Stochastic PDEs
driven by space-time white noise
with two reflecting walls
and related problems
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Microscopic phenomena

Each particle moves *randomly*.

There are some *interactions* among particles.
We observe *smooth* time evolution.
Or they are *stationary*.
discrete free interface model (static)

The interface is a function $\varphi : \mathbb{Z}^d \rightarrow \mathbb{R}$ (a “graph” of $\mathbb{R}$-valued function), of which energy on compact $\Lambda \subset \mathbb{Z}^d$ is given by the Hamiltonian

$$H_\Lambda(\varphi) = \frac{1}{2} \sum_{i,j \in \Lambda} V(\varphi(j) - \varphi(i)) + \sum_{i \in \Lambda, j \notin \Lambda} V(\varphi(j) - \varphi(i)),$$

with $V : \mathbb{R} \rightarrow \mathbb{R}$ even, uniformly convex, $V(0) = 0$.

The statistical properties of the interface are described by Gibbs measure, a probability measure $P^{\psi, \beta}_\Lambda$ on $\mathbb{R}^{\mathbb{Z}^d} = \{ \varphi : \mathbb{Z}^d \rightarrow \mathbb{R} \}$

$$P^{\psi, \beta}_\Lambda(\varphi) = (Z^{\psi, \beta}_\Lambda)^{-1} \exp\{-\beta H_\Lambda(\varphi)\} \prod_{i \in \Lambda} d\varphi_{\delta} \prod_{j \notin \Lambda} \delta_{\psi(j)}(d\varphi_{j})$$
dynamical point of view

“Gibbs states become more interesting when they are viewed as the equilibrium state of a dynamical system and, in addition, the dynamics often provides a natural approach to the analysis of Gibbs state.”—D.W. Stroock, *Logarithmic Sobolev Inequalities for Gibbs States*, LNM 1563, pp. 194–228 (1993).

There may be several dynamical systems which posesses the Gibbs state as equilibrium.

In this talk, we investigate a continuous interface on one-dimensional continuum fields.

\[
\text{Interface} = \varphi : \mathbb{R} \to \mathbb{R}.
\]

We will choose stochastic PDEs as a dynamic model (very natural, I believe!).
references

- T. Funaki, Lectures on probability theory and statistics, LNM 1869. (http://www.ms.u-tokyo.ac.jp/~funaki/)
linear SDE

\[ dX(t) = A(t)X(t)dt + \sigma(t)dW(t). \]

The solution is

\[ X(t) = \Phi(t) \left( X(0) + \int_0^t \Phi(s)^{-1}\sigma(s)dW(s) \right). \]

Φ(t) solves \( d\Phi(t) = A(t)\Phi(t)dt \), \( \Phi(0) = \text{Id} \).

\[ m(t) := E[X(t)]. \]

\[ \rho(s, t) := E[(X(s) - m(s)) \otimes (X(t) - m(t))]. \]

\[ V(t) := \rho(t, t). \]

\[ \dot{m}(t) = A(t)m(t) \]

\[ \dot{V}(t) = A(t)V(t) + \sigma(t)\sigma(t)^\dagger + V(t)^\dagger A(t)^\dagger \quad (1) \]
stationary solutions to SDEs

Assume $A(t) \equiv A$, $\sigma(t) \equiv \sigma$. $(\Phi(t) = \exp\{tA\})$.

equilibrium $\Rightarrow V(t) \equiv \text{Const.}$

\[
V(t) = e^{tA}V(0)e^{tA\dagger} + \int_0^t e^{sA}\sigma\sigma\dagger e^{sA\dagger}ds
\]

It is needed

\[
e^{tA}V(0)e^{tA\dagger} = \int_t^\infty e^{sA}\sigma\sigma\dagger e^{sA\dagger}ds
\]

