_N\}$ *is strictly decreasing and satisfies (1.2).* $\lim_{n\to\infty} \dfrac{w_{n+s}}{w_n} \prod_{k=1}^{s} \delta_{n+k} = 1$ *uniformly in* $s \in \mathbf{Z}_+$, *where* $\delta_k \stackrel{\text{def}}{=} \dfrac{2(1+a_k)}{2+b_k} > 1$. *If* $b_N = a_N$, $N \in \mathbf{Z}$, *it also holds that* $I = (0,1)$ *and* $\lim_{N\to-\infty} w_N = 1$.

The ratio of the rate for a off-horizontal to horizontal jump of X_{N,w_N} is w_N, hence (1.2) means that on small scales the process favors horizontal moves, while $w_N > 0$ means that the process span the whole fractal space and is not confined in a line. If $b_N = a_N$, $N \in \mathbf{Z}$, we have $\lim_{N\to-\infty} w_N = 1$, which implies that isotropy is asymptotically restored.

The fractals may be regarded to have 'obstacles' or holes in the space, when compared to uniform Euclidean spaces. Intuitively, a random walker that favors horizontal motion performs a one-dimensional random walk between a pair of obstacles, and eventually is forced to move in off-horizontal direction before they could move further horizontally. There are obstacles of various scales (sizes), separated by distances of the same order as their scales, hence globally, the random walker is scattered almost isotropically [4]. This phenomena is absent on regular spaces such as Euclidean spaces. The intuition implicitly guided the studies in [16, 17], but in spite of the generality in the intuition, it was not clear how to obtain such diffusions for fractals which lack exact self-similarity. Also the statements for the diffusion in that work were not referring to the properties which explicitly embodied the picture. It is the purpose of this paper to report some positive answers (Theorem 1.2) to these points.

We construct the diffusion as a weak limit of $X_{N,w_N}([L_N t])$, $N \in \mathbf{Z}$, for a time-scaling constant L_N. A key step in the construction is an asymptotic estimate of number of steps of X_{N,w_N}, whose expectation value is L_N. We apply a limit theorem for discrete-time multi-type supercritical branching processes with singular and irregular environment. We need multi-type branching processes because the horizontal and off-horizontal jumps have different transition probabilities. Branching rates change with the generation N (irregular environment) because the substructure of pre-gaskets varies with its scale N. Environment varies also because the transition probabilities of the random walk X_{N,w_N} vary with N. In particular, birth rates of types corresponding to the numbers of off-horizontal jumps approach 0 as $N \to \infty$ (singular environment). Compared with existing related results, there are two major complications arising from these requirements; criterion for supercriticality, and scaling factor for total number of descendants. For a construction of spatially symmetric diffusion on an exact self-similar finitely ramified fractal [22, 6, 24, 21], the expectation of off-spring for the associated branching process is a constant matrix independent of generation N. For a construction of asymptotically one-dimensional diffusion on an exact self-similar finitely ramified fractal [16, 18], the off-spring expectation matrix has a limit as $N \to \infty$. In these cases, the largest eigenvalue of the (limit of) off-spring expectation matrix gives the (asymptotic) growth rate of descendant expectations and governs supercriticality. A pioneering work for scale-irregular fractals by Hambly [10] deals with spatially symmetric diffusions on fractals called $HSG(\bar{\nu})$ (which have much in common with scale-irregular abb-gaskets with $b_N = a_N$, $N \in \mathbf{Z}$, as far as construction of diffusions are concerned). Due to the spatial symmetry, the associated branching process is of one-type, hence the off-spring expectation is one-dimensional, which gives the growth rate. In the present study, off-spring expectation is a multi-dimensional matrix, and neither is constant nor has a limit. Thus a criterion for supercriticality cannot be given in terms of growth rates. Furthermore, the ratios of expectations of the population between different types are unbounded, which obscures at first site, the existence of scaling factor for total descendant numbers.

Our approach is partly inspired by a study on multi-type branching processes in random environment by Cohn. Much of our proof of Proposition 2.1 follows the idea in [8]. In that work, a probability measure on environments is considered, and the assumptions on stationarity and ergodicity implicitly assured the last two assumptions in Proposition 2.1 (including supercriticality) to hold. The assumptions are not suitable for our purpose to consider singular environment, where some of the branching rates vanish in the limit. To formulate a sufficient condition of supercriticality in Theorem 2.2, we introduce a recursion relation in Appendix B which reflects recursive nature of branching processes. We apply this recursion relation also to prove continuity of limit distribution in Theorem 2.5. The idea of using recursion relation to prove continuity originally appeared in [16, Lemma 2.7], which we refine to handle irregular environments. It turns out in Proposition 2.4 that the existence of scaling factor for total descendant follows from a fact that the distribution of normalized population converge to a limit independent of types. Consideration on the branching processes may be interesting in its own respect, so we will discuss this in Section 2 independently of other sections.

To apply the general theory of branching processes to the diffusion, we consider in Section 3 estimates for generating functions. An algebraic part of our proof of estimates (Proposition 3.1) is computer-aided because it requires a routine work of rather lengthy calculations. We use these basic estimates to obtain estimates for number of steps of X_{N,w_N}, to which one can apply [16, Sect. 3].

Note added in Nov. 1995. While the present paper was being refereed, some new works on related subjects have appeared. There is now an alternative and a quite general convergence results for multitype branching processes (partly motivated by the present paper), which are very nicely applicable to the construction of the same diffusion [19]. An alternative and more general construction of the asymptotically one-dimensional (lower dimensional) diffusion on a subclass of nested fractals, together with some detailed studies such as the asymptotic estimates of the t dependence of the (diagonal) heat kernels $p_t(x,x)$ and the homogenization problems also appeared [13]. Estimates of the x, y dependence of $p_t(x,y)$ require tail structures of limit distribution of the branching processes, which we hear is now in progress by Hambly and Jones [12]. A characterization of asymptotically one-dimensional diffusions on the Sierpiński gasket by the exit distributions is given in [27]. Our standpoint is developed further in [5] where we deal with the Sierpiński carpet.

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2 Branching process with singular and irregular environment.

Let $d \ge 2$ be an integer and put $\mathcal{E} \stackrel{\text{def}}{=} \{1, 2, 3, \cdots, d\}$. Consider a discrete time d-type branching process $\vec{Z}_N = (Z_{N,j},\ j \in \mathcal{E})$, $N \in \mathbf{Z}_+$. Given $n \in \mathbf{Z}_+$ and $i \in \mathcal{E}$, let

$$\vec{Z}_{n,N,i} \stackrel{\text{def}}{=} (Z_{n,N,i,j}\ ,\quad j \in \mathcal{E}), \qquad N = n, n+1, \cdots,$$

be random vectors which give the number of descendant at time N from a single ancestor of type i at time n. We have, for $r \in \mathbf{Z}_+$ and $j \in \mathcal{E}$,

$$Z_{n+r,j} = \sum_{i \in \mathcal{E}} \sum_{u=1}^{Z_{n,i}} Z_{n,n+r,i,j,u}\ ,$$

where $(Z_{n,n+r,i,j,u}\ ,\ j \in \mathcal{E})$, $u \in \mathbf{Z}_+$, are i.i.d. copies of $\vec{Z}_{n,n+r,i}$ when conditioned on $Z_{n,i}$. Let $\{e_n\}$ be a sequence of non-negative real numbers.

Proposition 2.1. *Assume that following three conditions hold for each $i \in \mathcal{E}$.*

(1) Uniform estimates for second moments of $W_{nNij} \stackrel{\text{def}}{=} \dfrac{Z_{nNij}}{\mathrm{E}[\ Z_{nNij}\ ]}$;

$$v \stackrel{\text{def}}{=} \sup_{n \in \mathbf{Z}_+} \sup_{N \ge n+n_0} e_n\, \mathrm{E}[\ W_{nNij}^2\ ] < \infty, \quad j \in \mathcal{E}\ ,$$

$$\lim_{p \to \infty} \sup_{N \ge n+n_0} \mathrm{E}[\ W_{nNij}^2;\ W_{nNij} > p\ ] = 0, \quad j \in \mathcal{E}, n \in \mathbf{Z}_+\ ,$$

for some constant $n_0 \in \mathbf{Z}_+$.

(2) For each $n \in \mathbf{Z}_+$, $\gamma_{ni} \stackrel{\text{def}}{=} \lim_{N \to \infty} \dfrac{\mathrm{E}[\ Z_{nNij}\ ]}{\mathrm{E}[\ Z_{nN1j}\ ]} > 0$ exist, positive and independent of $j \in \mathcal{E}$.

(3) $\lim_{N \to \infty} \mathrm{Prob}[\ e_N\, Z_{Nj} \ge p\ ] = 1$, $j \in \mathcal{E}$, $p > 0$.

Then the sequence of normalized random vectors $(Z_{Nj}/\mathrm{E}[\ Z_{Nj}\ ],\ j \in \mathcal{E})$ *converges in* L_2 *as* $N \to \infty$ *to a random vector* $(W, W, \cdots, W)$ *with* $\mathrm{E}[\ W\ ] = 1$.

Proofs of the results in this section are postponed to the end of the section. Generalization to include e_N is for our application in Section 3. A simple sufficient condition for the existence of γ_{ni} is given in Appendix C, in terms of off-spring expectation matrices A_N ($A_{Nij} = \mathrm{E}[\ Z_{N-1,N,i,j}\ ]$). The last assumption states supercriticality. One of our main concern here is a useful condition for the last assumption to hold.

Definition. *A family of sequences of pairs of reals* $\{(x_{k,n}, y_{k,n}),\ n = 0, 1, \cdots, k\}$, $k \in \mathbf{Z}_+$, *is said to satisfy the assumption R, if there exist sequences of non-negative numbers* $\{a_n\}$, $\{w_n\}$, $\{w'_n\}$, $n \in \mathbf{Z}_+$, *satisfying* $2 \le \inf_n a_n$, $\sup_n a_n < \infty$, *and* $\max\{w_n, w'_n\} \le \min\{1,\ C_w \delta^{-n}\}$, $n \in \mathbf{Z}_+$, *for constants* $C_w > 0$ *and* $\delta > 1$, *such that*

$$\begin{aligned} x_{k,n} &\le x_{k,n+1}^{a_{n+1}} + w_{n+1} \min\{1 - x_{k,n+1}^{a_{n+1}}\ ,\ y_{k,n+1}\}\,, \\ y_{k,n} &\le x_{k,n+1} + w'_{n+1}\, y_{k,n+1}\,, \qquad 0 \le n \le k\,, \end{aligned}$$

hold for all $k \in \mathbf{Z}_+$. *Similarly,* $\{(x_{k,n}, y_{k,n}),\ n \in \mathbf{Z}_+\}$, $k \in \mathbf{Z}_+$, *is said to satisfy the assumption R, if similar relation hold for* $n \in \mathbf{Z}_+$ *and* $k \in \mathbf{Z}_+$.

Theorem 2.2. *Assume that for some* $j_0 \in \mathcal{E}$, $Z_{0,j} = 1$, $j = j_0$, *and* $Z_{0,j} = 0$, *otherwise. Let* $p > 0$ *and* $j \in \mathcal{E}$. *Suppose that there exists an integer* n_0 *and a non-empty subset* $\mathcal{E}' \subset \mathcal{E}$, *not equal to* $\mathcal{E}$, *such that the family of sequences* $\{(x_{k,n}, y_{k,n})\}$ *defined by*

$$\begin{aligned} x_{k,n} &= \max_{i \in \mathcal{E}'} \mathrm{Prob}[\ e_{n_0+k}\, Z_{n,n_0+k,i,j} < p\ ]\,, \\ y_{k,n} &= \max_{i \in \mathcal{E}\setminus\mathcal{E}'} \mathrm{Prob}[\ e_{n_0+k}\, Z_{n,n_0+k,i,j} < p\ ]\,, \qquad 0 \le n \le k,\ \ k \in \mathbf{Z}_+\,, \end{aligned}$$

satisfies the assumption R, and

$$\liminf_{k \to \infty,\ x_{k,k} \ne 0} \left\{ (-\log x_{k,k}) \prod_{\ell=0}^{k} a_\ell \right\}^{1/k} > 1\,. \tag{2.1}$$

Then

$$\lim_{N \to \infty} \mathrm{Prob}[\ e_N\, Z_{N,j} \ge p\ ] = 1\,.$$

The assumption (2.1) is an 'a priori estimate' that $\mathrm{Prob}[\ e_{n_0+k} Z_{k,n_0+k,i,j} < p\ ]$ is not too large. The Theorem then says that it is in fact small. Let us call the types $j \in \mathcal{E}'$ the dominant types, and the types $j \notin \mathcal{E}'$ the recessive types. The assumption R reflects the recursive nature of branching process. It is satisfied when the probability that recessive types appear in the off-springs of a parent at generation n vanishes exponentially as $n \to \infty$, and if recessive type do not appear in the off-springs, then at least one dominant type off-spring appears from a recessive parent, while at least a_{n+1} (no less than 2 but bounded) dominant off-springs appear from a dominant parent. $\{w_n\}$ represents singular environment, while n-dependence of $\{a_n\}$ implies irregular environment. Though the branching rate to recessive types vanish in the limit, the recessive types may contribute significantly to the growth of dominant types, because a recessive parent may give birth to exponentially many dominant type off-springs. Thus in general, we can not discard the recessive types from consideration for limit theorems. Note also that supercriticality is non-trivial because of the recessive types. Assumption on initial condition $Z_{0,j}$ is chosen to be simple, to avoid complications.

The following is useful in obtaining an a priori estimate of type (2.1) from moments of Z_{nNij}.

Proposition 2.3. *Let $p \in \mathbf{R}$ and X a real valued random variable. If* $\mathrm{E}[\ X\ ] > p$ *then* $\mathrm{Prob}[\ X \le p\ ] \le 1 - d + \sqrt{d^2 - 1}$, *where* $d = 1 + 2^{-1}\,\mathrm{V}[\ X\ ]\,(\mathrm{E}[\ X\ ] - p)^{-2}$.

The next statement is on the existence of norming factor for total descendant numbers.

Proposition 2.4. *Assume that the sequence* $(Z_{N,j}/\mathrm{E}[\ Z_{N,j}\ ],\ j \in \mathcal{E})$, $N \in \mathbf{Z}_+$, *converges in probability as* $N \to \infty$ *to a random vector* $(W, W, \cdots, W)$. *Then* $\dfrac{\sum_{j\in\mathcal{E}} Z_{N,j}}{\sum_{j\in\mathcal{E}} \mathrm{E}[\ Z_{N,j}\ ]}$ *converges in probability as* $N \to \infty$ *to* W.

We complete our list of the results with a sufficient condition for the continuity of the limit distribution, stated in terms of the assumption R. Let $W_{n,i}$, $n \in \mathbf{Z}_+$, $i \in \mathcal{E}$, be real valued random vectors, and let

$$\Phi_{n,i}(t) \overset{\mathrm{def}}{=} \mathrm{E}[\ \exp\left(\sqrt{-1}\, t\, W_{n,i}\right)\ ]\,,$$

denote the characteristic function. We assume an 'a priori' estimate of the form

$$|\Phi_{n,i}(t)| \le 1 - C_n\, t^2\,,\quad -t_n < t < t_n\,,\ n \ge n_0\,,\ i \in \mathcal{E}', \tag{2.2}$$

for some non-empty subset $\mathcal{E}' \subset \mathcal{E}$ not equal to $\mathcal{E}$, an integer n_0, and positive reals C_n and t_n.

Theorem 2.5. *Assume that* $\{t_k,\ k \in \mathbf{Z}_+\}$ *in (2.2) diverges to infinity as* $k \to \infty$ *exponentially fast at most* ($\lim_{k\to\infty} t_k = \infty$ *and* $\limsup_{k\to\infty} t_k^{1/k} < \infty$), *and satisfies*

$$\theta \overset{\mathrm{def}}{=} \inf_{k;\ \exists t_j < t_k}\ \sup_{j;\ t_j < t_k} \frac{t_j}{2\, t_k} > 0\,.$$

If for any sequence of reals $\{s_k,\ k \in \mathbf{Z}_+\}$ *the family of sequences* $\{(\tilde{x}_{k,n}, \tilde{y}_{k,n}),$ $n \in \mathbf{Z}_+\}$*,* $k \in \mathbf{Z}_+$*, defined by*

$$\begin{aligned} \tilde{x}_{k,n} &= \max_{i \in \mathcal{E}'} |\Phi_{n_0+n,i}(s_k)| \\ \tilde{y}_{k,n} &= \max_{i \in \mathcal{E} \setminus \mathcal{E}'} |\Phi_{n_0+n,i}(s_k)| \ , \quad n \in \mathbf{Z}_+ \ , \ k \in \mathbf{Z}_+ \ , \end{aligned}$$

satisfies the assumption R, and if

$$\liminf_{k \to \infty} \left(t_k^2 \, C_k \prod_{\ell=0}^{k-n_0} a_\ell \right)^{1/k} > 1 \, ,$$

holds with a_ℓ *as in the assumption R, then the distribution of* $W_{n,i}$ *is continuous for all* $n \in \mathbf{Z}_+$ *and* $i \in \mathcal{E}$*.*

The intuition for the assumptions is similar to those for Theorem 2.2, with $W_{n,i}$ being a weak limit of $\dfrac{\sum_j Z_{nNij}}{\sum_{k,j} \mathrm{E}[\ Z_{0Nkj}\ ]}$.

