Essential self-adjointness of Dirichlet operators on a path space with Gibbs measures via an SPDE approach

( joint work with Michael RÖCKNER )

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§1. Introduction (Problem)
• state space: infinite volume path space $C(\mathbb{R}, \mathbb{R}^d)$

• tangent space: $H := L^2(\mathbb{R}, \mathbb{R}^d)$

• underlying measure: Gibbs measure $\mu$

associated with the (formal) Hamiltonian

$$H(w) := \frac{1}{2} \int_{\mathbb{R}} |w'(x)|^2 dx + \int_{\mathbb{R}} U(w(x)) dx,$$

where $U : \mathbb{R}^d \to \mathbb{R}$ is a self-interaction potential.

Heuristically, $\mu$ is given by

$$\mu(dw) = Z^{-1} e^{-H(w)} \prod_{x \in \mathbb{R}} dw(x).$$
Consider a (pre-)Dirichlet form

\[ E(F, G) := \frac{1}{2} \int (D_H F(w), D_H G(w))_H \mu(dw) \]

for \( F, G \in \mathcal{FC}_b^\infty \) (smooth cylinder functions).

\[ \Rightarrow \] We can consider a (pre-)Dirichlet operator \((\mathcal{L}_0, \mathcal{FC}_b^\infty)\) through

\[ E(F, G) = - (\mathcal{L}_0 F, G)_{L^2(\mu)}. \]

**Our problem:** Is the pre-Dirichlet operator \((\mathcal{L}_0, \mathcal{FC}_b^\infty)\) essential self-adjoint in \( L^2(\mu) \)?
Related works for infinite-dimensional settings:

(i) • Takeda (’85), • Röckner-Zhang (’92),
    • Shigekawa (’95) etc.

⇒ Functional analytic approach (e.g. Malliavin calculus)
    under \( \mu(dw) = \rho(w)\mathcal{W}(dw) \)

(ii) • Albeverio-Kondratiev-Röckner (’95〜) etc.

⇒ (Finite dimensional) approximation approach with
    stochastic analysis (stochastic flow)

(iii) • Da Prato (2000〜), • Da Prato-Röckner (2002) etc.

⇒ SPDE approach
§2. Framework and Results
At the beginning, we introduce some notations and objects we will working with.

- **weight function** \( \rho_r \in C^\infty(\mathbb{R}, \mathbb{R}) \), \( r \in \mathbb{R} \), is defined by \( \rho_r(x) := e^{r|x|} \chi(x), x \in \mathbb{R} \), where \( \chi \) is a convex even smooth function with \( \chi(x) = |x| \) for \( |x| \geq 1 \). \( \rho_r(x) \approx e^{r|x|} \)

- \( E := L^2(\mathbb{R}, \mathbb{R}^d; \rho_{-2r}(x)dx) \), \( r > 0 \) fixed) with \( (X, Y)_E := \int_{\mathbb{R}} (X(x), Y(x))_{\mathbb{R}^d} \rho_{-2r}(x)d\mu \).

- \( H := L^2(\mathbb{R}, \mathbb{R}^d) \)
Before giving a Gibbs measure, we impose some conditions on the potential function $U$.

(U1): $U \in C^1(\mathbb{R}^d, \mathbb{R})$ and $\exists K_1 \in \mathbb{R}$ s.t.

$$ (\nabla U(z_1) - \nabla U(z_2), z_1 - z_2)_{\mathbb{R}^d} \geq -K_1|z_1 - z_2|^2_{\mathbb{R}^d} \text{ for } z_1, z_2 \in \mathbb{R}^d. $$

(U2): $\exists K_2 > 0, \exists p > 0$ s.t.

$$ |\nabla U(z)|_{\mathbb{R}^d} \leq K_2(1 + |z|^p_{\mathbb{R}^d}) \text{ for } z \in \mathbb{R}^d. $$

(U3): $\lim_{|z|_{\mathbb{R}^d} \to \infty} U(z) = \infty.$

Example: $U(z) = a(|z|^4_{\mathbb{R}^d} - |z|^2_{\mathbb{R}^d}), a > 0$
Under (U1) and (U3), we can construct a Gibbs measure on $C(\mathbb{R}, \mathbb{R}^d)$ in the following manner:

- Consider a Schrödinger operator $H := -\frac{1}{2}\Delta + U$ on $L^2(\mathbb{R}^d, \mathbb{R})$. $H$ has purely discrete spectrum and a complete set of eigenfunctions.