$\Rightarrow V(0) = \int_0^\infty e^{sA}\sigma\sigma\dagger e^{sA\dagger}ds$

$\Rightarrow V(t) \equiv V$, $AV + \sigma\sigma\dagger + VA\dagger = 0$

$A$ must be negative definite (all e.v.'s of $A < 0$).
Let us consider the following SDE.

\[
\begin{align*}
    dX(t) &= AX(t) \, dt + \sigma \, dW(t) \\
    X(0) &= \xi
\end{align*}
\]

Then we have Mehler’s formula:

\[
P(X(t) \in dx) = \frac{1}{\sqrt{(2\pi)^d |V|}} \exp \left\{ -\frac{1}{2} \left< x - m(t), V(t)^{-1}(x - m(t)) \right> \right\},
\]

where \( V(t) \) is given by (1).
Suppose that all the eigenvalues of $A$ have negative real parts and $\xi$ is a Gaussian random variable with zero-mean and covariance $V = \int_0^\infty e^{sA} \sigma \sigma^\dagger e^{sA^\dagger} ds$. Then $X(t)$ is a stationary, zero-mean Gaussian process of which covariance function is given by
\[
\rho(s, t) = \begin{cases} 
e^{(s-t)A}V, & 0 \leq t \leq s < \infty \\
Ve^{(t-s)A^\dagger}, & 0 \leq s \leq t < \infty. \end{cases}
\]

- linear case: all quantity are computable!
- General definitions of equilibrium?
Invariant measures

Let $X(t)$ be a $S$-valued process on $(\Omega, \mathcal{F}, P)$.

A probability measure $\mu$ on $(S, \mathcal{S})$ is called invariant if $X(0)$ is $\mu$-distributed ($P(X(0))^{-1} = \mu$) then $X(t)$ is also $\mu$-distributed, that is,

$$P_\mu(X(t) \in A) = \mu(A) \quad (= P_\mu(X(0) \in A)).$$

$P_x$: prob. meas on $(\Omega, \mathcal{F})$ s.t. $P_x(X(0) = x) = 1.$

$P_\mu(A) := \int_S P_x(A) \mu(dx)$, i.e., a probability (law) on $(\Omega, \mathcal{F})$ that the Markov process $X(t)$ has initial distribution $\mu.$
Usful formulations

\[ P_x(x(t) \in A) = E_x[1_A(X(t))] \] leads us . . .

A prob. meas. \( \mu \) on \((S, \mathcal{F})\) is invariant

\[ \iff \int_S E_x[F(X(t))] \mu(dx) = \int_S F(x) \mu(dx). \]

Similary, we may formulate

A prob. meas. \( \mu \) on \((S, \mathcal{F})\) is invariant

\[ \iff E_\mu[F(X(t))] = E_\mu[F(X(0))]. \]

The test function \( F \) may be taken from a measure determining family of \((S, \mathcal{F})\) \((C_b(S) \text{ etc.})\).
Let $X(t)$ be a $S$-valued process on $(\Omega, \mathcal{F}, P)$.

A probability measure $\mu$ on $(S, \mathcal{S})$ is called reversible if $X(0)$ is $\mu$-distributed ($P(X(0))^{-1} = \mu$) then

$$X(0) \in A \rightarrow X(t) \in B$$
$$X(t) \in A \rightarrow X(0) \in B$$

occurs in the same probability for every $t > 0$, that is,

$$P_\mu(X(0) \in A \land X(t) \in B) = P_\mu(X(0) \in B \land X(t) \in A).$$

The reversible measure $\mu$ is also an invariant measure for $X(t)$. We can reformulate the above by

$$\int_A P_x(X(t) \in B) \mu(dx) = \int_B P_x(X(t) \in A) \mu(dx).$$
Useful formulations

In a similar manner to the case of invariant measures...