The rest of this section is devoted to the proofs of the stated results.

Proof of Proposition 2.1. We follow [8] and prove first that $Z_{N,j}/\mathrm{E}[\ Z_{N,j}\ ]$ converges weakly. Since $W_{nNij} \geq 0$ and $\mathrm{E}[\ W_{nNij}\ ] = 1$, the family of random variables $\{W_{nNij}\ ,\ N = n, n+1, \cdots\}$ is tight, hence there exists a subsequence of integers $\{k_N\}$ such that $W_{n,k_N,i,j}$ converges weakly as $N \to \infty$ to a random variable $\tilde{W}_{nij}$. By assumption $\{W_{nNij}\}$ is uniformly integrable. Weak convergence and uniform integrability imply convergence of expectations;

$$\mathrm{E}[\ \tilde{W}_{n,i,j}\ ] = \lim_{N \to \infty} \mathrm{E}[\ W_{n,k_N,i,j}\ ] = 1 \, . \tag{2.3}$$

$$\sup_{n \in \mathbf{Z}_+} e_n \, \mathrm{E}[\ \tilde{W}^2_{n,i,j}\ ] \leq v \, . \tag{2.4}$$

Put $W_{n,N,i,j,u} \stackrel{\text{def}}{=} \dfrac{Z_{n,N,i,j,u}}{\mathrm{E}[\ Z_{n,N,i,j,u}\ ]}$. $(W_{n,N,i,j,u}\ ,\ j \in \mathcal{E})$, $u \in \mathbf{Z}_+$, are i.i.d. copies of $(W_{n,N,i,j}\ ,\ j \in \mathcal{E})$ when conditioned on $Z_{n,i}$. Hence $(W_{n,k_N,i,j,u},\ j \in \mathcal{E},$ $u \in \mathbf{Z}_+)$, conditioned on $Z_{n,i}$, converges weakly as $N \to \infty$ to a random vector $(\tilde{W}_{n,i,j,u},\ j \in \mathcal{E},\ u \in \mathbf{Z}_+)$, where $(\tilde{W}_{n,i,j,u}\ ,\ j \in \mathcal{E})$, $u \in \mathbf{Z}_+$, are i.i.d. copies of $(\tilde{W}_{n,i,j}\ ,\ j \in \mathcal{E})$ when conditioned on $Z_{n,i}$. Hence (2.3) and (2.4) imply, for positive integer p,

$$\begin{aligned} &\mathrm{E}[\ \left(Z_{ni}^{-1} \sum_{u=1}^{Z_{ni}} (\tilde{W}_{niju} - 1) \right)^2 ;\ Z_{ni} \geq p\ ] = \sum_{q=p}^{\infty} q^{-1} \mathrm{E}[\ (\tilde{W}_{nij1} - 1)^2 ;\ Z_{ni} = q\ ] \\ &\leq (v/e_n + 1)/p \, . \end{aligned}$$

This implies, with the assumption on supercriticality and Chebyshev's inequality, that for $\eta > 0$ and $\epsilon > 0$, there exists n_0 such that if $n \geq n_0$ then

$$\mathrm{Prob}[\ \left| Z_{ni}^{-1} \sum_{u=1}^{Z_{ni}} (\tilde{W}_{niju} - 1) \right| \geq \epsilon\ ]$$
$$\leq \mathrm{Prob}[\ \left| Z_{ni}^{-1} \sum_{u=1}^{Z_{ni}} (\tilde{W}_{niju} - 1) \right| \geq \epsilon\,,\ Z_{ni} \geq \frac{2(v + e_n)}{\eta\, \epsilon^2\, e_n}\ ] + \eta/2\ \leq\ \eta\,.$$

Convergence in probability follows;

$$\lim_{n\to\infty} \mathrm{Prob}[\ \left| Z_{ni}^{-1} \sum_{u=1}^{Z_{ni}} \tilde{W}_{niju} - 1 \right| \geq \epsilon\ ] = 0\,,\quad \epsilon > 0,\ i \in \mathcal{E}. \tag{2.5}$$

The second assumption in the statement and a property of branching process

$$\mathrm{E}[\ Z_{N,j}\ ] = \sum_{k\in\mathcal{E}} \mathrm{E}[\ Z_{n,k}\ ]\,\mathrm{E}[\ Z_{n,N,k,j}\ ] \tag{2.6}$$

implies that the limit

$$\beta_{n,i} \stackrel{\mathrm{def}}{=} \lim_{N\to\infty} \frac{\mathrm{E}[\ Z_{n,k_N,i,j}\ ]}{\mathrm{E}[\ Z_{k_N,j}\ ]} = \frac{\gamma_{n,i}}{\sum_k \mathrm{E}[\ Z_{n,k}\ ]\,\gamma_{n,k}}\,,$$

exists, positive, independent of j, and satisfies $\sum_{i\in\mathcal{E}} \beta_{ni}\,\mathrm{E}[\ Z_{ni}\ ] = 1$. This, with (2.5) and the non-negativity of Z and $\tilde{W}$ implies convergence in probability,

$$\lim_{n\to\infty} \mathrm{Prob}[\ \left| \sum_{i\in\mathcal{E}} \left(\beta_{ni} \sum_{u=1}^{Z_{ni}} \tilde{W}_{niju} \right) - \sum_{i\in\mathcal{E}} \beta_{ni}\, Z_{ni} \right| > \epsilon\ ] = 0\,,\ \epsilon > 0. \tag{2.7}$$

Note that in (2.7) everything except possibly $\tilde{W}_{niju}$ is independent of the choice of subsequence $\{k_N\}$.

Put

$$\xi_n(x) \stackrel{\mathrm{def}}{=} \lim_{N\to\infty} \mathrm{Prob}[\ Z_{k_N,j}/\mathrm{E}[\ Z_{k_N,j}\ ] \leq x_j\,,\ j \in \mathcal{E} \mid \vec{Z}_n\ ]\,.$$

$\xi_n(x)$ is a bounded martingale, hence converges as $n \to \infty$ to a random vector $\xi(x)$ a.s. The definitions of $\tilde{W}_{niju}$ and β_{ni} with (2.6) imply

$$\xi_n(x) = \mathrm{Prob}[\ \sum_{i\in\mathcal{E}} \left(\beta_{ni} \sum_{u=1}^{Z_{ni}} \tilde{W}_{niju} \right) \leq x_j\,,\ j \in \mathcal{E} \mid \vec{Z}_n\ ]\,,$$

on set of continuity points. This with (2.7) implies that $\xi(x)$ is independent of the choice of subsequence $\{k_N\}$. In particular,

$$\mathrm{E}[\ \xi(x)\ ] = \lim_{N\to\infty} \mathrm{Prob}[\ Z_{k_N,j}/\mathrm{E}[\ Z_{k_N,j}\ ] \leq x_j\,,\ j \in \mathcal{E}\ ]$$

is independent of the subsequence, hence $\{Z_{N,j}/\mathrm{E}[\, Z_{N,j}\,],\ j \in \mathcal{E}\}$ converges weakly to a random vector with distribution function $\mathrm{E}[\, \xi(x)\,]$. Furthermore, (2.7) implies that this random vector has equal components. Convergence in probability, and then in L_2, is now proved exactly as in [8, step 4]. $\square$

Proof of Theorem 2.2. By definition $0 \le x_{k,n} \le 1$ and $0 \le y_{k,n} \le 1$ for all n and k. With the assumption R and (2.1), we see that $\{(x_{k,n}, y_{k,n})\}$ satisfies all the assumption of Theorem B.2. Theorem B.2 implies $\lim_{k\to\infty} \max\{x_{k,0},\ y_{k,0}\} = 0$, which gives $\lim_{N\to\infty} \mathrm{Prob}[\, e_N Z_{0Nij} \ge p\,] = 1$. $\square$

Proof of Proposition 2.3. Put $Y = X - \mathrm{E}[\, X\,]$, $v = \mathrm{V}[\, X\,] = \mathrm{V}[\, Y\,]$, $b = \mathrm{E}[\, X\,] - p > 0$, and $t = \mathrm{Prob}[\, Y > -b\,] = \mathrm{Prob}[\, X > p\,]$. $0 = \mathrm{E}[\, Y\,] \le \mathrm{E}[\, Y;\ Y \le -b\,] + \mathrm{E}[\, Y;\ Y > 0\,]$ implies $\mathrm{E}[\, Y;\ Y > 0\,] \ge b\,(1-t)$. Using Schwarz inequality we have

$$t\,v \ge \mathrm{Prob}[\, Y > 0\,]\,\mathrm{E}[\, Y^2;\ Y > 0\,] \ge (\mathrm{E}[\, Y;\ Y > 0\,])^2 \ge b^2\,(1-t)^2\,.$$

The statement follows by solving this algebraic inequality in t. $\square$

Proof of Proposition 2.4.

$$\begin{aligned}
&\lim_{N\to\infty} \mathrm{E}[\ \min\{1,\ \left|\frac{\sum_{j\in\mathcal{E}} Z_{N,j}}{\sum_{j\in\mathcal{E}} \mathrm{E}[\, Z_{N,j}\,]} - W\right|\}\ ] \\
&\le \lim_{N\to\infty} \mathrm{E}[\ \sum_j \frac{\mathrm{E}[\, Z_{N,j}\,]}{\sum_k \mathrm{E}[\, Z_{N,k}\,]} \min\{1,\ \left|\frac{Z_{N,j}}{\mathrm{E}[\, Z_{N,j}\,]} - W\right|\}\ ] \\
&\le \sum_{j\in\mathcal{E}} \lim_{N\to\infty} \mathrm{E}[\ \min\{1,\ \left|\frac{Z_{N,j}}{\mathrm{E}[\, Z_{N,j}\,]} - W\right|\}\ ] = 0\,.
\end{aligned}$$

The assumption implies the last equality. Hence the statement follows. $\square$

Proof of Theorem 2.5. Note that the definition and assumption on θ imply $0 < \theta < 1/2$. Let $n_1 \ge n_0$, and let $\{s_k,\ k \in \mathbf{Z}_+\}$ be a sequence of reals satisfying

$$\theta\, t_{n_1+k} \le |s_k| \le t_{n_1+k}\,,\quad k \in \mathbf{Z}_+\,. \tag{2.8}$$

Put

$$x_{k,n} = \max_{i\in\mathcal{E}'} |\Phi_{n_1+n,i}(s_k)|$$
$$y_{k,n} = \max_{i\in\mathcal{E}\setminus\mathcal{E}'} |\Phi_{n_1+n,i}(s_k)| , \quad 0 \le n \le k,\ k \in \mathbf{Z}_+ .$$

The assumption R implies that $\{(x_{k,n}, y_{k,n})\}$ satisfies the recursion relation in the assumption of Theorem B.2. with $\{w_{n_1+n-n_0}\}$ and $\{a_{n_1+n-n_0}\}$ in place of $\{w_n\}$ and $\{a_n\}$. Also (2.8) and the assumption on a priori estimate (2.1) imply

$$\liminf_{k\to\infty} \left\{ -\log x_{k,k} \prod_{\ell=0}^{k} a_{n_1+\ell-n_0} \right\}^{\frac{1}{k}} \ge \liminf_{k\to\infty} \left\{ \theta^2 C_k t_k^2 \prod_{\ell=n_1-n_0}^{k-n_0} a_\ell \right\}^{\frac{1}{k-n_1}} > 1.$$

We see that $\{(x_{k,n}, y_{k,n})\}$ satisfies all the assumption of Theorem B.2, hence there exist positive constants C_1 and C_2 (which may depend on n_1 but not on k) such that

$$|\Phi_{n_1,i}(s_k)| \le C_1 \exp(-C_2\, k^2) , \quad k \in \mathbf{Z}_+ ,\ i \in \mathcal{E} . \tag{2.9}$$

The assumptions $\lim_k t_k = \infty$ and $\limsup_{k\to\infty} t_k^{1/k} < \infty$ imply that there exist constants $C_3 > 0$ and $C_4 > 1$ such that $t_{n_1+k} < C_3\, C_4^k$, $k \in \mathbf{Z}_+$. This and (2.8) and (2.9) imply

$$|\Phi_{n_1,i}(s_k)| \le C_5 \exp(-C_6\, (\log |s_k|)^2) ,$$

with positive constants C_5 and C_6 independent of i, k, and s_k. Note that s_k is an arbitrary number satisfying (2.8). Note also that for any $t \in \mathbf{R}$ satisfying $|t| > \min_k t_{n_1+k}$, there exists $j \in \mathbf{Z}_+$ satisfying

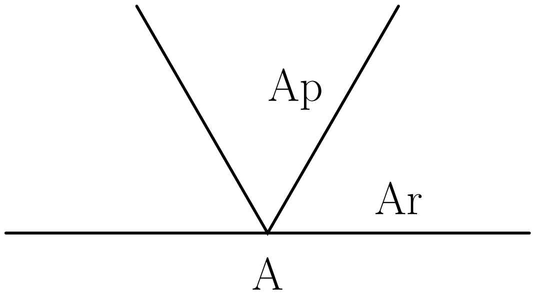
$$\theta\, t_{n_1+j} < |t| < t_{n_1+j} . \tag{2.10}$$

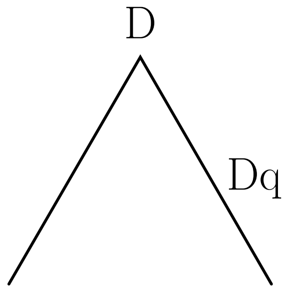
In fact, let $j = \min\{k \mid t_{n_1+k} > |t|\} - n_1$. (The assumption $\lim_{k\to\infty} t_k = \infty$ implies that the minimum exists.) The definition of θ implies that there exists $j' \in \mathbf{Z}_+$ such that $t_{n_1+j} > t_{n_1+j'} > \theta\, t_{n_1+j}$. Hence (2.10) follows.

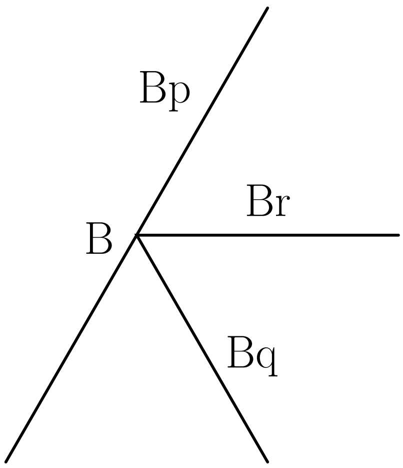
Therefore,

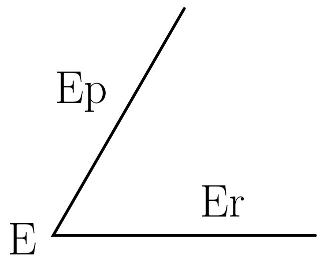
$$|\Phi_{n_1,i}(t)| \le C_5 \exp(-C_6\, (\log |t|)^2) , \tag{2.11}$$

for sufficiently large $|t|$. This implies $\Phi_{n,i} \in L_1(\mathbf{R})$, $n \ge n_0$, $i \in \mathcal{E}$. By the assumption of recursion relation, it follows inductively that $\Phi_{n,i} \in L_1(\mathbf{R})$ for any $n \in \mathbf{Z}_+$, which implies that $W_{n,i}$ is continuous. □


Ap
Ar
A


D
Dq


Bp
Br
B
Bq


Ep
Er
E

3 Convergence of path measures.

Consider a pre-gasket H_N and its vertices G_N. One needs to consider 4 types of vertices A, B, D, E, and 8 types of edges (as ordered pair of vertices) Ap, Ar, Bp, Bq, Br, Dq, Ep, Er, as in Figure 2 (see Appendix A for definitions). We put

$$\mathcal{E} \stackrel{\rm def}{=} \{Ap,\ Ar,\ Bp,\ Bq,\ Br,\ Dq,\ Ep,\ Er\}\,.$$

Let (a,b) be a pair of positive integers, and consider the case that H_N-substructure of the pre-gasket H_{N-1} is parametrized by (a,b): In the notation of Section 1 and Appendix A, $(a_N,b_N)=(a,b)$. Let $\tilde{\Omega}(a,b,i)$ be the set of walks on G_N whose starting point X and stopping point form an edge of type $i \in \mathcal{E}$ in H_{N-1}, and such that do not pass through points in $G_{N-1} \setminus \{X\}$:

$$\begin{aligned}\tilde{\Omega}(a,b,i) = \{\tilde{\omega} = (\tilde{\omega}(0),\cdots,\tilde{\omega}(L)) \subset G_N \text{ for some } L \mid (\tilde{\omega}(0),\tilde{\omega}(L)) \text{ is type } i,\\ \tilde{\omega}(k) \notin G_{N-1}\setminus\{\tilde{\omega}(0)\},\ \overline{\tilde{\omega}(k)\tilde{\omega}(k+1)} \in H_N,\ k=0,\cdots,L-1\}\,.\end{aligned}$$

For $i \in \mathcal{E}$ and $\tilde{\omega} \in \tilde{\Omega}(a,b,i)$, let $L_i(\tilde{\omega})$ be the number of steps in $\tilde{\omega}$ (ordered pairs of the form $(\tilde{\omega}(j),\tilde{\omega}(j+1))$) which are of type i. Define

$$F_i(a,b;u) \stackrel{\rm def}{=} \sum_{\tilde{\omega}\in\tilde{\Omega}(a,b,i)} \prod_{j\in\mathcal{E}} {u_j}^{L_j(\tilde{\omega})}\,, \quad u \in \mathbf{C}^{\mathcal{E}}\,.$$

Note that, by definition, there is no N-dependence in F_i. F_i is a generating function of number of steps of walks, hence is a rational function of u.