$\Rightarrow \cdot \lambda_0 (> \min U)$: the lowest eigenvalue of $H$,

$\cdot \Omega$: ground state of $H$ with $\|\Omega\|_{L^2(\mu)} = 1$ and $\Omega > 0$.

i.e., $H\Omega = \lambda_0\Omega$. $(e^{-tH}\Omega = e^{-t\lambda_0}\Omega)$
• $\mathcal{W}_{-T, z_1; T, z_2}$ ($T > 0, z_1, z_2 \in \mathbb{R}^d$): pinned BM measure with

$\mathcal{W}_{-T, z_1; T, z_2}(w(-T) = z_1, w(T) = z_2) = 1$.

• $p(t, z_1, z_2)$: transition probability of $d$-dim standard BM.

• $\sigma$-fields of the space $C(\mathbb{R}, \mathbb{R}^d)$:

$\mathcal{B} := \sigma(w(x); x \in \mathbb{R})$,

$\mathcal{B}_T := \sigma(w(x); -T \leq x \leq T)$,

$\mathcal{B}_{T,c} := \sigma(w(x); x < -T, x > T)$.
We define a probability measure on $C(\mathbb{R}, \mathbb{R}^d)$ by

$$
\mu(A) := e^{2T\lambda_0} \int_{\mathbb{R}^d} dz_1 \Omega(z_1) \int_{\mathbb{R}^d} dz_2 \Omega(z_2)
$$

$$
\times p(2T, z_1, z_2) \mathbb{E}^{\mathcal{W},T,z_1;T,z_2} \left[ e^{-\int_{-T}^T U(w(x)) dx} ; A \right]
$$

for $A \in \mathcal{B}_T$ and by extending the above to a measure on $\mathcal{B}$.

Remark: $p(2T, z_1, z_2) \mathbb{E}^{\mathcal{W},T,z_1;T,z_2} \left[ e^{-\int_{-T}^T U(w(x)) dx} \right]$ is equal to $e^{-2TH}(z_1, z_2)$. (Feynman-Kac formula)
We can obtain the estimate

\[ \int ( \int_{\mathbb{R}} |w(x)|^{2m}_{\mathbb{R}^d} \rho_{-2r}(x) \,dx) \,\mu(dw) \]

\[ \leq \frac{1}{r} \int_{\mathbb{R}^d} |z|^{2m}_{\mathbb{R}^d} \Omega(z)^2 \,dz < \infty, \quad m \in \mathbb{N}. \]

Then we notice that \( \mu(C) = 1 \), where

\[ C := \bigcap_{r>0} \{ w \in C(\mathbb{R}, \mathbb{R}^d); \|w\|_{r,\infty} < \infty \}. \]

( \( \|w\|_{r,\infty} := \sup_{x \in \mathbb{R}} |w(x)|^{r}_{\mathbb{R}^d} \rho_{-r}(x) \) )

\( \Rightarrow \) Since \( C \hookrightarrow E \) is continuous, we can regard \( \mu \) as a probability measure on \( E \).
DLR-equation:  
For $\forall T \in \mathbb{N}$, $\mu$-a.e. $\xi \in C(\mathbb{R}, \mathbb{R}^d)$:

$$\mu(dw|\mathcal{B}_{T,c})(\xi) = Z_{T,\xi}^{-1} e^{-\int_{-T}^{T} U(w(x)) dx} \times \mathcal{W}_{-T,\xi(-T);T,\xi(T)}(dw).$$