A prob. meas. $\mu$ on $(S, \mathcal{S})$ is reversible $\iff$

$$\int_S F(x)E_x[G(X(t))] \mu(dx) = \int_S G(x)E_x[F(X(t))] \mu(dx).$$

Similarly, we may formulate

A prob. meas. $\mu$ on $(S, \mathcal{S})$ is reversible $\iff$

$$E_\mu[F(X(0))G(X(t))] = E_\mu[G(X(0))F(X(t))].$$

The test function $F$ may be taken from a measure determining family of $(S, \mathcal{S})$ ($C_b(S)$ etc.).
Langevin’s equation again

\[ dX(t) = \frac{1}{2}AX(t) \, dt + dB(t), \quad A: \text{negative definite.} \]

Suppose also that \( A \) is symmetric.

This time, \( AV = VA \) and \( 2A = -V^{-1} \) for \( V = \int_0^\infty e^{2sA} \, ds \).

The reversible measure \( \mu \) on \( \mathbb{R}^d \) for Langevin’s dynamics is given by

\[ \mu(dx) = \frac{1}{Z} \exp \left\{ \frac{1}{2} \langle Ax, x \rangle \right\} \, dx. \]
perturbations of Langevin’s Dynamics

\[dX(t) = \frac{1}{2} (AX(t) - \nabla U(X(t))) \, dt + dB(t).\]

To avoid the difficulty from the integrability, assume \(U\) is bounded with bounded derivatives.

\[dY(t) = \frac{1}{2} AY(t) \, dt + dB(t).\]

Then the law \(Q\) of \(Y\) on \(C([0,T], \mathbb{R}^d)\) is given by Cameron–Martin–Maruyama–Girsanov.

The law \(R\) of \(X\) is also concretely given.

Easily compute \(dR/dQ\).

Using Itō formula allows us to escape the stochastic integral.

The reversible measure \(\mu\) for \(X(t)\) is given by

\[\mu(dx) = \frac{1}{Z} \exp \left\{ -U(x) + \frac{1}{2} \langle Ax, x \rangle \right\} \, dx.\]
Note that $\nabla (\frac{1}{2} \langle Ax, x \rangle) = Ax$.

Let $V : \mathbb{R}^d \to \mathbb{R}$ given (called, potential, energy, or Hamiltonian). A reversible measure of the dynamics obeying the following stochastic ordinary differential equation

$$dX(t) = -\frac{1}{2} \nabla V(X(t)) \, dt + dB(t)$$

is given by the following (Gibbs type) formula:

$$\mu(dx) = \frac{1}{Z} \exp\{-V(x)\} \, dx.$$ 

$Z$ is the normalizing constant (making $\mu$ probability measure) and is sometimes called a partition function.

However, is the reversible measure unique?
Analytic quantity

We fix an SDE: \( dX(t) = -\frac{1}{2} \nabla V(X(t)) \, dt + dB(t) \). The generator of \( X \) is defined by

\[
L := \frac{1}{2} \Delta - \frac{1}{2} \nabla V \cdot \nabla,
\]

namely

\[
(Lf)(x) = \frac{1}{2} (\Delta f)(x) - \frac{1}{2} \langle \nabla V(x), \nabla f(x) \rangle.
\]

Let \( \mu(dx) = Z^{-1} \exp\{-V(x)\} \, dx \) be a reversible measure of \( X \). Define a bilinear form \( \mathscr{E} \) by

\[
\mathscr{E}(f, g) := \frac{1}{2} \int_{\mathbb{R}^d} \langle \nabla f, \nabla g \rangle \, \mu(dx).
\]

for nice functions \( f, g : \mathbb{R}^d \to \mathbb{R} \).
Suppose $\lim_{|x| \to \infty} V(x) = +\infty$. Straightforward computation leads us to,

$$
\frac{1}{2} \int_{\mathbb{R}^d} \langle \nabla f, \nabla g \rangle \, \mu(dx) = - \int_{\mathbb{R}^d} (Lf)(x) g(x) \, \mu(dx)
$$

$$
= - \int_{\mathbb{R}^d} (Lg)(x) f(x) \, \mu(dx).
$$

\(\exists\) under \(\mu\), \(L\) can be considered as a usual second order differential operator.