Let $\Pi(w) = {}^t(\Pi_{Ap}(w),\cdots,\Pi_{Er}(w))$ be as in Table 1. The random walk

Table 1: Transition probabilities

i	Ap	Ar	Bp	Bq	Br	Dq	Ep	Er
$\Pi_i(w)$	$\frac{w}{2+2w}$	$\frac{1}{2+2w}$	$\frac{w}{1+3w}$	$\frac{w}{1+3w}$	$\frac{1}{1+3w}$	$\frac{1}{2}$	$\frac{w}{1+w}$	$\frac{1}{1+w}$

X_{N,w_N} on G_N defined in Section 1 is specified by a positive number w_N, defined in (1.3) and (1.4). It is easy to see from Figure 2 that the (one-step) jump probability of X_{N,w_N} for a jump of type i is $\Pi_i(w_N)$. The definitions of Π and F together with (1.4) imply

$$\Pi(f_{(a,b)}(w)) = F(a,b;\Pi(w))\,. \tag{3.1}$$

Define $\sharp\mathcal{E}$-dimensional matrix $A(a,b,w)$ by

$$A(a,b,w)_{ij} \stackrel{\rm def}{=} \frac{\partial F_i}{\partial u_j}(a,b;u=\Pi(w))\,.$$

It turns out that $A(a,b,w)_{Dq,j}$ for $j \neq Ap$, Ar, Dq diverge as $w \to 0$. We therefore define $\sharp\mathcal{E}$-dimensional diagonal matrix $S(w) = \mathrm{diag}(S_i(w),\ i \in \mathcal{E})$ by

$$S_{Dq}(w) = w\,, \quad S_i(w) = 1\,,\ i \neq Dq\,, \tag{3.2}$$

and define rational functions $\tilde{F}_i(a,b,w;u)$, $i \in \mathcal{E}$, $u \in \mathbf{C}^{\mathcal{E}}$, by

$$\tilde{F}_i(a,b,w;u) \stackrel{\mathrm{def}}{=} S_i(f_{(a,b)}(w))\, F_i(a,b;S^{-1}(w)u)\,. \tag{3.3}$$

Also we define a vector $\tilde{\Pi}(w)$ and a matrix $\tilde{A}(a,b,w)$ by

$$\begin{aligned} \tilde{\Pi}(w) &\stackrel{\mathrm{def}}{=} S(w)\,\Pi(w)\,, \\ \tilde{A}(a,b,w)_{ij} &\stackrel{\mathrm{def}}{=} \frac{\partial \tilde{F}_i}{\partial u_j}(a,b,w;u=\tilde{\Pi}(w)) = S_i(f_{(a,b)}(w))\, A(a,b,w)_{ij}\, S_j^{-1}(w)\,. \end{aligned}$$

Proposition 3.1. *Let a, a', b, and b' be integers no less than 2, and let I be the interval defined in (1.3). Then the elements of matrix $\tilde{A}(a,b,w)$ are positive for $w \in I$ and*

$$\sup_{w\in I} \tilde{A}(a,b,w)_{ij} < \infty\,, \quad i \in \mathcal{E},\ j \in \mathcal{E}\,, \tag{3.4}$$

$$\inf_{w,w'\in I} \left(\tilde{A}(a',b',w')\tilde{A}(a,b,w)\right)_{ij} > 0\,, \quad i \in \mathcal{E},\ j \in \mathcal{E}\,, \tag{3.5}$$

$$\tilde{A}(a,b,0)_{Ar,Ar} \geq (a+1)^2\,. \tag{3.6}$$

For each $i \in \mathcal{E}$ put $g_{a,b,i}(w,h) = \tilde{F}_i(a,b,w;\tilde{\Pi}(w)+w\,h)$. Then $g_{a,b,i}$ is a rational function in $h \in \mathbf{C}^{\mathcal{E}}$ and w, analytic at $h=0$ for $w \in I$, and for each $j_1,\cdots,j_4 \in \mathcal{E}$,

$$\sup_{w\in I}\left|\frac{1}{w}\frac{\partial^2 g_{a,b,j_1}}{\partial h_{j_2}\partial h_{j_3}}(w,h=0)\right| < \infty\,, \quad \textit{and} \quad \sup_{w\in I}\left|\frac{1}{w}\frac{\partial^3 g_{a,b,j_1}}{\partial h_{j_2}\partial h_{j_3}\partial h_{j_4}}(w,h=0)\right| < \infty\,.$$

Proof. $A(a,b,w)$ has non-negative elements because it is an expectation matrix for number of steps. Graphical considerations shows that every type j of steps appear with positive probability for any i, hence they are positive. $\tilde{F}$ is related to the generating function for number of steps of random walks (see also Proposition 3.2 below), from which we see that $g_{a,b,i}(w,h)$ are rational functions both in h and w, and analytic at $h=0$. The parameter w is the relative jump rate of the random walk. Therefore for $w \in I$ there are no singularities. The only possible relevant singularities of $\tilde{F}$ are at $w=0$. The estimates in the statement are proved by explicit calculation of $\tilde{F}$ with aid of computer. We give in Appendix D explicit form of $\tilde{A}(a,b,w=0)$ obtained as the first derivatives of $\tilde{F}$ using REDUCE. The explicit formula implies (3.4), (3.5), and (3.6). The estimates on higher derivatives of $\tilde{F}$ at $w=0$ is also obtained using REDUCE. See Appendix D for more information on computer aided proof of this Proposition. □

Note that (3.4) and (3.5) imply

$$\inf_{w\in I}\sum_{k\in\mathcal{E}}\tilde{A}(a,b,w)_{kj}>0,\quad j\in\mathcal{E}\,. \tag{3.7}$$

We go back to the gasket and look into the N-dependence. Assume that $\zeta \stackrel{\text{def}}{=} \{(a_N, b_N),\ N \in \mathbf{Z}\}$ is a bounded sequence of pairs of integers satisfying (1.1), which determines the gasket G. For $N \in \mathbf{Z}$, let $F_N = (F_{N,Ap}, \cdots, F_{N,Er})$ be a $\mathbf{C}^{\mathcal{E}}$-valued function in $\sharp\mathcal{E}\ (= 8)$ variables defined by

$$F_{N,i}(u) \stackrel{\text{def}}{=} F_i(a_N, b_N; u)\,,\ i\in\mathcal{E}. \tag{3.8}$$

Also define a diagonal matrix, $\Pi_N \stackrel{\text{def}}{=} \mathrm{diag}(\Pi(w_N))$. Using F_N and Π_N, we can write the generating functions for the number of steps of X_{N,w_N}. Let $i\in\mathcal{E}$, $j\in\mathcal{E}$, and $n\le N$. Let Z_{nNij} be the random variable which counts the number of steps of type $j\in\mathcal{E}$ between the times $T_{n,1}(X_{N,w_N})$ and $T_{n,0}(X_{N,w_N})$, under the condition that $(X_{N,w_N}(T_{n,0}(X_{N,w_N})), X_{N,w_N}(T_{n,1}(X_{N,w_N})))$ forms an edge of type i in H_n. $T_{n,i}$ is a hitting time of G_n defined in Section 1. By strong Markov property of simple random walks, the distribution of Z_{nNij} is independent of the starting point of X_{N,w_N}, and the random variables which count the number of steps of type $j\in\mathcal{E}$ between the times $T_{n,k+1}(X)$ and $T_{n,k}(X)$, $k=0,1,2,\cdots$, are independent and equal in distribution to Z_{nNij}, under similar conditions. By definition, $Z_{nnij}=1\ (j=i),\ \ =0\ (j\ne i)$.

Proposition 3.2. *Fix $n\in\mathbf{Z}$ and $i\in\mathcal{E}$. $(Z_{nNij}\,,\ j\in\mathcal{E})$, $N=n,n+1,\cdots$, is a multi-type branching process whose generating functions $\phi_{nN}=(\phi_{nN,Ap},\cdots,\phi_{nN,Er})$ defined by $\phi_{nNi}(z)\stackrel{\text{def}}{=}\mathrm{E}[\ \prod_{j\in\mathcal{E}} z_j^{Z_{nNij}}\ ]$, satisfy, for $n<N$,*

$$\phi_{nN}(z)=\Pi_n^{-1}(F_{n+1}\circ\cdots\circ F_N)(\Pi_N z)\,,\ z\in\mathbf{C}^{\mathcal{E}}\,. \tag{3.9}$$

Proof. The strong Markov property of simple random walks and the finite ramifiedness of the fractal imply that $\{Z_{nNij}\}$ is a branching process. In particular,

$$\phi_{nN}(z)=\phi_{n,N-1}(\phi_{N-1,N}(z))\,,\ N>n, \tag{3.10}$$

holds. The definitions of Π_N and F_N imply

$$\phi_{n,n+1}(z)=\Pi_n^{-1}F_{n+1}(\Pi_{n+1}z)\,, \tag{3.11}$$

which, together with (3.10) implies (3.9). □

Proposition 1.3 implies that some off-spring branching rates vanish as $N\to\infty$, hence we are considering branching process with singular environment.

For integers n and N satisfying $n \le N$, define $\sharp\mathcal{E}$-dimensional matrices

$$\tilde{\Pi}_N \stackrel{\rm def}{=} \mathrm{diag}(\tilde{\Pi}(w_N)), \quad \tilde{A}_N \stackrel{\rm def}{=} \tilde{A}(a_N, b_N, w_N), \quad \tilde{B}_{nN} \stackrel{\rm def}{=} A_{n+1}A_{n+2}\cdots A_N\,.$$

Then (3.9), (3.1), and (1.4) imply

$$\mathrm{E}[\; Z_{nNij}\;] = \left(\tilde{\Pi}_n^{-1}\,\tilde{B}_{nN}\,\tilde{\Pi}_N\right)_{ij}\,, \quad i\in\mathcal{E},\ j\in\mathcal{E},\ N\ge n\,. \tag{3.12}$$

Elementwise positivity of A_N and hence of B_{nN} were noted in Proposition 3.1;

$$\tilde{A}_{Nij} > 0,\ \tilde{B}_{nNij} > 0,\quad i,j\in\mathcal{E},\ N>n\,. \tag{3.13}$$

Proposition 3.3. *(1) For each $n_0\in\mathbf{Z}$ there exist positive constants C_1 and C_2 such that if $n\in\mathbf{Z}$ and $N\in\mathbf{Z}$ satisfy $N-2\ge n\ge n_0$, then*

$$\begin{aligned}\tilde{B}_{nNij} &\ge C_1\prod_{k=n+1}^{N}(a_k+1)^2\,,\\ \tilde{B}_{nNij}\,\frac{w_N^2}{w_n^2} &\ge C_2\prod_{k=n+1}^{N}\left(1+\frac{b_k}{2}\right)^2\,,\quad i\in\mathcal{E},\ j\in\mathcal{E}.\end{aligned}$$

(2) There exist limits

$$\gamma_{ni}\stackrel{\rm def}{=}\lim_{N\to\infty}\frac{\tilde{B}_{nNij}}{\tilde{B}_{nN1j}}\,,\quad n\in\mathbf{Z},\ i\in\mathcal{E},\ j\in\mathcal{E}\,,$$

independent of j, which satisfy, for each $n_0\in\mathbf{Z}$,

$$0<\inf_{n\ge n_0, i\in\mathcal{E}}\gamma_{ni}\le\sup_{n\ge n_0, i\in\mathcal{E}}\gamma_{ni}<\infty\,.$$

(3) For each $n_0\in\mathbf{Z}$ there exists a positive constant C_3 such that if integers n, m, and N satisfy $m-1\ge n\ge n_0$ and $N\ge\max\{m,n+2\}$, then

$$\sum_{k\in\mathcal{E}}\tilde{B}_{n,m-1,i,k}\sum_{k'\in\mathcal{E}}\tilde{B}_{mNk'j}\le C_3\,\tilde{B}_{nNij}\,,\quad i\in\mathcal{E},\ j\in\mathcal{E}\,.$$

Proof. Since F is rational in w, $\tilde{A}$ is also rational. Therefore (3.6) implies $\tilde{A}_{k22}\ge(a_k+1)^2+C_4\,w_k,\ k\in\mathbf{Z}$, where C_4 is a positive constant. Proposition 1.3 implies that $\sum_{k\ge n}w_k<\sum_{k\ge n_0}w_k<\infty,\ n\ge n_0$, hence we obtain the first estimate for $\tilde{B}_{nNij}$. Proposition 1.3 implies that for each n_0 there exists a constant $C_5>0$ such that $\displaystyle\frac{w_N}{w_n}\ge C_5\prod_{k=n+1}^{N}\delta_k^{-1},\ N\ge n\ge n_0$. With the first estimate, we have the second estimate. The estimates (3.4) and (3.5) imply that the elementwise positive matrices $\tilde{A}_N$, $N \in \mathbf{Z}$, satisfy the assumption of Theorem C.1 in Appendix C with $q = 2$. Theorem C.1 then implies the second assertion. $\zeta \stackrel{\text{def}}{=} \{(a_N, b_N),\ N \in \mathbf{Z}\}$ is a bounded sequence, hence contains finite number of distinct pairs; as far as ζ is concerned, taking supremum or infimum in N is taking maximum or minimum among finite possibilities. Assume $N \geq m + 2$. Then

$$\begin{aligned}\tilde{B}_{nNij} &\geq \sum_{k\in\mathcal{E}} \tilde{B}_{n,m-1,i,k} \sum_{k'\in\mathcal{E}} \tilde{A}_{mkk'} \min_{k''\in\mathcal{E}} \tilde{B}_{mNk''j} \\ &\geq \sum_{k} \tilde{B}_{n,m-1,i,k} \sum_{k'} \tilde{B}_{mNk'j} \inf_{\ell\in\mathcal{E},m'\geq n_0} \sum_{\ell'} \tilde{A}_{m'\ell\ell'} \frac{\inf_{k''}\{\tilde{B}_{mNk''j}/\tilde{B}_{mN1j}\}}{\sharp\mathcal{E}\sup_{k''}\{\tilde{B}_{mNk''j}/\tilde{B}_{mN1j}\}}.\end{aligned}$$

It is now easy to see that the second assertion and (3.7) imply the third assertion. The cases $N = m$ and $N = m + 1$ can be proved similarly. □

Put $W_{nNij} \stackrel{\text{def}}{=} \dfrac{Z_{nNij}}{\mathrm{E}[\ Z_{nNij}\ ]}$.

Proposition 3.4. *Let $i \in \mathcal{E}$ and $j \in \mathcal{E}$. For each $n_0 \in \mathbf{Z}$ there exists a positive constant C such that for all N and $n \geq n_0$ satisfying $N \geq n + 2$,*

$$\mathrm{E}[\ W_{nNij}^2\ ] \leq C\,\tilde{\Pi}_{nii}\,w_n^{-1}\,.$$

Also, for each $n \in \mathbf{Z}$ the third moment is bounded in N; $\sup_{N\geq n} \mathrm{E}[\ W_{nNij}^3\ ] < \infty$.