(Definition of Gibbs measures)

Betz-Lörinczi (’03) · · · If $\exists a > 2$, $U(z)$ grows at infinity faster than $|z|^a_{\mathbb{R}^d}$ but slower than $|z|^{2a-2}_{\mathbb{R}^d}$  
$\implies$ there is a unique Gibbs measure on $C(\mathbb{R}, \mathbb{R}^d)$. 
Quasi-invariance:

For every \( k \in C_0^\infty (\mathbb{R}, \mathbb{R}^d) \),

\[
\mu \sim \mu(k + \cdot) \text{ and } \mu(k + dw) = \Lambda(k, w) \mu(dw),
\]

where

\[
\Lambda(k, w) = \exp \left\{ \int_{\mathbb{R}} \left( U(w(x)) - U(w(x) + k(x)) 
- \frac{1}{2} |k'(x)|^2 + (w(x), \Delta_x k(x))_{\mathbb{R}^d} \right) dx \right\}
\]

and \( \Delta_x := d^2 / dx^2 \).
• the space of smooth cylinder functions:

\[ FC^\infty_b := \{ F(w) = f(\langle w, \varphi_1 \rangle, \cdots, \langle w, \varphi_n \rangle); n \in \mathbb{N}, f \in C^\infty_b(\mathbb{R}^n, \mathbb{R}), \varphi_1, \cdots, \varphi_n \in C^\infty_0(\mathbb{R}, \mathbb{R}^d) \}, \]

where \( \langle w, \varphi_i \rangle := \int_\mathbb{R} (w(x), \varphi_i(x))_{\mathbb{R}^d} \, dx, \ w \in E. \)

♣ \( FC^\infty_b \hookrightarrow L^2(\mu) \) (dense)

• \( H \)-Fréchet derivative \( D_H F : E \to H: \)

\[ D_H F(w)(\cdot) := \sum_{i=1}^n \partial_i f(\langle w, \varphi_1 \rangle, \cdots, \langle w, \varphi_n \rangle) \varphi_i(\cdot) \]

for \( F \in FC^\infty_b. \)
Define a (pre-)Dirichlet form \((\mathcal{E}, \mathcal{F}C_b^\infty)\) by
\[
\mathcal{E}(F, G) := \frac{1}{2} \int_{\mathcal{E}} (D_H F(w), D_H G(w)) \mu(dw)
\]
for \(F, G \in \mathcal{F}C_b^\infty\). By the quasi-invariance of \(\mu\), we obtain
\[
\mathcal{E}(F, G) = -\langle \mathcal{L}_0 F, G \rangle_{L^2(\mu)}, \quad F, G \in \mathcal{F}C_b^\infty, \cdots (\dagger)
\]
where
\[
\mathcal{L}_0 F = \frac{1}{2} \text{Tr}(D_H^2 F(w)) + \frac{1}{2} \left\{ \langle w, \Delta_x D_H F(w(\cdot)) \rangle - \langle \nabla U(w(\cdot)), D_H F(w) \rangle \right\}
\]
By (†), \((\mathcal{L}_0, \mathcal{F}C_b^\infty)\) is dissipative on \(L^2(\mu)\), i.e.,
\[(\mathcal{L}_0 F, F)_{L^2(\mu)} \leq 0 \text{ for } F \in \mathcal{F}C_b^\infty.\]

\[\implies \exists \text{ self-adjoint extension of } (\mathcal{L}_0, \mathcal{F}C_b^\infty).\]

(Friedrichs extension)

\[\uparrow \uparrow\]

\[\bullet (\mathcal{E}, \mathcal{D}(\mathcal{E})) : \text{the closure of } (\mathcal{E}, \mathcal{F}C_b^\infty) \text{ w.r.t } \mathcal{E}_1^{1/2}-\text{norm}\]

(Minimal Dirichlet form)
Theorem 1  (i) The pre-Dirichlet operator \((L_0, \mathcal{FC}_b^\infty)\) is essentially self-adjoint in \(L^2(\mu)\), i.e., 
\((\overline{L}_2, \text{Dom}(\overline{L}_2)) : \text{closure of } (L_0, \mathcal{FC}_b^\infty) \text{ in } L^2(\mu)\) is self-adjoint.