\(\exists\) \(L\) makes it possible to execute the calculus on a Gibbs state \(\mu\).
entropy

\[ \mu, \nu: \text{two probability measures on } \mathbb{R}^d. \]

Define a relative entropy of \( \mu \) with respect to \( \nu \) by

\[
H(\mu|\nu) := \begin{cases} 
\int_{\mathbb{R}^d} \frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu} \nu(dx) & \mu \ll \nu \\
\infty & \text{otherwise}
\end{cases}
\]

If \( \mu(dx) = f(x) \nu(dx) \), \( H(\mu|\nu) = \int f(x) \log f(x) \nu(dx) \).

Somebody may assert that this \( H \) must be called “negative” entropy!
log-Sobolev inequality

If $\mu(dx) = f(x)\nu(dx)$, we have $H(\mu|\nu) \leq C\mathcal{E}(\sqrt{f}, \sqrt{f})$, namely,

$$\int_{\mathbb{R}^d} f(x) \log f(x) \nu(dx) \leq C \int_{\mathbb{R}^d} \left\langle \nabla \sqrt{f(x)}, \nabla \sqrt{f(x)} \right\rangle \nu(dx).$$

Define $(P_t f)(x) := \mathbb{E}_x[f(X(t))]$ and $g_t(x) := P_t f(x)$.

Check that

$$\int f(x) \log f(x) \nu(dx) = -\int_0^\infty \frac{d}{dt} \int g_t(x) \log g_t(x) \nu(dx) dt.$$ 

Note that $\frac{d}{dt} P_t = LP_t$.

It is easy to see

$$\|\nu - \mu\|_{\text{total var}}^2 \leq 2H(\mu|\nu).$$
convergence of dynamics

$\mu_t(dx)$: distribution of $X(t)$ on $\mathbb{R}^d$.

Assume $\mu_t(dx) = f_t(x)\mu(dx)$ and $H(\mu_0|\mu) < \infty$.

Assume moreover that $\mu_t(dx)$ is absolutely continuous with respect to $dx$. Then we have

$$\frac{d}{dt} H(\mu_t|\mu) = -4\mathcal{E}(\sqrt{f_t}, \sqrt{f_t}).$$

Combining with log-Sobolev inequality, we have

$$H(\mu_t|\mu) \leq e^{-4t/C} H(\mu_0|\mu).$$

That is, the law of $X(t)$ converges to the reversible distribution exponentially fast.
Infinite dimensional case

Define a linear operator \( Af(x) = \frac{d^2}{dx^2}f(x) \) on \( L^2(0,1) \) with a domain the completion of \( D(A) := \{ f \in C^2(0,1); f(0) = f(1) = 0 \} \).

Then the eigen space of \( A \) is clearly \( \{ \sin n\pi x \}_{n=1}^\infty \) and the eigen values are \( \{-n^2\pi\} \), namely \( A \) is a strictly negative definite (unbounded) operator.

\( \star \) If \( A \) is considered with Neumann conditions ("1" is an eigen function), or over \( \mathbb{R} \), \( A \) is NOT negative.

\( \star \) Today I always assume \( A \) is considered with Dirichlet conditions. Such a twice differential operator will be simply denoted by \( \Delta \). (\( \Delta \) always denotes the closed Laplacian with Dirichlet boundary conditions).
Stochastic Partial Differential Equations

Let us consider the following SDE:

\[ dX(t) = \frac{1}{2} \Delta X(t) \ dt + dB(t). \]

This may be called a “stochastic partial differential equation”. \( \Delta \) is unbounded (non continuous). The Itô formula may fail. How to handle such an operator?

B(t) is “\( L^2 \)-valued”-Brownian motion? Does it mean Gaussian distributed on \( L^2 \)? In infinite dimension, we need to pay attentions to handle such measures.