Proof. By taking derivatives of ϕ_{nN} in Proposition 3.2 we obtain

$$\begin{aligned}(3.14)\quad & w_n\tilde{\Pi}_{nii}^{-1}\mathrm{E}[\ W_{nNij}^2\ ] \\ &= w_n\tilde{\Pi}_{nii}^{-1}\mathrm{E}[\ Z_{nNij}\ ]^{-2}\Big\{ \sum_{k_1,k_2,k_3\in\mathcal{E}} \sum_{m=n+1}^{N} \\ &\quad \left(\tilde{\Pi}_n^{-1}\tilde{B}_{n,m-1}\right)_{i,k_1} \frac{\partial^2 \tilde{F}_{k_1}}{\partial u_{k_2}\partial u_{k_3}}(a_m, b_m, w_m; \tilde{\Pi}(w_m)) \\ &\quad \times \left(\tilde{B}_{m,N}\tilde{\Pi}_N\right)_{k_2,j} \left(\tilde{B}_{m,N}\tilde{\Pi}_N\right)_{k_3,j} + \left(\tilde{\Pi}_n^{-1}\,\tilde{B}_{n,N}\,\tilde{\Pi}_N\right)_{i,j}\Big\} \\ &= \sum_{k_1,k_2,k_3\in\mathcal{E}} \sum_{m=n+1}^{N} \frac{w_n}{w_m}\,\tilde{B}_{n,m-1,ik_1}\,\frac{1}{w_m}\,\frac{\partial^2 g_{a_m,b_m,k_1}}{\partial h_{k_2}\partial h_{k_3}}(w_m, 0) \\ &\quad \times \frac{\tilde{B}_{mNk_2j}}{\tilde{B}_{nNij}}\,\frac{\tilde{B}_{mNk_3j}}{\tilde{B}_{nNij}} + \frac{w_n}{\tilde{B}_{nNij}\,\tilde{\Pi}_{Njj}}\,,\end{aligned}$$

where we used (3.12) and the definition of g in Proposition 3.1.

Table 1 and (3.2) imply that there exists a positive constant C_6 independent of n, N, i, j, such that

$$\frac{w_n}{\tilde{B}_{nNij}\,\tilde{\Pi}_{Njj}} \le C_6 \frac{w_n}{w_N \tilde{B}_{nNij}} \le C_6 C_2 \frac{w_N}{w_n \prod_{k=n+1}^{N}\left(1+\frac{b_k}{2}\right)},$$

where, in the last inequality, we used the first assertion in Proposition 3.3. $b_k > 0$ and $w_N < w_n$ (Proposition 1.3) therefore imply that the second term in the right hand side of (3.14) is bounded.

Next note that Proposition 3.1 implies that there exists a positive constant C_7 independent of n, N, i, j, such that the first term in the right hand side of (3.14) is bounded from above by

$$C_7 \sum_{k_1,k_2,k_3\in\mathcal{E}} \sum_{m=n+1}^{N} \frac{w_n}{w_m} \frac{\tilde{B}_{n,m-1,ik_1}\,\tilde{B}_{mNk_2j}}{\tilde{B}_{nNij}} \frac{\tilde{B}_{mNk_3j}}{\tilde{B}_{nNij}}.$$

Using the third, second, and first assertions of Proposition 3.3 in turn, we see that this quantity is further bounded from above by

$$\begin{aligned} & C_7 C_3 \sum_{m=n+1}^{N} \frac{w_n}{w_m} \sum_{k_3\in\mathcal{E}} \frac{\tilde{B}_{mNk_3j}}{\tilde{B}_{nNij}} \\ \le\ & C_8 \sum_{m=n+1}^{N} \frac{w_n}{w_m} \frac{1}{\sum_{\ell\in\mathcal{E}} \tilde{B}_{nmij\ell}} \\ \le\ & C_8 \sum_{m=n+1}^{\infty} \frac{w_m}{w_n \prod_{k=n+1}^{m}\left(1+\frac{b_k}{2}\right)^2}, \end{aligned}$$

for some positive constant C_8 independent of n, N, i, j. Using $w_m < w_n$ (Proposition 1.3) and $b_k \ge 2$, we see that this quantity is bounded. Therefore we have the bound in the statement of the proposition for the second moment $\mathrm{E}[\,W_{nNij}^2\,]$. The bound on the third moment is proved in a similar way. □

Let $\mathcal{E}' \subset \mathcal{E}$ be the set of horizontal edges;

$$\mathcal{E}' \stackrel{\text{def}}{=} \{Ar,\ Br,\ Er\}\,. \tag{3.15}$$

Note that Table 1 and (3.2) imply that there exist positive constants C and C' such that for $w \in I$ we have

$$C \le \tilde{\Pi}_i(w) \le C'\,, \quad i \in \mathcal{E}'\,, \qquad C\,w \le \tilde{\Pi}_i(w) \le C'\,w\,, \quad i \in \mathcal{E}\setminus\mathcal{E}'\,. \tag{3.16}$$

Define, for $N \in \mathbf{Z}_+$,

$$L_N \stackrel{\text{def}}{=} \sum_{i\in\mathcal{E}} \sum_{j\in\mathcal{E}} \mathrm{E}[\,Z_{0Nij}\,]\,. \tag{3.17}$$

The next Theorem shows that L_N is the appropriate scaling factor for the random walk $X = X_{N,W_N}$ on H_N. Note that $T_{n,1}(X) - T_{n,0}(X) = \sum_{j\in\mathcal{E}} Z_{nNij}$, where i is the type of edge formed by the endpoints of X in this time interval.

Theorem 3.5. *Let $m \in \mathbf{Z}$ and $i \in \mathcal{E}$. $(W_{mNij},\ j \in \mathcal{E})$, $N = m, m+1, \cdots$, converges in L_2 (hence in probability and in law) as $N \to \infty$. The limit is a random vector with equal components $(W_{mi}, \cdots, W_{mi})$, satisfying* $\mathrm{E}[\ W_{mi}\ ] = 1$ *and* $\sup_{n\ge m} w_n \mathrm{E}[\ W_{ni}^2\ ] < \infty$. *$L_N^{-1} \sum_{j\in\mathcal{E}} Z_{mNij}$ converges in probability as $N \to \infty$ to $W'_{mi} \stackrel{\mathrm{def}}{=} \dfrac{\gamma_{mi}\, W_{mi}}{\sum_{j,k} \mathrm{E}[\ Z_{0mjk}\ ]\, \gamma_{mk}}$, with γ_{mi} as in Proposition 3.3. The distribution of W'_{mi} is continuous.*

Proof. As noted in Proposition 3.2, $(Z_{mNij},\ j \in \mathcal{E})$, $N = m, m+1, \cdots$, is a branching process. The number of descendant at time N from a single ancestor of type k at time m is equal in distribution to that of $(Z_{mNkj},\ j \in \mathcal{E})$. Fix $j \in \mathcal{E}$. Proposition 3.4 implies, with (3.16),

$$\sup_{n\ge m} \sup_{N\ge n+2} w_n\, \mathrm{E}[\ W_{nNij}^2\ ] < \infty\,.$$

The uniform bound for $\mathrm{E}[\ W_{nNij}^3\ ]$ in Proposition 3.4 implies

$$\lim_{p\to\infty} \sup_{N\ge n+2} \mathrm{E}[\ W_{nNij}^2;\ W_{nNij} > p\ ] = 0\,,\quad n \ge m\,.$$

Proposition 3.3 with (3.12) implies that for each $n \ge m$, $\lim_{N\to\infty} \dfrac{\mathrm{E}[\ Z_{nNij}\ ]}{\mathrm{E}[\ Z_{nN1j}\ ]}$ exists, positive and independent of $j \in \mathcal{E}$. Hence, if we prove that $w_N\, Z_{mNij}$ diverges in probability to infinity, then all the assumptions of Proposition 2.1 will be satisfied, with e_N replaced by w_{m+N} and $n_0 = 2$. Proposition 2.1 then will imply that $(W_{mNij},\ j \in \mathcal{E})$ converges in L_2 as $N \to \infty$, to a random vector with equal components.

By definition, $Z_{mmij} = 1$ $(j = i)$ and $= 0$ $(j \ne i)$. Fix $p > 0$ and $j \in \mathcal{E}$, and define a family of sequences $\{(x_{k,n}, y_{k,n})\}$ $0 \le n \le k$, $k \in \mathbf{Z}_+$, by

$$\begin{aligned} x_{k,n} &= \max_{i'\in\mathcal{E}'} \mathrm{Prob}[\ w_{m+n_0(k)+k}\, Z_{m+n,m+n_0(k)+k,i',j} \le p\ ]\,,\\ y_{k,n} &= \max_{i'\in\mathcal{E}\setminus\mathcal{E}'} \mathrm{Prob}[\ w_{m+n_0(k)+k}\, Z_{m+n,m+n_0(k)+k,i',j} \le p\ ]\,, \end{aligned}$$

where $\mathcal{E}'$ is defined in (3.15). $n_0(k)$ is an arbitrary function of k taking non-negative integer values (to be specified later). Define a sequence $\{\tilde a_n,\ n \in \mathbf{Z}_+\}$ by $\tilde a_n = a_{m+n} + 1$, and $\{\tilde w_n,\ n \in \mathbf{Z}_+\}$ by

$$\tilde w_n \stackrel{\mathrm{def}}{=} \max_{i'\in\mathcal{E}'} \mathrm{Prob}[\ \sum_{k\in\mathcal{E}\setminus\mathcal{E}'} Z_{m+n-1,m+n,i',k} \ge 1\ ]\,.$$

$\tilde{w}_n$ is the largest probability among $i' \in \mathcal{E}'$ that the random walk $X = X_{N,w_N}$ with $N = m + n$ jumps off-horizontally at least once in the time interval $[T_{N-1,1}(X),\ T_{N-1,0}(X)]$, under the condition that the endpoints of X for this time interval forms an edge of type i' in H_{N-1}. By definition, $3 \le \inf_n \tilde{a}_n \le \sup_n \tilde{a}_n < \infty$. Also $0 \le \tilde{w}_n \le 1$ and is of order w_{m+n}, for which Proposition 1.3 implies $\tilde{w}_n \le C_1\, \delta^{-n}$ for some constant $C_1 > 0$ and $\delta = \min_k \delta_k = \min_k \dfrac{2(1+a_k)}{2+b_k} > 1$ (recall that there are only finite number of distinct pairs (a_k, b_k)).

A graphical consideration shows that

$$\begin{aligned} x_{k,n} \quad &\le \quad \max_{i' \in \mathcal{E}'} \Bigg\{ \mathrm{Prob}[\sum_{j' \in \mathcal{E} \setminus \mathcal{E}'} Z_{m+n,m+n+1,i',j'} = 0\,]\, x_{k,n+1}{}^{\tilde{a}_{n+1}} \\ &\qquad + \mathrm{Prob}[\sum_{j' \in \mathcal{E} \setminus \mathcal{E}'} Z_{m+n,m+n+1,i',j'} \ge 1\,]\, y_{k,n+1} \Bigg\}\,. \end{aligned}$$

(The first term in the outmost parenthesis corresponds to those paths whose $(m+n+1)$-decimated walks do not contain jumps of type $j' \in \mathcal{E} \setminus \mathcal{E}'$, while the second term corresponds to those with at least one such jumps.) We may either use $y_{k,n} \le 1$ or use $\mathrm{Prob}[\sum_{j' \in \mathcal{E} \setminus \mathcal{E}'} Z_{m+n,m+n+1,i',j'} = 0\,] \le 1$, to conclude that $x_{k,n}$ satisfies the inequality in the definition of the assumption R in Section 2, with $\{w_n\}$ and $\{a_n\}$ replaced by $\{\tilde{w}_n\}$ and $\{\tilde{a}_n\}$, respectively. Similarly, we find

$$\begin{aligned} y_{k,n} \quad &\le \quad \max_{i' \in \mathcal{E} \setminus \mathcal{E}'} \Bigg\{ \mathrm{Prob}[\sum_{j' \in \mathcal{E}'} Z_{m+n,m+n+1,i',j'} \ge 1\,]\, x_{k,n+1} \\ &\qquad + \mathrm{Prob}[\sum_{j' \in \mathcal{E}'} Z_{m+n,m+n+1,i',j'} = 0\,]\, y_{k,n+1} \Bigg\}\,, \end{aligned}$$

hence we see that $y_{k,n}$ also satisfies the inequality of the assumption R.

For $k \in \mathbf{Z}_+$ and $i' \in \mathcal{E}'$ put

$$\begin{aligned} e_{k,i'} \quad &\stackrel{\mathrm{def}}{=} \quad \mathrm{E}[\, w_{m+n_0(k)+k}\, Z_{m+k,m+n_0(k)+k,i',j}\,]\,, \\ v_{k,i'} \quad &\stackrel{\mathrm{def}}{=} \quad \mathrm{V}[\, w_{m+n_0(k)+k}\, Z_{m+k,m+n_0(k)+k,i',j}\,]\,. \end{aligned}$$

Proposition 3.3, (3.12), (3.16), and $b_k \ge 2$ imply $e_{k,i'} \ge C_2\, w_{m+k}^2\, 4^{n_0(k)}$, where $C_2 > 0$ is a constant independent of $n_0 \in \mathbf{Z}_+$ and $k \in \mathbf{Z}_+$. For each k, define $n_0(k)$ to be sufficiently large so that $e_{k,i'} \ge 2p$ for all $k \in \mathbf{Z}_+$. Proposition 2.3 then implies that $\mathrm{Prob}[\, w_{m+n_0(k)+k}\, Z_{m+k,m+n_0(k)+k,i,j} \le p\,] < 1 - 1/(2d)$, where $d \le 1 + 2\, v_{k,i'}\, e_{k,i'}^{-2}$. Applying Proposition 3.4 and (3.16) we see that $x_{k,k} < 1 - C_3\, w_{m+k}$, where C_3 is a positive constant independent of k. Proposition 1.3 implies $w_{m+k} \ge C_4\, w_m \prod_{\ell=0}^{k} \delta_{m+\ell}^{-1}$ with $\delta_k = 2(1+a_k)/(2+b_k)$, and for a positive constant C_4 independent of k. Hence

$$(-\log x_{k,k}) \prod_{\ell=0}^{k} \tilde{a}_\ell > C_3\, C_4\, w_m \prod_{\ell=0}^{k} (1 + b_{m+\ell}/2)\,,$$

which implies $\liminf_{k\to\infty,\ x_{k,k}\ne 0} \left\{ (-\log x_{k,k}) \prod_{\ell=0}^{k} \tilde{a}_\ell \right\}^{1/k} \ge 2 > 1$. We see that all the assumptions in Theorem 2.2 are satisfied, with e_N replaced by w_{m+N}, and $Z_{N,j}$ by $Z_{m,m+N,i,j}$. Theorem 2.2 then implies $\lim_{N\to\infty} \mathrm{Prob}[\, w_N\, Z_{mNij} \ge p\,] = 1\,,\ \ p > 0$, which, as we noted in the first part of the proof, proves the convergence of $(W_{mNij},\ j \in \mathcal{E})$ to $(W_{mi}, \cdots, W_{mi})$. Weak convergence and uniform integrability imply convergence in expectations. Therefore, from what we have proved, we obtain the statements on $\mathrm{E}[\, W_{mi}\,]$ and $\mathrm{E}[\, W_{mi}^2\,]$.

Let $n \in \mathbf{Z}_+$. The j independence of γ_{ni} in Proposition 3.3 implies, with (3.12),

$$\lim_{N\to\infty} \frac{\sum_j \mathrm{E}[\, Z_{nNij}\,]}{\sum_j \mathrm{E}[\, Z_{nN1j}\,]} = \gamma_{ni}\,.$$

Also from (3.12) one sees, for $m \le n \le N$,

$$\mathrm{E}[\, Z_{mNij}\,] = \sum_{k\in\mathcal{E}} \mathrm{E}[\, Z_{mnik}\,]\, \mathrm{E}[\, Z_{nNkj}\,]\,.$$

Hence, with (3.17), we see that $\lim_{N\to\infty} L_N^{-1} \sum_j \mathrm{E}[\, Z_{nNij}\,] = \dfrac{\gamma_{ni}}{\sum_{j,k} \mathrm{E}[\, Z_{0njk}\,]\,\gamma_{nk}}$.