(ii) \(e^{t\overline{L}_2} F(w) = P_tF(w), \ \mu\text{-a.s. } w, \ F \in L^2(\mu),\)

where \(\{P_t\}_{t \geq 0}\) is the transition semigroup corresponding to the parabolic SPDE

\[
dX_t(x) = \frac{1}{2} \left\{ \Delta_x X_t(x) - \nabla U(X_t(x)) \right\} dt + dB_t(x), \ x \in \mathbb{R}, \ t > 0, \cdots \quad (GL)
\]

where \(\{B_t\}_{t \geq 0}\) is a \(H\)-cylindrical Brownian motion.
As a consequence of Theorem 1, we can also obtain the Markov uniqueness.

**Theorem 2** (Markov uniqueness) The Dirichlet form \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is the unique extension of \((\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)\).

- \((\mathcal{E}, \text{Dom}(\mathcal{E}))\): Dirichlet form in \(L^2(\mu)\)
  
  is an extension of \((\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)\).

\[\iff\]

- \(\mathcal{F}\mathcal{C}_b^\infty \subset \text{Dom}(\mathcal{E}),\)
- \(\mathcal{E}(F, G) = (-\mathcal{L}_0 F, G)_{L^2(\mu)}\) holds for \(\forall F \in \mathcal{F}\mathcal{C}_b^\infty, \forall G \in \text{Dom}(\mathcal{E}).\)
• Albeverio-Kusuoka ('88),
• Albeverio-Kusuoka-Röckner ('90) etc.

Characterization of the maximal Dirichlet form

\((\mathcal{E}^+, \mathcal{D}(\mathcal{E}^+))\)

### Application
(Rademacher type theorem)

\[ F : E \to \mathbb{R} \text{ measurable s.t. for } \forall w \in E, \forall h \in H, \]
\[ |F(w + h) - F(w)| \leq C\|h\|_H \]

\[ \implies F \in \mathcal{D}(\mathcal{E}^+) \implies F \in \mathcal{D}(\mathcal{E}) \]

Kusuoka

Theorem 2

♣ If we consider "\(H\)-distance function", this plays a key role to give the upper bound of \((P_t1_A, 1_B)_{L^2(\mu)}\).

§3 Sketch of the Proof for the Main Theorem
Our approach is essentially based on Da Prato and Röckner’s one (2002).

- \((\mathcal{L}_0, \mathcal{FC}_b^\infty)\): dissipative \(\overset{\rightarrow}{\implies}\) closable

\[\Rightarrow (\bar{\mathcal{L}}_2, \text{Dom}(\bar{\mathcal{L}}_2))\): closure of \((\mathcal{L}_0, \mathcal{FC}_b^\infty)\)

(dissipativity also holds.)

\[
\downarrow
\]

**Aim:** \((\bar{\mathcal{L}}_2, \text{Dom}(\bar{\mathcal{L}}_2))\) : m-dissipative, i.e.,
\[
\exists \lambda > 0, \text{Range}(\lambda - \bar{\mathcal{L}}_2) = \mathcal{L}_2^2(\mu).
\]

\[\uparrow\] (Lumer-Phillips Theorem)

It is sufficient to show

\[
\exists \lambda > 0, \mathcal{FC}_b^\infty \subset \text{Range}(\lambda - \bar{\mathcal{L}}_2)(\subset \mathcal{L}_2^2(\mu)).
\]
Hence it is sufficient to show
\[ \exists \lambda > 0, \forall F \in \mathcal{F}C_\infty^b, \exists \Phi \in \text{Dom}(\overline{L}_2) \text{ s.t.} \]
\[ \lambda \Phi - \overline{L}_2 \Phi = F \cdots (\#) \]
(infinite-dimensional elliptic problem)

♣ Candidate:
\[ \Phi = \int_0^\infty e^{-\lambda t} P_tF dt, \quad \lambda > \frac{K_1}{2} + r^2 \]

Facts on the SPDE (GL) (Iwata, Funaki, \ldots)
(i) SPDE (GL) has a unique (pathwise) solution 
\( (X_t^w(\cdot))_{t \geq 0} \) living in \( C([0, \infty), \mathcal{C}) \) for an initial data \( w \in \mathcal{C} \).