If we were able to reach the “solution”, it may have a reversible measure \( \mu \) “defined” by

\[ \mu(dw) = \frac{1}{Z} \exp \left\{ \frac{1}{2} \langle \Delta w, w \rangle \right\} dw, \]

where \( dw \) denotes the Feynman measure, possibly infinite dimensional analogue of Lebesgue measure.
Gaussian measure on a Banach space $B$

We call $\mu$, a probability measure on $B$, Gaussian if for every $\phi \in B^*$, considered as a random variable on $(B, \mathcal{B}(B), \mu)$, the law of $\phi$ is Gaussian measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

A linear subspace $H \subset B$, with Hilbert norm $| \cdot |_H$, is said to be a reproducing kernel space for $\mu$ if $H$ is complete, continuously embedded in $B$ such that, for every $\phi \in B^*$, the law of $\phi$ is zero-mean Gaussian with covariance $|\phi|^2_H$.

For every symmetric Gaussian measure $\mu$ on $B$, there exists a unique reproducing kernel space $H$. 
Let $E$ be another Banach space $E \hookrightarrow B$ continuously as a Borel set. If $\mu$ is a symmetric Gaussian on $B$ and $E$, then the reproducing kernel space w.r.t. $B$ and $E$ are the same.

$B$ given
$\mu$ given
$H \hookrightarrow B$ uniquely determined.

E.g., $B := C([0,T], \mathbb{R}^d)$, $\mu$ : Wiener measure. Then $H$ is so-called Cameron–Martin space,

$$H := \{ f : [0,T] \rightarrow \mathbb{R}^d; \text{abs. conti.}, f'(t) \in L^2 \}.$$

uniqueness of reproducing kernel space
Abstract Wiener Space

$B$: separable Banach space.
$H$: Hilbert space ($H \hookrightarrow B$ conti., densely embedded).
$\mu$: Gaussian measure on $B$ such that
\[
\int_B \exp \left\{ \sqrt{-1} \langle w, \phi \rangle \right\} \mu(dw) = \exp \left\{ -\frac{1}{2} |\phi|_H^2 \right\}
\]
for every $\phi \in B^* \subset H^* = H$.

The triplet $(B, H, \mu)$ is called an abstract Wiener space.

$\mu(\cdot - h)$ is equivalent to $\mu$ iff. $h \in H$, and
\[
\frac{d\mu(\cdot - h)}{d\mu}(w) = \exp \left\{ -\frac{1}{2} |h|_H^2 + \langle h, w \rangle \right\}.
\]
A function $F : B \to \mathbb{R}$ is $H$-differentiable on $w \in B$ if there exists $DF(w) \in H$ such that
\[
\frac{d}{dt}F(w + th)\bigg|_{t=0} = \langle DF(w), h \rangle_H
\]
is satisfied for every $h \in H$.

\[ F(w + th) = F(w) + t \langle DF(w), h \rangle + o(|t|). \]

If $F(w) = f(\langle w, \phi_1 \rangle, \langle w, \phi_2 \rangle, \ldots, \langle w, \phi_n \rangle)$, $\phi_i \in B^*$, then
\[
DF(w) = \sum_{i=1}^{n} \partial_i f(\ldots) \phi_i(x) \in H.
\]
Dirichlet form theory on $B$

Using $H$-derivative, we define a bilinear form

$$\mathcal{E}(F,G) := \frac{1}{2} \int_B \langle DF(w), DG(w) \rangle_H \mu(dw)$$

on an AWS $(B,H,\mu)$ with $D(\mathcal{E}) = \mathbb{D}^2_1(B)$. Then $(\mathcal{E}, D(\mathcal{E}))$ is a regular Dirichlet form (closed symmetric Markovian form).

We can prove (S. Kusuoka, *Dirichlet forms and diffusion processes on Banach Space*, 82) that there exists a diffusion process on $B$ which (weakly) solves to

$$dX(t) = -\frac{1}{2}X(t)\,dt + dB(t),$$

where $B(t)$ is $B$-valued Brownian motion.