Convergence of $(W_{nNij},\ j \in \mathcal{E})$ and Proposition 2.4 imply that $\dfrac{\sum_j Z_{nNij}}{\sum_j \mathrm{E}[\, Z_{nNij}\,]}$ converges in probability to W_{ni}. Therefore we have the convergence in probability of $L_N^{-1} \sum_{j\in\mathcal{E}} Z_{nNij}$ to W'_{ni}. With $\mathrm{E}[\, W_{ni}\,] = 1$ and Proposition 3.3, we have

$$\mathrm{E}[\, W'_{ni}\,] = \frac{\gamma_{ni}}{\sum_{j,k} \mathrm{E}[\, Z_{0njk}\,]\,\gamma_{nk}} \ge C_5\, \Big(\sum_{j,k} \mathrm{E}[\, Z_{0njk}\,]\Big)^{-1}\,, \tag{3.18}$$

for some positive constant C_5 independent of $n \ge 0$ and $i \in \mathcal{E}$. Similarly, there exists a positive constant C_6 such that

$$\mathrm{E}[\, {W'_{ni}}^2\,] \le C_6\, \Big(\sum_{j,k} \mathrm{E}[\, Z_{0njk}\,]\Big)^{-2}\, w_n^{-1}\,, \quad n \ge 0,\ i \in \mathcal{E}\,. \tag{3.19}$$

Let

$$\Phi_{n,i}(t) \stackrel{\rm def}{=} \mathrm{E}[\ \exp\left(\sqrt{-1}\,t\,W'_{n,i}\right)\ ]\,,\quad t\in\mathbf{R}\,,$$

denote the characteristic function. With obvious bound $0 \le 1-\Re\Phi_{ni}(t) \le t^2\mathrm{E}[\ {W'_{ni}}^2\ ]/2$ and $|\Im\Phi_{ni}(t)-t\mathrm{E}[\ W'_{ni}\ ]| \le t^2\mathrm{E}[\ {W'_{ni}}^2\ ]/2$, $t\in\mathbf{R}$, we can proceed as in the first half of the proof of [16, (2.45)], using random walk representation [16, (2.30)] for $\Phi_{ni}(t)$, to obtain

$$|\Phi_{n,i}(t)| \le 1-\tilde{C}_n\,t^2\,,\quad -t'_n<t<t'_n\,,\ n\ge 0,\ i\in\mathcal{E}'\,,$$

with $\tilde{C}_n = C_7\,w_{n+1}^{-1}\min_j \mathrm{E}[\ W'_{n+1,j}\ ]^2$ and $t'_n = \dfrac{\min_j \mathrm{E}[\ W'_{n+1,j}\ ]}{\max_j \mathrm{E}[\ {W'_{n+1,j}}^2\ ]}$, where C_7 is a positive constant independent of $n\ge 0$. (Replace $6^k t$ in [16, (2.45)] by t and $(3/4)^{n+k}$ by w_n.) We may use a narrower interval $(-t_n,t_n)$ with $t_n\le t'_n$ for the estimate above, in applying Theorem 2.5. Put

$$t_n \stackrel{\rm def}{=} C_5{C_6}^{-1}\,w_n\,\sum_{j,k}\mathrm{E}[\ Z_{0njk}\ ]\,.$$

Then (3.18) and (3.19) imply $t_n\le t'_n$. With (3.18), Proposition 1.3, and Proposition 3.3, we see that an assumption of Theorem 2.5

$$\liminf_{k\to\infty}\left(t_k^2\tilde{C}_k\prod_{\ell=0}^{k-n_0}(a_{n_0+\ell+1}+1)\right)^{1/k} > 1\,,$$

is satisfied with a_ℓ replaced by $a_{n_0+\ell+1}+1$.

Proposition 3.3 and (3.12) imply $\lim_{k\to\infty} t_k=\infty$, while boundedness of $\tilde{A}_{nij}$ implied in (3.4) with Proposition 1.3 and (3.16) gives $\sup_{k\ge 0} t_k^{1/k} < \infty$. Let $n\ge 0$ and $m\ge 0$. Proposition 3.3, Proposition 1.3, (3.12), (3.16), and $b_k\ge 2$ imply

$$\frac{t_n}{t_{n+m}} < \frac{C_8\,w_n}{w_{n+m}\,\min_\ell\sum_{k\in\mathcal{E}'}B_{n,n+m,\ell,k}} < C_8\,4^{-m}\,,$$

where C_8 is a positive constant independent of n and m. Therefore there exists an m_0 such that $\dfrac{t_n}{t_{n+m_0}}<1$ for all $n\ge 0$. With the boundedness of A_{nij} we also see $\inf_{n\ge 0}\dfrac{t_n}{t_{n+m_0}}>0$. Hence all the assumptions for $\{t_k\}$ in Theorem 2.5 hold.

Using [16, (2.30)], we can proceed with similar arguments as we did for $\mathrm{Prob}[\ w_{m+n_0(k)+k}\,Z_{m+n,m+n_0(k)+k,i',j}\le p\ ]$, from which we see that $\Phi_{n,i}$ satisfies the assumption R condition of Theorem 2.5. We have now proved that $\Phi_{n,i}$ satisfies all the assumption of Theorem 2.5, which implies that the distribution of $W'_{n,i}$ is continuous. □

Let $D \stackrel{\text{def}}{=} D([0,\infty);G)$ be the set of cadlag paths on the scale-irregular *abb*-gasket G. For $n \in \mathbf{Z}$ and $x \in G_n$ we define a family of probability measures $P_x^{(N)}[\ \cdot\ ]$, $N = n, n+1, \cdots$, on D, by $P_x^{(N)}[\ w(0) = x\ ] = 1$ and

$$P_x^{(N)}[\ w(t_i) = x_i,\ i = 1,2,\cdots,r\ ] = \mathrm{Prob}[\ X_{N,w_N,x}([L_N\, t_i]) = x_i,\ i = 1,\cdots,r\ ],$$

where $X_{N,w_N,x}$ is the random walk X_{N,w_N} with starting point x; $X_{N,w_N,x} = x$. We use abbreviations such as

$$P_x^{(N)}[\ w(0) = x\ ] \stackrel{\text{def}}{=} P_x^{(N)}[\ \{w \in D \mid w(0) = x\}\ ]\,,$$

and write $E_x^{(N)}[\ \cdot\ ]$ for the expectations with respect to $P_x^{(N)}[\ \cdot\ ]$. Define $T_{n,i}(w)$, $w \in D$, similarly as we did in Section 1 for processes, and put $W_{n,i} \stackrel{\text{def}}{=} T_{n,i+1} - T_{n,i}$. Let $N \geq n$, $x \in G_N$, $i \in \mathbf{Z}_+$, and let $x_0, x_1, \cdots, x_i$ be a sequence of points in G_n such that each adjoining pair is an n-neighbor pair and x_0 and x are in a unit triangle of H_n. Consider the distribution of $W_{n,j}$, $j = 0,1,\cdots,i-1$, under the conditional probability

$$P_x^{(N)}[\ \cdot \mid w(T_{n,0}(w)) = x_0, w(T_{n,1}(w)) = x_1,\ \cdots,\ w(T_{n,i}(w)) = x_i\ ]\,.$$

Since the probability is based on random walks, this distribution is a direct product of the distributions of each $W_{n,j}$, and as we noted before Theorem 3.5, the distribution of each $W_{n,j}$ under the conditional probability is equal to that of $L_N^{-1} \sum_{\ell \in \mathcal{E}} Z_{nNk\ell}$, if (x_j, x_{j+1}) forms an edge of type $k \in \mathcal{E}$, and is independent of i, j, x, and x_j's. We denote this distribution of $W_{n,j}$ by $Q_{n,k}^{(N)}[\ \cdot\ ]$, and their limit distributions as $N \to \infty$ by $Q_{n,k}[\ \cdot\ ]$, $k \in \mathcal{E}$. Theorem 3.5 implies

$$\lim_{N\to\infty} Q_{n,k}^{(N)}[\ s \mid a < s < b\ ] = Q_{n,k}[\ s \mid a < s < b\ ]\,,\quad 0 \leq a < b \leq \infty\,. \tag{3.20}$$

We need a following type of uniformity to handle processes starting from 'irrational' points.

Proposition 3.6. *Let N, M, n be non-negative integers satisfying $N \geq M \geq n$, and let $x \in G_M$ and $y \in G_n$ such that x and y are in a unit triangle of H_n. Then there exists a positive constant C_1 independent of x, y, n, M, and M, such that*

$$E_x^{(N)}[\ T_{n,0}(w) \mid w(T_{n,0}(w)) = y\ ] \leq C_1 \prod_{\ell=1}^{n} \left(1 + \frac{b_\ell}{2}\right)^{-2}\,.$$

Proof. By similar arguments for the proof of [16, (3.2)], we see that there exist positive constants C_2 and C_3 such that for $X_m = X_{m,w_m,x'}$

$$\mathrm{E}[\ T_{m-1,0}(X_m) \mid X_m(T_{m-1,0}(X_m)) = y'\ ] \leq C_1 + \frac{C_2}{w_m}\,, \tag{3.21}$$

for all $m \in \mathbf{Z}_+$, $x' \in G_m \backslash G_{m-1}$, and $y' \in G_{m-1}$, with x' and y' in a unit triangle of H_{m-1}. For an m-neighbor pair (u, v) forming a type k edge, Proposition 3.3, (3.17), and (3.12) imply

$$\begin{aligned} &L_N^{-1} \mathrm{E}[\, W_{m,i}(X_{N,w_N}) \mid X_{N,w_N}(T_{m,i}) = u, X_{N,w_N}(T_{m,i+1}) = v \,] \\ &= L_N^{-1} \sum_{\ell \in \mathcal{E}} Z_{nNk\ell} \le C_3 \,\tilde{\Pi}_{nk}^{-1} \,(\sum_{\ell,\ell'} \tilde{B}_{0.n-1,\ell,\ell'})^{-1} , \end{aligned} \tag{3.22}$$

where C_3 is a positive constant independent of m, N, u and v. Proposition 3.3, (3.21), and (3.22) imply

$$E_{x'}^{(N)}[\, T_{m-1,0}(w) \mid w(T_{m-1,0}(w)) = y' \,] \le C_4 \prod_{\ell=1}^{m} (1 + b_\ell/2)^{-2} , \tag{3.23}$$

for all $N \ge m \ge 0$, $x' \in G_m \setminus G_{m-1}$, and $y' \in G_{m-1}$, with x' and y' in a unit triangle of H_{m-1}. C_4 is a constant. The estimate (3.23), combined with the strong Markov property of the random walks, implies for $N \ge M \ge n$, $x \in G_M$, $y \in G_n$, with x and y in a unit triangle of H_n,

$$\begin{aligned} &E_x^{(N)}[\, T_{n,0}(w) \mid w(T_{n,0}(w)) = y \,] \\ &= \sum_{\{y_i\}} \sum_{i=n+1}^{M} E_{y_i}^{(N)}[\, T_{i-1,0}(w) \mid w(T_{i-1,0}(w)) = y_{i-1} \,] \\ &\times P_x^{(N)}[\, w(T_{i,0}(w)) = y_i,\ n+1 \le i \le M-1 \mid w(T_{n,0}) = y \,] \\ &\le C_1 \prod_{\ell=1}^{n} (1 + b_\ell/2)^{-2} , \end{aligned}$$

where the first summation is taken over $\{y_i\} = (y_n, y_{n+1}, \cdots, y_M)$ with $y_i \in G_i$, $y_n = y$, $y_M = x$, such that y_i and y_{i-1} are in a unit triangle of H_{i-1}, for $i = n+1, \cdots, M$. □

The following result is used to prove that an N-decimated walk of a diffusion, obtained as the continuum limit $N \to \infty$ of a sequence of random walks, is equal to the original random walk.

Proposition 3.7. *For $N \in \mathbf{Z}_+$, let X_N be a simple random walk on G_N with N-neighbor jumps. Assume that there exists a sequence L_N diverging to infinity as $N \to \infty$ such that, $\tilde{X}_N(\cdot) \stackrel{\mathrm{def}}{=} X_N([L_N \cdot])$ converges almost surely as $N \to \infty$ to some continuous strong Markov process $X(\cdot)$ on G. Let $n \in \mathbf{Z}_+$. If for each $N \ge n$, the n-decimated walk (defined in Section 1) of X_N is equal in law to X_n, then the n-decimated walk of X is also equal in law to X_n.*

Proof. Fix $x \in G_n$ and $y \in G_n$. Denote by $P^{(x)}[\ \cdot\ ]$ the conditional probability with condition $X_N(0) = x$, $N \in \mathbf{Z}_+$, $X(0) = x$, and let $E^{(x)}[\ \cdot\ ]$

be expectation with respect to $P^{(x)}[\ \cdot\ ]$. For $N \geq n$ and a positive integer q, define $\sigma_{N,q} \stackrel{\text{def}}{=} \inf\{t \geq 0 \mid d(\tilde{X}_N(t), G_n \setminus \{x\}) \leq 1/q\}$, $\sigma_q \stackrel{\text{def}}{=} \inf\{t \geq 0 \mid d(X(t), G_n \setminus \{x\}) \leq 1/q\}$, $\sigma_{N,\infty} \stackrel{\text{def}}{=} \inf\{t \geq 0 \mid \tilde{X}_N(t) \in G_n \setminus \{x\}\}$, $\sigma_\infty \stackrel{\text{def}}{=} \inf\{t \geq 0 \mid X(t) \in G_n \setminus \{x\}\}$, where d is the metric on G. The almost sure convergence of $\tilde{X}_N$ to X implies

$$\sigma_q \leq \liminf_{N\to\infty} \sigma_{N,q} \leq \limsup_{N\to\infty} \sigma_{N,q} \leq \sigma_{q+1}\ ,\ \ a.s., \quad q > 0\,. \tag{3.24}$$

Define a harmonic function $h:\ G \to [0,1]$ as follows; for $z \in G_\infty$, i.e., $z \in G_m$ for some $m \geq n$, define $h(z) \stackrel{\text{def}}{=} \text{Prob}[\ \tilde{X}_m(\sigma_{m,\infty}) = y \mid \tilde{X}_m(0) = z\ ]$. The assumption on the decimation property implies that $\text{Prob}[\ \tilde{X}_{m'}(\sigma_{m',\infty}) = y \mid \tilde{X}_{m'}(0) = z\ ]$ is constant for $m' \geq m$, hence h is well-defined on G_∞. We can see that [16, Proposition 3.2] holds in our case, which implies that h is continuous. In particular, h is uniquely extendable as continuous function to G. By definition, $h(y) = 1$ and $h(y') = 0$, $y' \in G_n \setminus \{y\}$. X_N is a simple random walk, and h, restricted on G_N, is an associated harmonic function. Therefore $h(\tilde{X}_N(t \wedge \sigma_{N,q}))$, $t \geq 0$, is a martingale ($a \wedge b \stackrel{\text{def}}{=} \min\{a,b\}$), hence $E^{(x)}[\ h(\tilde{X}_N(t \wedge \sigma_{N,q}))\ ] = h(x)$, $N \geq n$. This with (3.24), $\lim_{N\to\infty} \tilde{X}_N = X$, and continuity of h implies

$$E^{(x)}[\ \min_{\sigma_q \leq s \leq \sigma_{q+1}} h(X(t \wedge s))\ ] \leq h(x) \leq E^{(x)}[\ \max_{\sigma_q \leq s \leq \sigma_{q+1}} h(X(t \wedge s))\ ]\,.$$

Continuity of X implies that $\lim_{q\to\infty} \sigma_q = \sigma_\infty$. Hence we have $h(x) = E^{(x)}[\ h(X(t \wedge \sigma_\infty))\ ]$, $t \geq 0$. Since this is independent of t, we have $h(x) = E^{(x)}[\ h(X(\sigma_\infty))\ ] = \text{Prob}[\ X(\sigma_\infty) = y \mid X(0) = x\ ]$, which implies that the transition probability of n-decimated walk of X is equal to that of $X_n(0)$. □

Proof of Theorem 1.2. We can apply [16, Sect. 3], with [16, Theorems 2.5, 2.8] replaced by Theorem 3.5, [16, (3.1)] by (3.20), and [16, Proposition 3.1(1)] by Proposition 3.6. Then for $x_N \in G_N$, $N \in \mathbf{Z}_+$, satisfying $\lim_{N\to\infty} x_N = x$, the sequence of measures $P^{(N)}_{x_N}[\ \cdot\ ]$ (the distribution of $X_{N,w_N,x_N}([L_N t])$), $N \in \mathbf{Z}$, converges weakly as $N \to \infty$ to a symmetric Feller process X. Skorokhod's Theorem implies that there exists a probability space and G_N valued processes X_N, $N \in \mathbf{Z}$, such that X_N is equal in law to X_{N,w_N,x_N} and converges almost surely to a process equal in law to X. Proposition 3.7 implies that the n-decimated walk of this process is equal in law to the original random walk X_{n,w_n}. That this random walk has the asymptotically one-dimensional (and isotropy restoration) properties, is proved in Proposition 1.3. □

Appendix A. The scale-irregular *abb*-gasket.

The scale-irregular pre-*abc*-gasket as a graph.

A mathematical definition of a wide class of pre-fractals, including pre-*abc*-gaskets, is given in [15, Section 5.1]. The definition of a scale-irregular pre-*abc*-gasket as a graph is an easy scale-irregular extension. For convenience to the readers, we reproduce relevant part of the definition in [15], with implementation of scale-irregularity for the scale-irregular pre-*abc*-gaskets.