(ii) For \( F \in \mathcal{F}\mathcal{C}_b^\infty \), we set
\[
P_t F(w) := \mathbb{E}[F(X_t^w)], \quad w \in \mathcal{C}, \quad t \geq 0.
\]
Then \( (P_t)_{t \geq 0} \) can be regarded as a \( C_0 \)-contraction symmetric semigroup on \( L^2(\mu) \).

(iii) Its infinitesimal generator is an extension of the (pre-)Dirichlet operator \( (L_0, \mathcal{F}\mathcal{C}_b^\infty) \).

\( \leftarrow \) an easy consequence of Itô’s formula)
• Difficulty: It is **difficult** to show $\Phi \in \text{Dom}(\overline{L}_2)$ directly!!

↓ How to show?

We insert a tractable space which corresponds to the Ornstein-Uhlenbeck (OU-)operator. i.e., We want to understand as $\overline{L}_2 = (\text{OU-operator}) + (\text{perturbation})$.

**Formulation of the OU operator**

**Step 1.** Take $\kappa > 0$ s.t. $\kappa > 2r^2$

$$(\rightarrow \omega := \frac{\kappa}{2} - r^2 > 0)$$

Set $S_t w(x) := e^{-\kappa t/2} \int_{\mathbb{R}} g(t, x, y) w(y) dy$
\begin{align*}
\Rightarrow (S_t)_{t \geq 0} & : C_0\text{-contraction semigroup on } E. \\
& \quad \text{(Note it is not symmetric!)} \\
\Rightarrow (A, \text{Dom}(A)) & : \text{infinitesimal generator of } (S_t). \quad \left( A = \frac{1}{2}(\Delta_x - \kappa) \right)
\end{align*}

**Step 2.** Consider a parabolic SPDE

\[ dY_t(x) = \frac{1}{2}\left\{ \Delta_x Y_t(x) - \kappa Y_t(x) \right\} dt \]
\[ + dB_t(x), \quad x \in \mathbb{R}, \quad t > 0 \quad \cdots \text{(OU)} \]

with an initial data \( w \in E. \)
We can write down the solution of (OU) as

\[ Y_t^w = S_t w + \int_0^t S_{t-s} \sqrt{Q} dW_s, \quad t \geq 0, \cdots (\star) \]

where

- \( Q \in L(E, E) \) : \( Qw := \rho_{-2r} \cdot w \)
- \( (W_t)_{t \geq 0} \) : \( E \)-cylindrical Brownian motion.

Remark: (mean of (\( \star \))) = \( S_t w \),
\( (\text{covariance of } (\star)) = \int_0^t S_{t-s}^* Q S_{t-s} ds (=: Q_t) \)

⇒ We easily see \( Q_t : E \rightarrow E \) is a trace class operator.
Define the **OU-semigroup** \((R_t)_{t \geq 0}\) by

\[
R_t F(w) := \mathbb{E}[F(Y_t^w)] = \int_E F(S_t w + y) N_{Q_t}(dy)
\]

How should we choose a good domain for \((R_t)_{t \geq 0}\)?

