A note on a scaling limit of successive approximation for a differential equation with moving singularity.

Tetsuya Hattori*and Hiroyuki Ochiai†

ABSTRACT

We study a method of successive approximation to $\dfrac{d\phi}{dx}(x) = -\phi(x)^2$, a simplest first order non-linear ordinary differential equation whose solutions have moving singularities. We give a sufficient condition for a series of approximate solutions ϕ_n, $n = 0, 1, 2, \cdots$, to have a scaling limit, namely, $\phi(x) = \lim\limits_{n\to\infty} q_n^{-1}\phi_n(q_n^{-1}x)$ exists, where $q_n = \phi_n(0)$. In other words, the approximation sequence $\phi_n(x)$, $n = 0, 1, 2, \cdots$, approaches the exact solution x^{-1} in an asymptotically conformal way: $\phi_n(x) \approx q_n\phi(q_n x)$ as $n \to \infty$.

The crucial condition for the scaling limit to exist is the existence of the limit $\lim\limits_{n\to\infty} q_{n+1}/q_n$. For a certain choice of ϕ_0, the problem is related to a problem of random sequential bisection, through which we find $\lim\limits_{n\to\infty} q_n^{1/n} = e^{1/c}$, where $x = c$ is the unique positive solution to $x \log \dfrac{2e}{x} = 1$, $x > 1$. Numerical calculations suggest that all the conditions for the scaling limit to exist are satisfied for this choice of ϕ_0.

1 Introduction.

Consider a simplest first order non-linear ordinary differential equation

(1.1) $\dfrac{d\phi}{dx}(x) = -\phi(x)^2,\ x > 0\,.$

A solution $\phi(x) = (x - c)^{-1}$ has a singularity, whearas (1.1) has no singularities. The singularity point $x = c$ is an aribtrary constant, hence it is called a moving singularity. We may, without loss of generality put $c = 0$, to fix the arbitrary constant c. Equivalently, we may impose boundary condition at infinity:

(1.2) $\phi(x) = x^{-1} + o(x^{-2}),\ x \to \infty,$

to obtain a unique solution $\phi(x) = x^{-1}$ in $C^1((0,\infty))$.

Applying the method of successive approximation, we obtain a sequence ϕ_n, $n = 0, 1, 2, \cdots$, of functions defined recursively, by

$$\frac{d\phi_{n+1}}{dx}(x) = -\phi_n(x)^2,\ x > 0,\ \phi_n(x) = x^{-1} + o(x^{-2}),\ x \to \infty,\quad n = 0, 1, 2, \cdots,$$

*Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan. e-mail: hattori@math.nagoya-u.ac.jp

†Department of Mathematics, Kyushu University, Hakozaki, Fukuoka 812-8581, Japan. e-mail: ochiai@math.kyushu-u.ac.jp

or, equivalently,

(1.3) $\phi_{n+1}(x) = \int_x^\infty \phi_n(x')^2\,dx',\ x>0,\ n=0,1,2,\cdots,$

with an initial approximation

(1.4) $\phi_0(x) = x^{-1} + o(x^{-2}),\ x\to\infty.$

Successive approximation (1.3) gives a sequence of functions converging to a solution ϕ of the differential equation. We are interested in the 'rate of convergence' of successive approximation to a solution near the moving singularity. We go into this problem by studying the scaling limit, namely, the existence problem of $\bar{\phi}(z) = \lim_{n\to\infty} q_n^{-1}\phi_n(q_n^{-1}z)$, for a sequence of positive numbers q_n, $n=0,1,2,\cdots$, which diverges to ∞ as $n\to\infty$. (The scaling factor for x and the scaling factor for ϕ_n should be equal, because $\phi_n(x)\approx x^{-1}$ for very large n and x somewhat near 0.) Possibility of existence of scaling limits for successive approximations to differential equations with moving singularities seems not to have been studied.

To be specific, let us denote by $\mathcal{C}$ a set of entire functions $\phi:\ \mathbf{C}\to\mathbf{C}$, satisfying $\phi(x) = x^{-1}+o(x^{-2})$, $x\to+\infty$, whose coefficients in the Maclaurin series have alternating signs;

$$\phi(z) = \sum_{k=0}^{\infty}(-1)^k a_k z^k,\ a_k\ge 0,\ k=0,1,2,\cdots.$$

In Section 2 we prove the following.