Denote a set of positive integers by $\mathbf{N}$, and a set of non-negative integers by $\mathbf{Z}_+$. For $\sigma = (a,b,c) \in \mathbf{N}^3$, define an equivalence relation $\overset{\sigma}{\sim}$ on $\mathbf{Z}_+^2$, parametrized by σ, by the defining relations

$$\begin{cases} (i,1) \overset{\sigma}{\sim} (i+1,0) & 0 \le i < a, \\ (i,2) \overset{\sigma}{\sim} (i+1,1) & a \le i < a+b, \\ (i,0) \overset{\sigma}{\sim} (i+1,2) & a+b \le i < a+b+c-1, \\ (a+b+c-1,0) \overset{\sigma}{\sim} (0,2). \end{cases}$$

Let $\Sigma_\infty = (\sigma_1, \sigma_2, \sigma_3, \cdots)$, $\sigma_n = (a_n, b_n, c_n)$, $n \in \mathbf{N}$, be a sequence in $\mathbf{N}^3$. Write $\Sigma_0 = \phi$ and $\Sigma_n = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ for $n \in \mathbf{N}$.

For $n \in \mathbf{Z}_+$, the *finite* scale-irregular pre-*abc*-gasket at n-th stage construction $\tilde{H}_n(\Sigma_n)$, parametrized by Σ_n, is a triplet

$$\tilde{H}_n(\Sigma_n) = (V(\Sigma_n), B(\Sigma_n), P(\Sigma_n)),$$

of a set of vertices $V(\Sigma_n)$, a set of edges (a set of unordered pairs of vertices) $B(\Sigma_n)$, and a set of three vertices $P(\Sigma_n) = \{p_{n0}, p_{n1}, p_{n2}\} \subset V(\Sigma_n)$, defined inductively as follows.

$\tilde{H}_0(\Sigma_0)$ is defined by $V(\Sigma_0) = \{0,1,2\}$, $B(\Sigma_0) = \{\{0,1\},\{1,2\},\{2,0\}\}$, and $P(\Sigma_0) = \{p_{00}, p_{01}, p_{02}\}$, where $p_{0i} = i$, $i = 0,1,2$.

Assume that $\tilde{H}_{n-1}(\Sigma_{n-1})$ is defined for an $n \in \mathbf{N}$. Define an equivalence relation $\sim$ on a set of pairs

$$\{(m,v) \mid m \in \mathbf{Z}_+,\ v \in V(\Sigma_{n-1})\},$$

by the defining relation

$$\text{(A.1)} \qquad \begin{array}{l} (m,v) \sim (m',v') \ \text{ if and only if } \ v = p_{n-1,i},\ v' = p_{n-1,j}, \\ \text{ for some } i,j, \ \text{ and } \ (m,i) \overset{\sigma_n}{\sim} (m',j). \end{array}$$

$V(\Sigma_n)$ is then defined by

$$V(\Sigma_n) = \{(m,v) \mid m = 0,1,2,\cdots,a_n+b_n+c_n-1,\ v \in V(\Sigma_{n-1})\}/\sim\ .$$

Denote the equivalence class of (m, v) by $((m, v))$. $B(\Sigma_n)$ is defined by

$$\begin{aligned} B(\Sigma_n) &= \{\{((m,v)),((m,w))\} \mid \\ &\quad m = 0, 1, \cdots, a_n + b_n + c_n - 1, \quad \{v, w\} \in B(\Sigma_{n-1})\}, \end{aligned}$$

and $P(\Sigma_n) = \{p_{n0}, p_{n1}, p_{n2}\}$ is defined by

$$\text{(A.2)} p_{n0} = ((0, p_{n-1,0})), \quad p_{n1} = ((a_n, p_{n-1,1})), \quad p_{n2} = ((a_n + b_n, p_{n-1,2})).$$

For each $n \in \mathbf{N}$, there is an injection $\iota: \ V(\Sigma_{n-1}) \to V(\Sigma_n)$ defined by

$$\iota: \ V(\Sigma_{n-1}) \ni v \mapsto ((0, v)) \in V(\Sigma_n).$$

ι maps a bond $\{v, v'\} \in B(\Sigma_{n-1})$ to a bond $\{((0, v)), ((0, v'))\} \in B(\Sigma_n)$. We can therefore identify $(V(\Sigma_{n-1}), B(\Sigma_{n-1}))$ as a subset of $(V(\Sigma_n), B(\Sigma_n))$. Define a graph $\tilde{H}_\infty(\Sigma_\infty) = (V(\Sigma_\infty), B(\Sigma_\infty))$ by

$$\tilde{H}_\infty(\Sigma_\infty) = \bigcup_{n \in \mathbf{Z}_+} (V(\Sigma_n), B(\Sigma_n)),$$

with the identification induced by ι assumed.

Note that with the identification ι, $p_{n0} = p_{00} = 0$ holds for any $n \in \mathbf{N}$. We call $0 \in \tilde{H}_\infty(\Sigma_\infty)$ the origin, and also use the notation O.

For $\sigma = (a, b, c) \in \mathbf{N^3}$, define $R(\sigma)$ by $R(\sigma) = (a, c, b)$, and for a sequence $\Sigma_\infty = (\sigma_1, \sigma_2, \sigma_3, \cdots)$ in $\mathbf{N^3}$, define $R(\Sigma_\infty)$ by

$$R(\Sigma_\infty) = (R(\sigma_1), R(\sigma_2), R(\sigma_3), \cdots).$$

Define also an equivalence relation $\overset{R}{\sim}$ by $(+, O) \overset{R}{\sim} (-, O)$. A graph $\tilde{H}'_\infty(\Sigma_\infty) = (V'(\Sigma_\infty), B'(\Sigma_\infty))$ (scale-irregular pre-*abc*-gasket as a graph) is defined by

$$V'(\Sigma_\infty) = \Big(\{(+, v) \mid v \in V(\Sigma_\infty)\} \bigcup \{(-, v) \mid v \in V(R(\Sigma_\infty))\}\Big) \,/\, \overset{R}{\sim},$$

and

$$\begin{aligned} B'(\Sigma_\infty) &= \{\{((+,v)),((+,w))\} \mid \{v,w\} \in B(\Sigma_\infty)\} \\ &\quad \bigcup \{\{((-,v)),((-,w))\} \mid \{v,w\} \in B(R(\Sigma_\infty))\}, \end{aligned}$$

where $((+, v))$ denotes the equivalence class of $(+, v)$. Again, we write O for $((+, O)) = ((-, O)) \in V'(\Sigma_\infty)$ and call it the origin.

Metric on the scale-irregular pre-*abc*-gasket.

Metrics on the pre-*abc*-gaskets and *abc*-gaskets, i.e. for the case without scale-irregularity, are given in [17]. We extend the definition to allow for scale-irregularity.

Let $s \in \{+,-\}$, N and n be integers satisfying $n \geq N \geq 0$, for each $k \in \{n, n-1, \cdots, N+1\}$, m_k be an integer satisfying $0 \leq m_k < a_k + b_k + c_k$, and $i \in \{0,1,2\}$. Then the sequence

$$(s, m_n, m_{n-1}, \cdots, m_{N+1}, N, i) \tag{A.3}$$

determines an element $x \in V'(\Sigma_\infty)$ by the sequence of equivalence classes

$$\begin{aligned} &x = ((s,v)),\ v = ((m_n, v_{n-1})),\ v_{n-1} = ((m_{n-1}, v_{n-2})), \\ &v_{n-2} = ((m_{n-2}, v_{n-3})),\ \cdots,\ v_{N+1} = ((m_{N+1}, p_{N,i})). \end{aligned}$$

We take (A.3) as a representation of x and write

$$x = (s, m_n, m_{n-1}, \cdots, m_{N+1}, N, i) \in V'(\Sigma_\infty). \tag{A.4}$$

As a convention, we write $x = (s, n, i)$ with $N = n$ for $x = ((s, p_{ni}))$.

Fix $N \in \mathbf{Z}_+$, and define $G_{-N}(\Sigma_\infty) \subset V'(\Sigma_\infty)$ as a set of vertices $x \in V'(\Sigma_\infty)$ which has a representation (A.4). $P(\Sigma_0) = V(\Sigma_0)$ implies that each element $x \in V'(\Sigma_\infty)$ has a representation of the form (A.3) with $N = 0$, hence, $G_0(\Sigma_\infty) = V'(\Sigma_\infty)$. Also (A.2) implies

$$G_0(\Sigma_\infty) = V'(\Sigma_\infty) \supset G_{-1}(\Sigma_\infty) \supset G_{-2}(\Sigma_\infty) \supset \cdots .$$

For each $N \in \mathbf{Z}_+$ define a shift $\tilde{\tau}_N$ on the space of sequences in $\mathbf{N^3}$ by

$$\tilde{\tau}_N((\sigma_1, \sigma_2, \sigma_3, \cdots)) = (\sigma_{N+1}, \sigma_{N+2}, \sigma_{N+3}, \cdots). \tag{A.5}$$

Let $x \in G_0(\Sigma_\infty) \setminus \{O\}$. x may have more than one representations. However, (A.1) implies that for each fixed N,

$$x = (s, m_n, m_{n-1}, \cdots, m_{N+1}, N, i) = (s', m'_n, m'_{n-1}, \cdots, m'_{N+1}, N, i')$$

if and only if $(m_{N+1}, i) \overset{\sigma_{N+1}}{\sim} (m'_{N+1}, i')$ and $s = s'$, $m_k = m'_k$, $k = n, n-1, \cdots, N+2$. Hence there is an injection

$$\tilde{\tau}_N^* :\ G_0(\tilde{\tau}_N(\Sigma_\infty)) \to G_{-N}(\Sigma_\infty)$$

defined by

$$\tilde{\tau}_N^*(s, m_n, m_{n-1}, \cdots, m_1, 0, i) = (s, m_n, m_{n-1}, \cdots, m_1, N, i)\,.$$

Put

$$E_{-N}(\Sigma_\infty) = \{\{\tilde{\tau}_N^*(x), \tilde{\tau}_N^*(y)\} \mid \{x, y\} \in B'(\tilde{\tau}_N(\Sigma_\infty))\}.$$

Let x and y be elements of $G_0(\Sigma_\infty)$. Denote by $path(x, y)$ the collection of finite sequences

$$z = \{z_0 = x, z_1, \cdots, z_\kappa = y\}, \quad \text{for some } \kappa = \kappa_z \in \mathbf{Z}_+,$$

which has a property that for each $i = 0, 1, \cdots, \kappa_z - 1$, $\{z_i, z_{i+1}\} \in E_{-\nu_z(i)}(\Sigma_\infty)$ for some $\nu_z(i) \in \mathbf{Z}_+$.

For $z \in path(x, y)$ put

$$L(z) = \sum_{i=0}^{\kappa_z - 1} \prod_{n=1}^{\nu_z(i)} (\min\{a_n, b_n, c_n\} + 1).$$

where, $\sigma_n = (a_n, b_n, c_n)$, and $\Sigma_\infty = (\sigma_1, \sigma_2, \cdots)$. (By convention, we define the product in the definition of L to be 1, if $\nu_z(i) = 0$.) Then the metric $\tilde{d}(\Sigma_\infty)$ is defined by $\tilde{d}(\Sigma_\infty)(x, y) = \inf_{z \in path(x,y)} L(z)$. It is straightforward to see that $\tilde{d}$ is a metric, and, in particular,

$$\text{(A.6)} \qquad \tilde{d}(\Sigma_\infty)(x, y) = \prod_{n=1}^{N} (\min\{a_n, b_n, c_n\} + 1)\,, \quad \{x, y\} \in E_{-N}(\Sigma_\infty).$$

In considering (anisotropic) random walks on $\tilde{H}'_\infty(\Sigma_\infty)$, it is convenient to have the notion of vertex types and edge types [16, 17]. One sees [17] that a vertex $x \in V'(\Sigma_\infty)$ is classified into 6 types; A, B, C, D, E, F, and an edge as an ordered pair of vertices is classified into 18 types; X_y with $X = A, B, C$ and $y = p, q, r, s$, X_y with $X = E, F$ and $y = p, r$, and D_p, D_q, by the following rule.

(1) The origin O is of type A.

(2) A vertex which has two representations of the forms $(s, m_n, \cdots, m_1, 0, i)$ and $(s, m_n, \cdots, m'_1, 0, i')$ for some s, n, i, i', and m_k's, satisfying $(m_1, i) \overset{\sigma}{\sim} (m'_1, i')$ is of type A, B, or C, if $(i, i') = (1, 0)$, $(0, 2)$, or $(2, 1)$, respectively.

(3) Any other vertex with a representation of the form $(s, m_n, \cdots, m_1, 0, i)$ is of type D, E, or F, if $i = 2, 0$, or 1, respectively.

(4) Let $\{x, y\} \in B'(\Sigma_\infty)$. Then x and y have representations of the form $x = (s, m_n, \cdots, m_1, 0, i)$ and $y = (s, m_n, \cdots, m_1, 0, i')$. If x is of type A then (x, y) as an ordered pair is of type A_p, A_q, A_r, or A_s, if $(i, i') = (0, 2)$, $(1, 2)$, $(0, 1)$, or $(1, 0)$, respectively. If x is of type B then (x, y) is of type B_p, B_q, B_r, or B_s, if $(i, i') = (0, 2)$, $(2, 1)$, $(0, 1)$, or $(2, 0)$, respectively. If x is of type C then (x, y) is of type C_p, C_q, C_r, or C_s, if $(i, i') = (1, 2)$, $(2, 0)$, $(1, 0)$, or $(2, 1)$, respectively. If x is of type D then (x, y) is of type D_p or D_q, if $(i, i') = (2, 0)$ or $(2, 1)$, respectively. If x is of type E then (x, y) is of type E_p or E_r, if $(i, i') = (0, 2)$ or $(0, 1)$, respectively. If x is of type F then (x, y) is of type F_p or F_r, if $(i, i') = (1, 2)$ or $(1, 0)$, respectively.

Inspired by the Sierpiński gasket, we call the edges of types X_r with $X = A, B, C, E, F$, and A_s, the 'horizontal' edges.

The scale-irregular abc-gasket.

Fix $S:\ \mathbf{Z}\to\mathbf{N^3}$. For $N\in\mathbf{N}$, define $S_N=(S(N),S(N-1),S(N-2),\cdots)$ (note that the numbers are now in decreasing orders), and put $G_N=G_0(S_N)$. G_N has a graph structure with the edge set

$$E_N \stackrel{\rm def}{=} B'(S_N). \tag{A.7}$$

Define a metric d_N on G_N by

$$d_N(x,y) = \tilde{d}(S_N)(x,y)\prod_{n=1}^{N}\frac{1}{\min\{a_n,b_n,c_n\}+1}, \quad x,y\in G_N, \tag{A.8}$$

where we wrote $S(n)=(a_n,b_n,c_n)$. (We define the product to be 1, if $N=0$.)

For each pair of non-negative integers N, N', satisfying $N\le N'$, there is an injection from $G_N=G_0(S_N)$ to $G_{N'}=G_0(S_{N'})$ defined by

$$(s,m_n,m_{n-1},\cdots,m_0,0,i)\mapsto(s,m_n,m_{n-1},\cdots,m_0,N'-N,i). \tag{A.9}$$

We identify G_N with a subset of $G_{N'}$ with this injection;

$$G_0(S_0)\subset G_1\subset G_2\subset G_3\subset\cdots.$$

Let $G_\infty=\bigcup_{N\in\mathbf{Z}_+} G_N$ with this identification assumed. Using (A.8) and (A.6), we see that if $N'\ge N$

$$d_{N'}(x,y)=d_N(x,y),\quad x,y\in G_N\subset G_{N'}. \tag{A.10}$$

For any x and y in G_∞, define $d(x,y)$ as follows. There exists $N\in\mathbf{Z}_+$ such that $x,y\in G_N$. Then define $d(x,y)=d_N(x,y)$. With (A.10) we see that d is a well-defined metric.

The scale-irregular abc-gasket G is the completion of G_∞ by d.

A subset $G_N\subset G$ has a graph structure with the vertex set G_N and the edge set E_N given by (A.7). We use the notation $H_N=(G_N,E_N)$ $(=\tilde{H}'_\infty(S_N))$ to refer to the graph structure, and call it a scale-irregular pre-abc-gasket (of scale N). For $x\in G_N$, we call a vertex $y\in G_N$ an N-neighbor (of x) if $\{x,y\}\in E_N$. We use, for H_N, the notion of vertex types and edge types, A_p, A_r, etc., and the terminology 'horizontal (edge)', in accordance with the corresponding notations for $\tilde{H}'_\infty(\Sigma_\infty)$.

If S has a property $b_N=c_N$ for all N, where $S(N)=(a_N,b_N,c_N)$, we call G the scale-irregular abb-gasket and G_N the scale-irregular pre-abb-gasket. For a scale-irregular pre-abb-gasket, we identify the types $A_p=A_q$, $A_r=A_s$, $B_s=B_q$, $C=B$, $D_p=D_q$, and $F=E$. Hence for a scale-irregular abb-gasket, there are 4 vertex types A, B, D, E, and 8 edge types A_p, A_r, B_p, B_q, B_r, D_p, E_p, E_r.