However, our goal is still far away...
Gaussian measure on Hilbert space

$B$: separable Hilbert space.

$Q: B \rightarrow B$, strictly positive symmetric nuclear operator, $\text{Ker} \, Q = \{0\}$

$\mu$: Gaussian measure on $B$ with covariance operator $Q$.

$H := Q^{1/2}(B)$, $\langle f, g \rangle_H := \left\langle Q^{-1/2}f, Q^{-1/2}g \right\rangle_B$.

\[
\int_B \exp \left\{ \sqrt{-1} \, \langle w, \phi \rangle_B \right\} \mu(dw) = \exp \left\{ -\frac{1}{2} \langle Q\phi, \phi \rangle_B \right\}.
\]

\[
\mu(dw) = \frac{1}{Z} \exp \left\{ -\frac{1}{2} \langle Q^{-1}w, w \rangle_B \right\} \, dw.
\]
on $B$. 

“computations” based on Feynman measure

\[
\int_B \langle DF(w), DG(w) \rangle_H \mu(dw)
\]

\[
= \int_B \langle DF(w), DG(w) \rangle Z^{-1} \exp \left\{ -\frac{1}{2} \langle Q^{-1}w, w \rangle_B \right\} \, dw
\]

\[
= \int_B \langle DF(w), DG(w) \rangle Z^{-1} \exp \left\{ -\frac{1}{2} \langle w, w \rangle_H \right\} \, dw
\]

\[
\leftrightarrow dX(t) = dW(t) - \frac{1}{2} X(t) \, dt.
\]

Now, we shall take a concrete Hilbert space \( B := H^{-1}(0, 1), \)
\( H := L^2(0, 1), \) and \( Q := (-\Delta)^{-1}. \)

Recall that we want to consider

\[
\mu(dw) = \frac{1}{Z} \exp \left\{ \frac{1}{2} \langle \Delta w, w \rangle \right\} \, dw,
\]

on \( L^2. \)
SPDE on Feynman measure

Recall that (in finite dim)

\[ \int_{H=B} \langle DF(w), DG(w) \rangle_H Z^{-1} \exp \left\{ -\frac{1}{2} \langle (-\Delta) w, w \rangle_H \right\} dw \]

\[ \longleftarrow dX(t) = \frac{1}{2} \Delta X(t) dt + dB(t). \]

\[ = \int_B \langle DF(w), DG(w) \rangle_H Z^{-1} \exp \left\{ -\frac{1}{2} \langle (-\Delta) (-\Delta) w, w \rangle_B \right\} dw \]

\[ = \int_B \langle DF(w), DG(w) \rangle_H Z^{-1} \exp \left\{ -\frac{1}{2} \langle (-\Delta) w, w \rangle_H \right\} dw \]

\[ = \int_B \langle DF(w), DG(w) \rangle_H Z^{-1} \exp \left\{ -\frac{1}{2} |w|_{H_0^1}^2 \right\} dw \]

\[ = \int_H \langle \nabla F(w), \nabla G(w) \rangle_H \beta(dw) \]
A Gaussian process (pinned B.m.) $X(t) \equiv X^{a\rightarrow b}(t)$,
\[ dX(t) = dB(t) + \frac{b - X(t)}{1 - t} dt, \quad X(0) = a \]
has a covariance function $\rho(s, t) = (s \wedge t) - st$, which verifies
\[ ((-\Delta)^{-1} f)(x) = \int_0^1 \rho(x, y) f(y) dy. \]

A pinned Wiener measure $\beta$, extended to $L^2(0,1)$, the law of $X^{0\rightarrow 0}$ is a Gaussian measure on $L^2(0,1)$ with covariance operator $(-\Delta)^{-1}$. 
Let us go to the *strong* formulation of SPDE.

The white noise on $[0, \infty)$ is a centered Gaussian random variable $W(t)$ with covariance
\[ E[W(t)W(s)] = \delta_{t-s}. \]

It is easy to see that $W(t) = dB(t)/dt$ (Itô derivative, or Schwartz sense).