Theorem 1.1. *Let $\phi_0\in\mathcal{C}$ and let ϕ_n, $n=0,1,2,\cdots$, be a sequence defined recursively by (1.3) on $[0,\infty)$. Then for each n, ϕ_n are analytically continued to $\mathbf{C}$ and $\phi_n\in\mathcal{C}$ holds.*

Furthermore, let the coefficients $a_{0,k}$, $k=0,1,2,\cdots$, in

$$\phi_0(z) = \sum_{k=0}^{\infty}(-1)^k a_{0,k} z^k,$$

satisfy $a_{0,k}\le a_{0,0}^{k+1}$, $k=1,2,3,\cdots$. Then if the sequence q_n, $n=0,1,2,\cdots$, defined by

(1.5) $q_n = \phi_n(0),\ n=0,1,2,\cdots.$

is increasing in n, and the sequence of the ratios of successive terms has a limit greater than 1;

(1.6) $\exists\rho = \lim_{n\to\infty}\dfrac{q_{n+1}}{q_n} > 1,$

then the sequence of the entire functions defined by

(1.7) $\bar{\phi}_n(z) = q_n^{-1}\phi_n(q_n^{-1}z),\ n=0,1,2,\cdots,$

converges uniformly on compact sets in $\mathbf{C}$ to an entire function $\bar{\phi}(z) = \sum_{k=0}^{\infty}(-1)^k\alpha_k z^k$ defined by

(1.8) $\alpha_0 = 1,\ \alpha_k = \dfrac{1}{k\rho^{k+1}}\sum_{j=1}^{k}\alpha_{k-j}\alpha_{j-1}, k=1,2,3,\cdots.$

This Theorem gives a sufficient condition for the sequence of successive approximations to exhibit a scaling limit.

The conditions on the coefficients in the definition of the class $\mathcal{C}$ and in Theorem 1.1 may be stronger than necessary. However, as we will see below, we have examples of interest satisfying these conditions, so we impose them to avoid technical complications. The condition (1.6) on q_n, on the other hand, seems more essential and deserve further study. In fact, existence of the limit in (1.6) is still open, but we have some results for the value of ρ, assuming its existence.

For $a > 1$ let $\mathcal{O}_a$ be a set of non-negative valued functions $\phi : [0, \infty) \to [0, \infty)$ defined on non-negative reals which has an expression

$$(1.9)\ \phi(x) = \int_0^\infty e^{-xt}(1 - F(t))\, dt\,,\ x \geq 0,$$

where $F : [0, \infty) \to [0, 1]$ is an increasing right continuous function satisfying $F(x) \leq Cx^a$, $x > 0$, for some positive constant C. (In particular, we impose $\phi(0) < \infty$, which implies $\lim_{x\to\infty} F(x) = 1$.)

In Section 3 we prove the following.

Theorem 1.2. *(i) Let $a > 1$ and $\phi_0 \in \mathcal{O}_a$, and let ϕ_n, $n = 0, 1, 2, \cdots$, be a sequence defined recursively by (1.3) on $[0, \infty)$. If $0 < \ell < \ell_s(a)$, then $\lim_{n\to\infty} \ell^{-n}\phi_n(\ell^{-n}x) = x^{-1}$, $x > 0$, and $\lim_{n\to\infty} \ell^{-n}\phi_n(0) = \infty$ hold. Here*

$$(1.10)\ell_s(a) = \begin{cases} ((a+1)/2)^{1/a}, & 1 < a < c-1, \\ e^{1/c}, & a \geq c-1, \end{cases}$$

and c is the unique solution to $c \log \dfrac{2e}{c} = 1$ with $c > 1$.

(ii) Let $1 < a < c - 1$ and $\phi_0 \in \mathcal{O}_a$, and assume that there exist positive constants C', a', δ, satsfying $a' \geq a$, such that the function F in (1.9) correpsonding to $\phi = \phi_0$ satisfies $F(x) \geq C'x^{a'}$ for $0 \leq x \leq \delta$, then for $\ell > \ell_s(a')$, and for each $x \geq 0$, $\lim_{n\to\infty} \ell^{-n}\phi_n(\ell^{-n}x) = 0$.

Numerically, $c - 1 = 3.31107040700\cdots$ and $e^{1/c} = 1.2610704868\cdots$.

Theorem 1.1 and Theorem 1.2 imply the following result, which describes the meaning of Theorem 1.2 in terms of Theorem 1.1.

Corollary 1.3. *Assume that $\phi_0 : \mathbf{C} \to \mathbf{C}$ is in $\mathcal{C}$ and its restriction on $[0, \infty)$ is in $\mathcal{O}_a$ for some $a > 1$. Let ϕ_n, $n = 0, 1, 2, \cdots$, be a sequence defined recursively by (1.3) on $\mathbf{C}$. Then if the assumptions in Theorem 1.1 hold, then $\rho \geq \ell_s(a)$.*

If, in addition, $1 < a < c-1$ and there exist positive constants C' and δ such that the function F in (1.9) correpsonding to $\phi = \phi_0$ satisfies $F(x) \geq C'x^a$ for $0 \leq x \leq \delta$, then $\rho = \ell_s(a)$.

Proof of Corollary 1.3 assuming Theorem 1.1 and Theorem 1.2. The assumption (1.6) implies

$$\lim_{n\to\infty} \phi_n(0)^{1/n} = \lim_{n\to\infty} q_n^{1/n} = \rho.$$

Theorem 1.2 for $x = 0$ therefore implies $\rho \geq \ell$, if $0 < \ell < \ell_s(a)$, hence $\rho \geq \ell_s(a)$.

The second part is proved similarly using the second part of Theorem 1.2. $\square$

In Section 3 we also give a sufficient condition for the increasing property of q_n, $n = 0, 1, 2, \cdots$ (Proposition 3.3).