We also use the notion of a 'unit triangle'. By a unit triangle of H_N (or a unit triangle of scale N) we mean a closure (in G with respect to the metric d) of a set

$$\bigcup_{N' \geq N} \{(s, m_n, \cdots, m_0, m_{-1}, \cdots, m_{-N'+N}, 0, i) \in G_0(S_{N'}) \mid i = 0, 1, 2,$$
$$m_{-k} = 0, 1, 2, \cdots, a_{N+k} + b_{N+k} + c_{N+k} - 1,\ k = 1, 2, \cdots, N' - N\} \ (\subset G)$$

for some fixed s, m_n, $\cdots$, m_0. $(s, m_n, \cdots, m_0, 0, i) \in G_0(S_N) = G_N$, $i = 0, 1, 2$, are defined to be the three vertices of the triangle.

If S is a constant map defined by $S(0) = (a, b, c)$, then G is an *abc*-gasket [15, 16, 17]. If, furthermore, $a = b = c = 1$, then G is the Sierpiński gasket.

Remark. Assume that S is a bounded map. As in [17], $(\min\{a_n, b_n, c_n\} + 1)$, $n \in \mathbf{Z}$, in the definitions of metrics can be replaced by ℓ_n, $n \in \mathbf{Z}$, satisfying $\ell_n \leq (\min\{a_n, b_n, c_n\} + 1)$, $n \in \mathbf{Z}$, and $\inf_n \ell_n > 1$. The first condition implies (A.6), with $(\min\{a_n, b_n, c_n\} + 1)$ replaced by ℓ_n. The second condition with the boundedness of the map S implies that there exists $C > 0$ such that if x and y is in a unit triangle of G_N then $d(x, y) \leq C \prod_{n=1}^{N} {\ell_n}^{-1}$.

Appendix B. Decay estimate from non-linear recursion relations.

The Lemma below gives a mild decay estimate from a non-linear recursion relation. We apply the Lemma to prove a Theorem which states a sharp decay estimate from another recursion relation with more involved assumptions.

Lemma B.1. *Let $\{w_n,\ n \in \mathbf{Z}_+\}$ be a sequence in $[0, 1]$ satisfying $\sum_n w_n < \infty$, and $\{a_n,\ n \in \mathbf{Z}_+\}$ a sequence satisfying $D \stackrel{\text{def}}{=} \inf_n a_n > 1$ and $\sup_n a_n < \infty$. For each $k \in \mathbf{Z}_+$ define a sequence $\{x_{k,n},\ n = k, k-1, \cdots, 0\}$ by a recursion relation*

$$x_{k,n} = (1 - w_{n+1})\, x_{k,n+1}^{a_{n+1}} + w_{n+1}\,, \quad n = k-1, k-2, \cdots, 0\,,$$

with initial condition $x_{k,k}$ satisfying $0 \leq x_{k,k} \leq 1$. If

$$\lim_{k \to \infty,\ x_{k,k} \neq 0} (-\log x_{k,k}) \prod_{\ell=0}^{k} a_\ell = \infty$$

holds, then there exist positive constants C_1 *and* k_1 *(independent of* n *and* k*) such that*

$$x_{k,n} \leq C_1 \sup_{\ell \geq n} w_\ell + \exp\left(-D^{f_k - n - 1}\right) , \quad 0 \leq n \leq f_k ,\ k \geq k_1 ,$$

where

$$f_k = \sup\{n \leq k-1 \mid \sqrt{D}\,(-\log x_{k,k}) \prod_{\ell=n+2}^{k} a_\ell > 1\} + 1 ,$$

with a convention $\prod_{\ell=k+1}^{k} a_\ell = 1$*, and* $f_k = k$ *if* $x_{k,k} = 0$*.*

Proof. Put $C_2 = \sup_n \exp(a_n)$. $\sum_n w_n < \infty$ and $D > 1$ imply that there exists a constant n_1 such that $\prod_{\ell \geq n} (1 - C_2\, w_\ell) \geq \sqrt{D}^{-1}$, $n \geq n_1$. By assumption, $\lim_{k\to\infty} f_k = \infty$, hence there exists a constant k_1 such that $f_k \geq n_1 - 1$, $k \geq k_1$. Let $k \geq k_1$ in the following. We first prove that

$$x_{k,f_k} \leq \exp\left(-D^{-1}\right) , \quad k \geq k_1 . \tag{B.1}$$

If $x_{k,k} = 0$ then (B.1) directly follows, so we assume $x_{k,k} > 0$. Put $u_{k,n} = -\log x_{k,n}$. The assumptions and the recursion relation imply $0 < x_{k,n} \leq 1$ for all n, hence $u_{k,n}$ exist and are non-negative. Furthermore,

$$u_{k,n} = a_{n+1} u_{k,n+1} - \log\left(1 + (x_{k,n+1}^{-a_{n+1}} - 1)\, w_{n+1}\right) \leq a_{n+1} u_{k,n+1} ,$$

which implies

$$u_{k,n} \leq u_{k,k} \left(\prod_{\ell=n+1}^{k} a_\ell \right) , \quad 0 \leq n \leq k . \tag{B.2}$$

The definition of f_k implies $f_k \leq k$ and the following three inequalities;

$$u_{k,k} \geq \sqrt{D}^{-1} , \quad \text{if } f_k = k , \tag{B.3}$$

$$\sqrt{D}\, u_{k,k} \left(\prod_{\ell=f_k+1}^{k} a_\ell \right) > 1 , \tag{B.4}$$

$$\sqrt{D}\, u_{k,k} \left(\prod_{\ell=n+2}^{k} a_\ell \right) \leq 1 , \quad f_k \leq n \leq k-1 . \tag{B.5}$$

The estimates (B.2), (B.5), and $D > 1$ imply $u_{k,n+1} < 1$, $f_k \leq n \leq k-1$, which, together with the recursion relation implies

$$\begin{aligned} u_{k,n} &= a_{n+1} u_{k,n+1} - \log(1 + \exp(u_{k,n+1} a_{n+1})\, w_{n+1} (1 - \exp(-u_{k,n+1} a_{n+1}))) \\ &\geq a_{n+1} u_{k,n+1} - \log\left(1 + C_2\, w_{n+1}\, u_{k,n+1}\, a_{n+1}\right) \\ &\geq a_{n+1} (1 - C_2 w_{n+1})\, u_{k,n+1} , \qquad f_k \leq n \leq k-1 . \end{aligned}$$

This with $k \geq k_1$ and (B.4) implies $u_{k,f_k} \geq u_{k,k}(\prod_{\ell=f_k+1}^{k} a_\ell)(\prod_{\ell=f_k+1}^{k}(1 - C_2 w_\ell)) \geq D^{-1}$, if $f_k \leq k-1$. If $f_k \geq k$ then $f_k = k$, hence (B.3) implies $u_{k,f_k} > D^{-1}$. Therefore we have (B.1).

Put $v_n = \sup_{\ell \geq n} w_\ell$. $\{v_n\}$ is decreasing, bounded above by 1, and $\lim_{n\to\infty} v_n = 0$. Define a sequence $\{z_{k,n},\ n = f_k, f_k - 1, \cdots, 0\}$ by $z_{k,n} = {z_{k,n+1}}^D + v_{n+1}$, $0 \leq n \leq f_k - 1$, and $z_{k,f_k} = \exp(-D^{-1})$. Then

$$x_{k,n} \leq z_{k,n}\,, \quad 0 \leq n \leq f_k\,. \tag{B.6}$$

Put

$$z_{k,n} = \exp\left(-D^{f_k-n-1}\right) + v_n\, r_{k,n}\,. \tag{B.7}$$

Taylor's formula implies $(\alpha+\beta)^D - \alpha^D \leq D\,\beta\,(\alpha+\beta)^{D-1}$, for any $\alpha > 0$, $\beta > 0$, and $D > 1$. If we put $\alpha = \exp\left(-D^{f_k-n-2}\right)$ and $\beta = v_{n+1}\, r_{k,n+1}$ we have, with ${z_{k,n+1}}^D = z_{k,n} - v_{n+1}$,

$$\begin{aligned} r_{k,n} &= v_n^{-1}\left(z_{k,n} - \exp\left(-D^{f_k-n-1}\right)\right) \\ &\leq \frac{v_{n+1}}{v_n}\, r_{k,n+1}\, D \left(\exp\left(-D^{f_k-n-2}\right) + v_{n+1}\, r_{k,n+1}\right)^{D-1} + \frac{v_{n+1}}{v_n}\,, \end{aligned}$$

which, with $v_{n+1} = \sup_{\ell \geq n+1} w_\ell \leq v_n$ and $D > 1$,

$$r_{k,n} \leq r_{k,n+1}\, D\,(e^{-1} + v_{n+1}\, r_{k,n+1})^{D-1} + 1\,, \quad 0 \leq n \leq f_k - 2\,. \tag{B.8}$$

Put $\rho = \frac{1}{2}(1 + D\,e^{-D+1})$. $D > 1$ implies $0 < D\,e^{-D+1} < \rho < 1$. Therefore there exists a constant k_2 defined by

$$k_2 = \inf\{n \geq 0 \mid D\,(e^{-1} + v_n\,(1-\rho)^{-1})^{D-1} < \rho\}\,.$$

Monotonicity of $\{v_n\}$ implies $D\,(e^{-1} + v_n\,(1-\rho)^{-1})^{D-1} < \rho$, $n \geq k_2$. If $f_k \geq k_2+1$ then we can prove by induction that $r_{k,n} \leq (1-\rho)^{-1}$, $k_2 \leq n \leq f_k$. In fact, we explicitly have $r_{k,f_k} = 0$ and $r_{k,f_k-1} = v_{f_k}/v_{f_k-1} \leq 1$. (The latter holds, because (B.7) implies $z_{k,f_k-1} = e^{-1} + v_{f_k-1}\, r_{k,f_k-1}$, while $z_{k,f_k-1} = {z_{k,f_k}}^D + v_{f_k} = e^{-1} + v_{f_k}$.) If $r_{k,n+1} \leq (1-\rho)^{-1}$ holds for some n with $k_2 \leq n \leq f_k - 2$, then (B.8) and the definition of k_2 implies $r_{k,n} \leq (1-\rho)^{-1}\rho + 1 = (1-\rho)^{-1}$. Thus if $f_k \geq k_2+1$, $r_{k,n}$ for $k_2 \leq n \leq f_k$ are bounded by a constant independent of n and k. k_2 is independent of n and k. Therefore $r_{k,n}$ for $0 \leq n \leq k_2$ are bounded by a constant independent of n and k. If $f_k < k_2+1$, similar argument shows, with $r_{k,f_k} = 0$, that $r_{k,n}$ for $0 \leq n \leq f_k$ are bounded by a finite number independent of n and k. This with (B.6) and (B.7) implies the statement. □

Theorem B.2. *Let $\{w_n\}$, $\{w'_n\}$, $n \in \mathbf{Z}_+$, be sequences in $[0,1]$ satisfying*

$$\max\{w_n,\ w'_n\} \le C_w \delta^{-n}\,, \quad n \in \mathbf{Z}_+\,,$$

for positive constants (independent of n) C_w and $\delta > 1$. Also let $\{a_n,\ n \in \mathbf{Z}_+\}$ be a sequence satisfying $\inf_n a_n \ge 2$ and $\sup_n a_n < \infty$. For each $k \in \mathbf{Z}_+$ consider a sequence in $[0,1]^2$

$$\{(x_{k,n}, y_{k,n})\,, \quad n = k, k-1, \cdots, 0\} \ \subset [0,1]^2\,,$$

and assume that it satisfies a recursive inequality

$$\begin{aligned} x_{k,n} &\le x_{k,n+1}^{a_{n+1}} + w_{n+1}\ \min\{1 - x_{k,n+1}^{a_{n+1}}\,,\ y_{k,n+1}\}\,, \\ y_{k,n} &\le x_{k,n+1} + w'_{n+1}\, y_{k,n+1}\,, \quad n = k-1, k-2, \cdots, 0\,. \end{aligned}$$

If

$$\liminf_{k\to\infty,\ x_{k,k}\ne 0} \left\{ (-\log x_{k,k}) \prod_{\ell=0}^{k} a_\ell \right\}^{1/k} > 1 \tag{B.9}$$

holds, then there exist positive constants C_1 and C_2 (independent of k) such that

$$\max\{x_{k,0},\ y_{k,0}\} \le C_1\, \exp(-C_2\, k^2)\,, \quad k \in \mathbf{Z}_+\,.$$

Proof. Define $\{\tilde{x}_{k,n},\ n = k, k-1, \cdots, 0\}$ by $\tilde{x}_{k,k} = x_{k,k}$ and

$$\tilde{x}_{k,n} = (1 - w_{n+1})\, \tilde{x}_{k,n+1}^{a_{n+1}} + w_{n+1}\,, \quad n = k-1, k-2, \cdots, 0\,,$$

Then the recursion relation for $x_{k,n}$ and the assumption $y_{k,n} \le 1$ imply

$$x_{k,n} \le \tilde{x}_{k,n}\,, \quad 0 \le n \le k,\ k \ge 0\,.$$

$\{\tilde{x}_{k,n}\}$ satisfies all the assumptions of Lemma B.1 with $D = 2$, hence there exist positive constants C_3 and k_1 (independent of n and k) such that

$$x_{k,n} \le C_3 \sup_{\ell \ge n} w_\ell + \exp\left(-2^{f_k - n - 1}\right)\,, \quad 0 \le n \le f_k,\ k \ge k_1\,, \tag{B.10}$$

where

$$f_k = \sup\{n \le k-1 \mid \sqrt{2}\,(-\log x_{k,k}) \prod_{\ell=n+2}^{k} a_\ell > 1\} + 1\,. \tag{B.11}$$

Since (B.9) implies $\lim_{k\to\infty} f_k = \infty$, there exists a constant $k_2 \ge k_1$ such that for $0 \le n \le f_k/2$ and $k \ge k_2$,

$$\exp\left(-2^{f_k - n - 1}\right) \le \exp\left(-2^{f_k/2 - 1}\right) \le C_w \delta^{-f_k/2} \le C_w \delta^{-n}\,.$$

This with (B.10) implies $x_{k,n} \le (C_3+1)\, C_w\, \delta^{-n}$, $0 \le n \le f_k/2$, $k \ge k_2$. Applying this estimate to the original recursion relations and using $w_n \le C_w \delta^{-n}$, $w'_n \le C_w \delta^{-n}$, and $a_n \ge 2$, we have

$$\begin{aligned} x_{k,n} &\le C_w\, \delta^{-n-1} \left((C_3+1)\, x_{k,n+1} + y_{k,n+1} \right), \\ y_{k,n} &\le x_{k,n+1} + C_w\, \delta^{-n-1}\, y_{k,n+1}\,, \quad 0 \le n \le f_k/2\,,\ k \ge k_2\,. \end{aligned}$$

Iterating once, we find

$$\max\{x_{k,n},\ y_{k,n}\} \le C_5\, \delta^{-n-1} \max\{x_{k,n+2},\ y_{k,n+2}\}\,, \quad 0 \le n \le f_k/2-1\,,\ k \ge k_2\,,$$

where C_5 is a positive constant independent of n and k. Iterating this $[f_k/4]$ times, where $[x]$ is the largest integer not exceeding x, and using $x_{k,n} \le 1$, $y_{k,n} \le 1$, we find

$$\text{(B.12)} \quad \max\{x_{k,0},\ y_{k,0}\} \le \exp\left\{ \left[\frac{f_k}{4}\right] (\log C_5) - \left[\frac{f_k}{4}\right]^2 (\log \delta) \right\}, \quad k \ge k_2\,.$$

The assumption (B.9) implies that there exist positive constants $k_3 \ge k_2$ and $\delta' > 1$ (independent of k) such that $(-\log x_{k,k}) \prod_{\ell=0}^{k} a_\ell > \delta'^k$, $k \ge k_3$. The definition (B.11) then implies $f_k \ge \min\left\{ \dfrac{\log \delta'}{\log \sup_\ell a_\ell}\, k-1,\ k \right\}$, $k \ge k_3$. Applying this to (B.12), increasing constants for terms with $k < k_3$ if necessary, we have the statement. □

Appendix C. Products of matrices with positive elements.

We present an elementary theorem on the existence of a limit of normalized products of matrices with positive elements. We assume no relation among matrices in the product, such as commutativity or stationarity. We also allow the infimums of some components to be zero.