Da Prato, Pliola, Tubaro etc. introduced the following subspaces of \(C(E)\):

- **\(UC_{b,2}(E)\)** ··· the set of all functions \(F : E \rightarrow \mathbb{R}\) with \(\frac{F(\cdot)}{1+\|\cdot\|^2_E}\) is uniformly continuous and bounded. This is a Banach space w.r.t the norm \(\|F\|_{b,2} := \sup_{w \in E} \frac{|F(w)|}{1+\|w\|^2_E}\).
\( C^1_{b,2}(E) \) \( \cdots \) the subspace of \( UC_{b,2}(E) \) of those functions \( F \) are continuously differentiable with
\[
\| DF \|_{b,2} := \sup_{w \in E} \frac{\| DF(w) \|_E}{1 + \| w \|_E^2} < \infty ,
\]
where \( DF : E \to E \) is the \( E \)-Fréchet derivative of \( F \).

Remark: \( D_H F = QDF \)

\( \implies (R_t)_{t \geq 0} \) : semigroup on \( UC_{b,2}(E) \)

\( \clubsuit \) It is not strongly continuous! But it is regarded as a \( \pi \)-semigroup in the sense of Da Prato and Priola.
Step 3. Define the OU-operator $L$ through the resolvent

$$(\lambda - L)^{-1} F(w)$$

$$= \Psi \lambda F(w)$$

$$:= \int_0^\infty e^{-\lambda t} R_t F(w) dt, \ \lambda > 0, w \in E,$$

and set

- $\mathcal{D}(L; UC_{b,2}(E)) := \Psi \lambda (UC_{b,2}(E)),$
- $\mathcal{D}(L; C_{b,2}^1(E)) := \Psi \lambda (C_{b,2}^1(E)).$

Remark: $\mathcal{D}(L; C_{b,2}^1(E)) \subset \mathcal{D}(L; UC_{b,2}(E))$
Key Proposition

(i) \( \mathcal{FC}_b^\infty \subset \mathcal{D}(L; C_{b,2}^1(E)) \subset \text{Dom}(\overline{\mathcal{L}}_2) \)

(ii) For \( F \in \mathcal{FC}_b^\infty \),

\[
LF(w) = \frac{1}{2} \text{Tr}(D_H^2 F(w)) + \frac{1}{2} \langle w, (\Delta_x - \kappa)D_H F(w(\cdot)) \rangle, w \in E.
\]

(iii) For \( F \in \mathcal{D}(L; C_{b,2}^1(E)) \),

\[
\overline{\mathcal{L}}_2 F = LF + (b(\cdot), DF)_E,
\]

where \( b : \text{Dom}(b) \subset E \rightarrow E \) is a measurable mapping with \( \text{Dom}(b) = \mathcal{C} \) defined by

\[
b(w)(\cdot) := \frac{1}{2} (\kappa w(\cdot) - \nabla U(w(\cdot))).
\]
Hence to solve the equation (♯), it is sufficient to show that our candidate

\[ \Phi = \int_0^\infty e^{-\lambda t} P_t F dt, \quad \lambda > \frac{K_1}{2} + r^2, \]

satisfies

(i) \( \Phi \in D(L; C_{b,2}^1(E)) \),

(ii) \( \lambda \Phi - L\Phi - (b(\cdot), D\Phi)_E = F \)

\[ \Rightarrow \quad \text{How to check the assertions (i) and (ii)?} \]
Fact (E. Priola, ’98)

\( F \in \mathcal{D}(L; C_{b,2}^1(E)) \) is equivalent to

(i-1): \( \sup_{t>0} \frac{1}{t} \| R_t F - F \|_{b,2} < \infty \)

(i-2): \( \exists G(= LF) \in C_{b,2}^1(E) \) s.t.

\[
\lim_{t \to 0} \frac{1}{t} (R_t F(w) - F(w)) = G(w), \ w \in E.
\]

\( \implies \) To show the item (i-2), we transform

\[
\frac{1}{t} (R_t \Phi(w) - \Phi(w))
\]

as follows:

- \( S(b)_t := \int_0^t S_{t-s} b(X_s^w) ds \)
\[
\frac{1}{t} (R_t \Phi(w) - \Phi(w)) \\
= \frac{1}{t} \mathbb{E} [\Phi(Y_t^w) - \Phi(w)] \\
= \frac{1}{t} \mathbb{E} [\Phi(X_t^w - S(b)_t) - \Phi(w)] \\
= \frac{1}{t} \mathbb{E} [\Phi(X_t^w) - \Phi(w)] \\
- \mathbb{E} \left[ \int_0^1 \left( D \Phi(X_t^w - \theta S(b)_t), \frac{1}{t} S(b)_t \right) E \, d\theta \right] \\
= \frac{1}{t} (P_t \Phi(w) - \Phi(w)) \\
- \int_0^1 \mathbb{E} \left[ \left( D \Phi(X_t^w - \theta S(b)_t), \frac{1}{t} S(b)_t \right) E \right] d\theta
\]
By letting $t \searrow 0$ on the right-hand side, we have

\[ \frac{1}{t} (P_t \Phi(w) - \Phi(w)) = \frac{e^{\lambda t} - 1}{t} \int_t^\infty e^{-\lambda s} P_s F(w) ds - \frac{1}{t} \int_0^t e^{-\lambda s} P_s F(w) ds \]

\[ \Rightarrow \lambda \Phi(w) - F(w) \]

\[ \int_0^1 \mathbb{E} \left[ \left( D\Phi(X_t^w - \theta S(b)_t), \frac{1}{t} S(b)_t \right)_E \right] d\theta \]

\[ \Rightarrow \int_0^1 \mathbb{E} \left[ (D\Phi(X_0^w), b(X_0^w))_E \right] d\theta \]

\[ = (D\Phi(w), b(w))_E \]
Hence it holds that

\[
\lim_{t \to 0} \frac{1}{t} (R_t \Phi(w) - \Phi(w)) = \lambda \Phi(w) - F(w) + (D\Phi(w), b(w))_E
\]

\[
\Downarrow
\]

To show \((\text{RHS}) \in C^1_{b,2}\), we need some regularities of the function \(D\Phi\).

♣ Representation formula (estimate) for the gradient of \(P_t \implies \Phi \in C^2_b(E)\)


\[\leftarrow \text{from the view point of stochastic flow}\]

\[\|D(P_tF)(w)\|_E \leq e^{\left(\frac{K_1}{2} + r^2\right)t} P_t(\|DF\|_E)(w)\]
The gradient estimate leads us to the estimate

\[ \| D\Phi \|_\infty \leq \int_0^\infty e^{-\lambda t} \| D(P_t F) \|_\infty \, dt \]

\[ \leq \| DF \|_\infty \int_0^\infty e^{\left(\frac{K_1}{2} + r^2 - \lambda\right)t} \, dt \]

\[ < \infty \quad \text{under} \quad \lambda > \frac{K_1}{2} + r^2 \]

By these arguments, we have shown the assertions (i) and (ii)!

**REMARK:** Above proof is not complete!
• $P_t F(w)$ and $P_t \Phi(w)$ are defined only on $\mathcal{C}$.
• To show $\Phi \in C^2_b(E)$, we need $U \in C^2(\mathbb{R}^d, \mathbb{R})$.

\[ \Downarrow \]

To give a complete proof, we should introduce approximation functions for the drift $b$.
• Yosida approximation $\rightarrow$ Lipschitz continuity
• Mollifization technique on infinite dimensions

$$ b_\beta(w) := \int_{E} e^{\beta B} b(e^{\beta B} w + y) N^{\frac{1}{2}} B^{-1}(e^{2\beta B} - 1) \, (dy) $$

($B$: negative definite self-adjoint op with $B^{-1}$ is of trace class)
Further Problems:

- Gibbs measures on $C(\mathbb{R}, \mathbb{R}^d)$ with two-body potentials (Osada-Spohn, Funaki, Hariya, etc)

\[ \tilde{H}(w) = H(w) + \int \int_{\mathbb{R}^2} W(x - y, w(x) - w(y)) \, dx \, dy \]

- $P(\phi)_2$-time evolution ??