The space-time white noise is a centered Gaussian field on $(x, t)$ with covariance $E[W(x, t)W(y, s)] = \delta_{x-y}\delta_{t-s}$.

- rigorous formulation is expected.
- prefer to fit to stochastic analysis.
Let \((\mathcal{S}, \mathcal{B}(\mathcal{S}))\) be the Schwartz space and \(\mu\) is a Gaussian measure on \((\mathcal{S}, \mathcal{B}(\mathcal{S}))\) with reproducing kernel space \(L^2\). Each element \(w \in (\mathcal{S}, \mathcal{B}(\mathcal{S}), \mu)\) is called white noise. Each stochastic quantity is a white noise functional.

- Itô calculus?
Easiest way—Brownian sheet approach


Let \( E := [0, \infty)^2 \), \( m \): Lebesgue measure on \( E \).

A random set function \( W \) on \( \mathcal{B}(E) \) is called a white noise if

1. \( W(A) \sim \mathcal{N}(0, m(A)) \)
2. \( A \cap B = \emptyset \Rightarrow W(A) \) and \( W(B) \) is independent and \( W(A \cup B) = W(A) + W(B) \).

A process \( \{B(x, t)\}_{(x,t) \in E} \) defined by \( B(x, t) := W((0, x] \times (0, t]) \) is called Brownian sheet.
properties of Brownian sheet

1. $E[B(x, t)B(y, s)] = (x \wedge y)(t \wedge s)$.
2. if $x$ is fixed, $\{B(x, t)\}_{t \geq 0}$ is a Brownian motion.
3. $M(t) := B(t, t)$ is a martingale, of (non stationary) independent increments, and is not a Brownian motion.

Define $\dot{B}(x, t) := \frac{\partial^2 B(x, t)}{\partial x \partial t}$ in the sense of Schwartz distribution, namely, for $\phi \in C^2_0(E)$,

$$\dot{B}(\phi) = \int_E B(x, t) \frac{\partial^2 \phi(x, t)}{\partial x \partial t}.$$  

If we may “expect” the existence of the Itô integral,  

$$\dot{B}(\phi)\text{must be} \iint \phi(x, t)W(dx, dt).$$
Itô integral with respect to Brownian sheet

Take $\phi(x, t) = 1_{[0,x] \times [0,t]}(x, t)$.

$$
\dot{B}(\phi) = \int_0^t \int_0^x \frac{\partial^2 B(x, t)}{\partial x \partial t}(y, s) \, dy \, ds
= B(x, t) = \iint \phi(x) W(dw, dt).
$$

It is certainly true; The theory of the Itô integral w.r.t. Brownian sheet can be constructed as a usual way:


We denote the space-time white noise by $\frac{\partial^2 B(x, t)}{\partial x \partial t}$ or $B(dx, dt)$ as a formal Itô derivative.
cylindrical approach—easily handled

A stochastic process $W(t)$ is called a cylindrical Brownian motion on $L^2$ or white noise process if $W \equiv \{W(t; \psi)\}$ is a family of $\mathbb{R}$-valued stochastic process with a parameter family $\psi \in L^2$ such that

1. $\forall \psi \in L^2, W(t; \psi)/\|\psi\|_{L^2}$ is a one-dimensional standard B.m.

2. $\forall \alpha, \beta \in \mathbb{R}, \psi, \varphi \in L^2,$

$$W(t; \alpha \varphi + \beta \psi) = \alpha W(t; \varphi) + \beta(t; \psi)$$

almost surely (for $\alpha, \beta, \varphi, \psi$).