Theorem C.1. *Let d and q be positive integers, $\mathcal{E} \stackrel{\text{def}}{=} \{1,2,\cdots,d\}$, and $\{A_N,\ N=1,2,3,\cdots\}$ be a sequence of d-dimensional matrices whose elements are positive and bounded, satisfying $\inf_{N,i,j} (A_N A_{N+1} \cdots A_{N+q-1})_{ij} > 0$. Then for $i \in \mathcal{E}$ and $j \in \mathcal{E}$, $\gamma_i \stackrel{\text{def}}{=} \lim_{N\to\infty} \dfrac{(A_1 \cdots A_N)_{ij}}{(A_1 \cdots A_N)_{1j}}$ exists, positive, and is independent of j.*

Proof. For $N > n \geq 0$ define

$$B_{nN} \stackrel{\text{def}}{=} A_{n+1}A_{n+2}\cdots A_N$$

and put

$$\gamma_{Nij} \stackrel{\text{def}}{=} \frac{B_{0Nij}}{B_{0N1j}}\,, \quad i \in \mathcal{E},\ j \in \mathcal{E}\,.$$

The elementwise positivity of A_{N+1} and B_{nN} imply for each i that $\{\min_k \gamma_{Nik}\}$ is increasing and $\{\max_k \gamma_{Nik}\}$ is decreasing in N, in particular, the sequence $\{\gamma_{Nij},\ N = 1, 2, \cdots\}$ is bounded. Therefore, for each i and j, and for any subsequence of positive integers there exists a further subsequence $\{a_N\}$ such that the limit

(C.1) $$\gamma_{ij}^{(a)} \stackrel{\text{def}}{=} \lim_{N\to\infty} \gamma_{a_N,ij} > 0\,.$$

exists and is positive.

For $0 < n < N$ and $i \in \mathcal{E}$, $j \in \mathcal{E}$, put

$$p_{nNij} \stackrel{\text{def}}{=} \frac{B_{0n1i}\,B_{nNij}}{B_{0N1j}}\,.$$

The definition and the elementwise positivity of B_{nNij} imply, for $0 < n < N$, $i, j \in \mathcal{E}$,

(C.2) $$0 < p_{nNij} < 1\,, \quad \sum_{k\in\mathcal{E}} p_{nNkj} = 1\,, \quad \gamma_{Nij} = \sum_{k\in\mathcal{E}} \gamma_{nik}\, p_{nNkj}\,.$$

We prove a couple of Lemma for p_{nNkj}.

Lemma C.2. *Fix $\{a_N\}$, and let $\gamma_{ij}^{(a)}$ be as above. If for every $i, j \in \mathcal{E}$ either $\inf_{n>0} \inf_{N>n+q} p_{nNij} > 0$ or $\inf_{n>0} \inf_{N>n+q} p_{nNji} > 0$ hold, then for every $i \in \mathcal{E}$, $\gamma_{ij}^{(a)}$ is independent of j.*

Proof. Put $n = a_M$ and $N = a_{M'}$ in (C.2). We see from (C.1) and (C.2) that for each $\epsilon > 0$ there is an integer M_0 such that for any integers M, M' satisfying $M' > M > M_0$ we have

(C.3) $$\left|\sum_{k\in\mathcal{E}} (\gamma_{ij}^{(a)} - \gamma_{ik}^{(a)}) p_{a_M,a_{M'},kj}\right| < \epsilon\,.$$

Now suppose that the Lemma is wrong; $\gamma_{ik_1}^{(a)} < \gamma_{ik_2}^{(a)}$ and $\gamma_{ik_1}^{(a)} \leq \gamma_{ij}^{(a)} \leq \gamma_{ik_2}^{(a)}$, $j \in \mathcal{E}$. If we put $j = k_2$ in (C.3) and keep $k = k_1$ term in the summation we have $\epsilon > (\gamma_{ik_2}^{(a)} - \gamma_{ik_1}^{(a)}) p_{a_M,a_{M'},k_2,k_1}$, while if we put $j = k_1$ and keep $k = k_2$ term we have $\epsilon > (\gamma_{ik_2}^{(a)} - \gamma_{ik_1}^{(a)}) p_{a_M,a_{M'},k_1,k_2}$. Since $\epsilon > 0$ is arbitrary, these inequalities contradicts the assumption of the Lemma. □

Lemma C.3.

$$\inf_{n>0}\inf_{N>n+q} p_{nNij} > 0\,, \quad i\in\mathcal{E},\ j\in\mathcal{E}\,.$$

Proof. Note that each p_{nNij} is positive by (C.2). Therefore it is sufficient to consider the cases where N and n are sufficiently large. For sufficiently large N,

$$p_{1Nij} \ge \frac{A_{11i}}{\sum_{k_1} A_{11k_1}} \frac{\sum_{k_2} A_{2,i,k_2}\, B_{2,N,k_2,j}}{\max_{k_1}\sum_{k_2} A_{2,k_1,k_2}\, B_{2,N,k_2,j}} \ge \frac{A_{11i}}{\sum_{k_1} A_{11k_1}} \min_{k_2} \frac{A_{2,i,k_2}}{\max_{k_1} A_{2,k_1,k_2}}\,,$$

where we used an inequality among non-negative numbers a_i, b_i, c_i, $i\in\mathcal{E}$; $\dfrac{\sum a_i c_i}{\sum b_i c_i} \ge \min_i \dfrac{a_i}{b_i}$. Hence $\inf_{N>1} p_{1Nij} > 0$. If we prove $\inf_{n>0}\inf_{N>n+q} \dfrac{p_{nNij}}{p_{1Nij}} > 0$, then the Lemma is proved. For sufficiently large N and n with $N-q>n$,

$$\frac{p_{nNij}}{p_{1Nij}} \ge \frac{\min_k A_{11k}B_{1nki}B_{nNij}}{\max_k A_{11i}B_{1nik}B_{nNkj}} \ge \min_{\{k_i\}} \frac{\min_k A_{11k}A_{2kk_1}B_{n-q,n,k_2i}B_{n,n+q,ik_3}}{A_{11i}A_{2ik_1}\max_k B_{n-q,n,k_2k}B_{n,n+q,kk_3}}\,.$$

Taking the infimum of both sides with respect to N and n, we see, with the assumptions of Theorem C.1, $\inf_{n>0}\inf_{N>n+q} \dfrac{p_{nNij}}{p_{1Nij}} > 0$. □

Let us continue the proof of the Theorem. Lemma C.2 and Lemma C.3 imply that $\gamma^{(a)}_{ij}$ of (C.1) is independent of j. Fix $i\in\mathcal{E}$, and consider two subsequences of positive integers. There are subsequences, $\{a_N\}$ and $\{b_N\}$, for each of the subsequences respectively, such that the limits

(C.4) $$\gamma^{(a)}_i \stackrel{\text{def}}{=} \lim_{N\to\infty}\gamma_{a_N,ij} > 0\,, \quad\text{and}\quad \gamma^{(b)}_i \stackrel{\text{def}}{=} \lim_{N\to\infty}\gamma_{b_N,ij} > 0\,,$$

exist, positive, and are independent of j. Put $n=b_M$ and $N=a_{M'}$ in (C.2);

(C.5) $$\gamma_{a_{M'},ij} = \sum_{k\in\mathcal{E}} \gamma_{b_M,ik}\, p_{b_M,a_{M'},kj}\,, \quad a_{M'} > b_M\,,\ i\in\mathcal{E},\ j\in\mathcal{E}\,.$$

The equations (C.4), (C.5), and (C.2) imply that for any positive ϵ there exists an integer N_0 such that if $a_{M'} > b_M > N_0$ hold, then

$$\left|\gamma^{(a)}_i - \gamma^{(b)}_i\right| = \left|\gamma^{(a)}_i - \left(\gamma^{(b)}_i \sum_{k\in\mathcal{E}} p_{b_M,a_{M'},kj}\right)\right| < \epsilon\,,\ j\in\mathcal{E}\,.$$

Hence $\gamma^{(a)}_i = \gamma^{(b)}_i$, which implies that the limit is independent of subsequences. Positivity of the limit also follows from (C.4). □

Appendix D. Estimates on generating function.

We give an explicit formula for the generating function

$$g_{a,b,i}(w,h) = \tilde{F}_i(a,b,w;\tilde{\Pi}(w) + w\,h)\,,$$

introduced in Section 3. As mentioned in the Introduction, an algebraic part of our proof of estimates (Proposition 3.1) is computer-aided, because it requires a routine work of lengthy calculations. A complete proof the formula is long and it would not be worthwhile to describe the details of the calculations. But it may be reasonable to specify which part of our estimates are computer-aided. In this Appendix, we summarize the notations we used for the computer calculations, and the results obtained by using REDUCE on computers. The derivations are basically as in [16, 17], to which we refer for further explanation.

For each $i \in \mathcal{E}$, $g_{a,b,i}$ has an expression $g_{a,b,i}(w,h) = Num_i/Den_i$, where

$$Den_i = \det W + \sum_{\alpha=1}^{3} \det \left(\begin{array}{c|c} 0 & O_\alpha \\ \hline I_{\alpha,i} & W \end{array} \right)\,,$$

$$Num_i = -\det \left(\begin{array}{c|c} 0 & O'_i \\ \hline I'_i & W \end{array} \right)\,.$$

The definitions of O_α, $I_{\alpha,i}$, O'_i I'_i, and W, in the above equations are as follows. Put $Z(i) = \Pi(w)_i + w\,S^{-1}(w)_{ii}\,h_i$, $i \in \mathcal{E}$. Then $O_1 = (Z(Ar), 0, 0, 0, 0, Z(Bq))$, $O_2 = (0, Z(Ar), Z(Bq), 0, 0, 0)$, and $O_3 = (0, 0, 0, Z(Bp), Z(Bp), 0)$. For $X \in \{A, B, D, E\}$ and $t \in \{p, q, r\}$, we write Xt to specify an element in $\mathcal{E}$, with an obvious rule. With this convention, $O'_{Xp} = O_3$, $X \in \{A, B, E\}$, $O'_{Dq} = O_1$, and $O'_i = O_2$, otherwise. $I_{1,Dq} = 0$, otherwise $I_{1,Xt} = {}^t(Z(Xr), 0, 0, 0, 0, Z(Xp))$. $I_{2,At} = {}^t(0, Z(Ar), Z(Ap), 0, 0, 0)$, otherwise $I_{2,Xt} = 0$. $I_{3,Xt} = 0$, if $X \in \{A, E\}$, otherwise $I_{3,Xt} = {}^t(0, 0, 0, Z(Xq), Z(Xq), 0)$. $I'_{Xq} = {}^tI_{3,Xq}$, while for $t \neq q$, $I'_{Xt} = I_{1,Xt}$. W is a 6 dimensional matrix given by

$$W = \mathrm{I} - \begin{pmatrix} W_\alpha(1) & W_\beta(1) & 0 & 0 & 0 & Z(Bq) \\ W_\beta(1) & W_\alpha(1) & Z(Bq) & 0 & 0 & 0 \\ 0 & Z(Ap) & W_\alpha(2) & W'_\beta(2) & 0 & 0 \\ 0 & 0 & W_\beta(2) & W_{\alpha'}(2) & Z(Br) & 0 \\ 0 & 0 & 0 & Z(Br) & W_{\alpha'}(2) & W_\beta(2) \\ Z(Ap) & 0 & 0 & 0 & W_{\beta'}(2) & W_\alpha(2) \end{pmatrix}.$$

For $j = 1, 2$, $W_\alpha(j) + \alpha'(j) = W_{\alpha'}(j) + \alpha(j) = 1 - \bar{\beta}(j)\,\dfrac{\Delta(j)_{n(j)}}{\Delta(j)_{n(j)-1}}$, $W_\beta(j) = \dfrac{\bar{\beta}(j)}{\Delta(j)_{n(j)-1}} \left(\dfrac{\beta(j)}{\beta'(j)}\right)^{n(j)/2}$, $W_{\beta'}(j) = \dfrac{\bar{\beta}(j)}{\Delta(j)_{n(j)-1}} \left(\dfrac{\beta'(j)}{\beta(j)}\right)^{n(j)/2}$, where $\bar{\beta}(j) =$

$\sqrt{\beta(j)\,\beta'(j)}$, $2\bar\alpha(j) = \alpha(j) + \alpha'(j)$, and $\Delta(j)_{n(j)} = \dfrac{x_+(j)^{n(j)+1} - x_-(j)^{n(j)+1}}{x_+(j) - x_-(j)}$, $x_\pm(j) = 1 + \delta(j) \pm \sqrt{\delta(j)\,(2+\delta(j))}$, $\delta(j) = \dfrac{1 - 2\bar\alpha(j) - 2\bar\beta(j)}{2\bar\beta(j)}$. Finally, $n(j)$, $\alpha(j)$, $\alpha'(j)$, $\beta(j)$, and $\beta'(j)$ are given by $n(1) = a-1$, $n(2) = b-1$, $\alpha(1) = \alpha'(1) = Z(Ap)\,Z(Dq)$, $\alpha(2) = Z(Br)\,Z(Er)$, $\alpha'(2) = Z(Bq)\,Z(Ep)$, $\beta(1) = \beta'(1) = Z(Ar) + Z(Ap)\,Z(Dq)$, $\beta(2) = Z(Bp) + Z(Br)\,Z(Ep)$, $\beta'(2) = Z(Bq) + Z(Bq)\,Z(Er)$.

With these explicit formula, we obtain the following order estimate. Define C_i, $i \in \mathcal{E}$, by $C_{Ar} = 1/2$, $C_{Br} = C_{Er} = 1$, and $C_i = 0$, otherwise.

Proposition D.1. *For all $i \in \mathcal{E}$, $w^{-3}\,Den_i$ and $w^{-4}\,(Num_i - C_i\,Den_i)$ are rational in w and h, analytic at $w = h = 0$.*

We also find by REDUCE calculation that $O(w^3)$ terms in Den_i do not vanish;

(D.1) $$\lim_{w\to 0} w^{-3}\,Den_i \neq 0\,, \quad i \in \mathcal{E}.$$

The matrix $\tilde A(a,b,w)$ defined in Section 3 is rational in w, and has no poles in $w \ge 0$. The explicit form of $\tilde A(a,b,w=0)$ given below is obtained by explicit calculation of the first derivatives of $\tilde F$ given above, using REDUCE.

Define, for notational simplicity, a matrix $M(a,b)$ by $M(a,b)_{ij} = (b+2)^2(a+1)^{-1}\tilde A(a,b,w=0)_{ij}$, $i,j \in \mathcal{E}$, and put $B_2 = b+2$. Then

$$M(a,b) = \left[\begin{array}{cccccc}
2B_2 & 0 & (b^3+9b^2+14b+12)/12 & b(b+4)(b-1)/6 & b(b+5)(b+1)/12 & 0 \\
(aB_2+b)B_2 & (a+1)B_2^2 & (b^2+4b+6)(b-1)/6 & b(b^2+6b+11)/3 & b(2b^2+9b+13)/12 & (a-1)B_2^2 \\
2B_2 & 0 & (b^3+9b^2+20b+24)/6 & b(b+4)(b-1)/3 & b(b+5)(b+1)/6 & 0 \\
0 & 0 & b(b+4)(b-1)/6 & (b^3+9b^2+14b+12)/3 & b(b+5)(b+1)/6 & 0 \\
2(aB_2-1)B_2 & 2aB_2^2 & b(b^2+4b+7) & 2b(b^2+3b+5) & (2b^2+5b+8)(b+1)/2 & 2(a-1)B_2^2 \\
0 & 0 & b(b+1)B_2/4 & b(b+1)B_2/2 & b(b+1)B_2/4 & B_2^2 \\
2B_2 & 0 & (b^3+9b^2+14b+12)/6 & b(b+4)(b-1)/3 & b(b+5)(b+1)/6 & 0 \\
2(aB_2-1)B_2 & 2aB_2^2 & (b^2+4b+6)(b-1)/3 & 2b(b^2+6b+11)/3 & b(2b^2+9b+13)/6 & 2(a-1)B_2^2
\end{array}\right]$$

$$\left.\begin{matrix} (b^2+10b+12)(b-1)/12 & b(b+7)(b-1)/12 \\ (b^2+4b+6)(b-1)/6 & (2b^2+11b+24)(b-1)/12 \\ (b^2+10b+12)(b-1)/6 & b(b+7)(b-1)/6 \\ b(b+4)(b-1)/6 & b(b+7)(b-1)/6 \\ (b^2+2b+2)(b-1) & (2b^2+3b+8)(b-1)/2 \\ b(b-1)B_2/4 & b(b-1)B_2/4 \\ b(b+8)(b+1)/6 & b(b+7)(b-1)/6 \\ b(b^2+6b+11)/3 & b(2b^2+15b+37)/6 \end{matrix}\right] .$$

It is straightforward to see that Proposition D.1 and (D.1) imply the estimates in Proposition 3.1 for second and third derivatives of g.

Remark. It may be interesting to summarize a possibility of proofs without computers. At present, the estimates for which REDUCE calculations are inevitable, are the proof of (D.1) and the explicit form of $\tilde{A}$. The required estimates in Section 3 concerning $\tilde{A}$ are (3.4), (3.5), and (3.6), among which (3.4) and (3.5) reflects a network structure of the (pre-) fractal, and (3.6) is actually an expectation with respect to one-dimensional simple random walk. It therefore suffices with relatively soft estimates of $\tilde{A}$. With these considerations, presumably, we may be able to avoid computer aided proof after all. For our purpose, rigorous derivation of the above results by REDUCE on computers is sufficient.

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