This time, the theory of the Itô integral is rather easy.

spece-time white noise is a formal Itô derivative of $W(t)$.
Itô integral for cylindrical B.m.

\[
\int_0^t \langle f(s), dW(s) \rangle_{L^2} := \sum_{k=1}^{\infty} \int_0^t \langle f(s), \psi_k \rangle dW(s; \psi_k).
\]

For \( \Phi(t) : L^2 \to H \), Hilbert–Schmidt s.t.

\[
E \left[ \int_0^T \| \Phi(t) \|^2_{HS} dt \right] < \infty,
\]

We define the stochastic integral \( \int_0^t \Phi(s) dW(s) \) by

\[
\left\langle \int_0^t \Phi(s) dW(s), \phi \right\rangle_H = \int_0^t \langle \Phi(s)^* \phi, dW(s) \rangle_{L^2} ds, \quad \forall \phi \in H.
\]

\( W(s) \) is not even an \( L^2 \)-valued process, the Itô integral is actually \( H \)-valued process.

distribution valued process (really easiest)

Let \((H^{-1}, L^2, \mu)\) be an AWS.

A Wiener space associated to this abstract Wiener space is called white noise process.

1. Actually it is the same with cylindrical approach.
stochastic partial differential equations

We consider stochastic partial differential equations

\[ dX(t) = \frac{1}{2} \Delta X(t) dt - F(t; X(t)) dt + \sigma(X(t)) dW(t), \]

with \( W(t) \) is a white noise process (\( dW(t)/dt \) is a space-time white noise), and Dirichlet boundary conditions on \((0,1)\).

Sometimes this equation is written in the following form:

\[ \frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} - f(x, t; u(x, t)) + \sigma(x, t; u(x, t)) \dot{W}(dx, dt). \]

Now we define its solution.
We call a function $X(t) \in L^2(0, 1)$ is a solution to the SPDE if

$$\langle X(t), \phi \rangle - \langle X(0), \phi \rangle = \int_0^t \langle X(s), \phi'' \rangle \, ds - \int_0^t \langle F(X(s)), \phi \rangle \, ds + \int_0^t \langle \sigma(X(s))\phi, dW(s) \rangle$$

is satisfied for every $\phi \in C_0^2(0, 1)$.

Under Lipschitz conditions on the coefficients, we can prove the existence and uniqueness result.

$X(t)$ actually stays in $C([0, 1])$ and $X(t)$ is $C([0, 1])$-valued diffusion.

The regularity is that $X(t)(x)$ is $(1/2 - \varepsilon)$-Hölder in $x$ and $(1/4 - \varepsilon)$-Hölder in $t$.

In higher space dimension, the solution does not stay in any function space.
Let $e^{t\Delta}$ be a semigroup. Then $X(t)$ is a solution to the SPDE if it satisfies

$$X(t) = e^{t\Delta}X(0) + \int_0^t e^{s\Delta}f(X(s))\,ds + \int_0^t e^{s\Delta}\sigma(X(s))\,dW(s).$$

A weak form solution is also a mild solution.

A mild form solution is also a weak form solution.
Let us consider an SPDE with an additive space-time white noise:

\[ dX(t) = \frac{1}{2} (\Delta X(t) - V'(\cdot, X(t))) \, dt + dW(t). \]

Then the reversible measure \( \mu \) for \( X(t) \) is given by

\[ \mu(dw) = \frac{1}{Z} \exp \left\{ - \int_0^1 V(x, w(x)) \, dx \right\} \beta(dw), \]

\( \beta \) is a Gaussian measure on \( C([0,1]) \) induced by a pinned Brownian bridge 0 to 0.
corresponding Dirichlet form

Define a bilinear form

\[ \mathcal{E}(F, G) := \frac{1}{2} \int_{L^2} \langle \nabla F(w), \nabla G(w) \rangle_{L^2} \mu(dw) \]

for \( F, G \in \mathscr{F}C_0^\infty \), where \( \nabla \) denotes the Fréchet derivative on \( L^2 \). Then the closure is a Dirichlet form corresponding to the SPDE (T. Funaki, *The reversible measures of multidimensional Ginzburg–Landau type continuum model*, Osaka J., 1991).

The Poincaré inequality and log-Sobolev inequality hold.