Corollary 1.3 essentially determines the value of ρ (assuming its existence) in (1.6) for the case $1 < a < c-1$, The case $a \geq c-1$ seems harder. However, the following considerations on a specific choice of ϕ_0 suggest that $\rho = e^{1/c}$ for large a. Put

(1.11) $$\phi_0(z) = \frac{1}{z}(1-\exp(-z)),\ z \in \mathbf{C}.$$

Note that this choice of ϕ_0 is both in $\mathcal{C}$ and in $\mathcal{O}_a$ for any $a > 1$, because

$$\frac{1}{z}(1-\exp(-z)) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(k+1)!} z^k,\ z \in \mathbf{C},$$

and

$$\frac{1}{x}(1-\exp(-x)) = \int_0^1 e^{-xt}\, dt,\ x > 0\,.$$

In fact, with this choice, all the assumptions in Theorem 1.1, except perhaps (1.6) are easily seen to hold. Though we were unable to find a proof for (1.6), we have the following result.

Theorem 1.4. *Let ϕ_n, $n = 0, 1, 2, \cdots$, be a sequence defined recursively by (1.3) on $[0, \infty)$, with ϕ_0 as in (1.11). Then q_n is increasing in n, and $\lim_{n\to\infty} q_n^{1/n} \geq e^{1/c}$, where c is as in Theorem 1.2.*

Moreover, if (1.6) holds, then $\lim_{n\to\infty} q_n^{1/n} = e^{1/c}$, in particular, $\rho = e^{1/c}$.

The specific choice (1.11) in this Theorem is motivated by studies in random sequential bisection of a rod and binary search trees [3, 7]. We give a proof of Theorem 1.4 in Section 4. Numerical calculations suggest that q_{n+1}/q_n is decreasing in n, hence (1.6) does hold. We give details in Appendix. Thus we conclude with the following Conjecture.

Conjecture 1.5. *Let ϕ_n, $n = 0, 1, 2, \cdots$, be a sequence defined recursively by (1.3) on $\mathbf{C}$, with ϕ_0 as in (1.11). Then $\lim_{n\to\infty} \phi_n(0)^{-1}\phi_n(\phi_n(0)^{-1}z) = \bar{\phi}(z)$ uniformly on compact sets in $\mathbf{C}$, where $\bar{\phi}(z) = \sum_{k=0}^{\infty}(-1)^k \alpha_k z^k$ is given by (1.8).*

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2 Scaling limit.

Here we prove Theorem 1.1.

First we prove by induction that $\phi_n \in \mathcal{C}$ for all $n = 0, 1, 2, \cdots$. Assume that $\phi_n \in \mathcal{C}$ for a non-negative integer n. $\phi_{n+1}(x) = x^{-1} + o(x^{-2})$, $x \to \infty$, follows directly from (1.3) and $\phi_n(x) = x^{-1} + o(x^{-2})$. Also $\int_0^{\infty} \phi_n(x)^2\, dx < \infty$ exists, and we can rewrite (1.3) as

(2.1) $$\phi_{n+1}(x) = \int_0^{\infty} \phi_n(x')^2\, dx' - \int_0^x \phi_n(x')^2\, dx',\ x \geq 0\,.$$

The integrand in the last term is entire, hence we can analytically continue ϕ_{n+1} to the whole complex plain $\mathbf{C}$ as an entire function, using this expression. Put

$$\phi_n(z) = \sum_{k=0}^{\infty} (-1)^k a_{n,k} z^k, \ z \in \mathbf{C}, \ n = 0, 1, 2, \cdots .$$

Inserting this in (1.3) we find

$$(2.2) \ \ a_{n+1,0} = \phi_{n+1}(0) = q_{n+1} = \int_0^{\infty} \phi_n(x)^2 dx \ (> 0), \ n = 0, 1, 2, \cdots ,$$

and

$$(2.3) \ \ a_{n+1,k} = \frac{1}{k} \sum_{j=1}^{k} a_{n,k-j} a_{n,j-1} \ (\geq 0), \ k = 1, 2, 3, \cdots , \ n = 0, 1, 2, \cdots .$$

This proves that $\phi_{n+1} \in \mathcal{C}$. By induction, $\phi_n \in \mathcal{C}$ for all n.

Let us introduce a notation which we use throughout this section. For $r > 0$, let M_r be a map on a space of infinite sequences

$$M_r : \ a = \{a_k \mid k = 0, 1, 2, \cdots\} \mapsto M_r(a) = \{M_r(a)_k \mid k = 0, 1, 2, \cdots\}$$

defined by $M_r(a)_0 = 1$ and

$$(2.4) \ \ M_r(a)_k = \frac{1}{k r^{k+1}} \sum_{j=1}^{k} a_{k-j} a_{j-1} \, , \ k = 1, 2, 3, \cdots .$$

For sequences $a = \{a_k \mid k = 0, 1, 2, \cdots\}$ and $b = \{b_k \mid k = 0, 1, 2, \cdots\}$ we write $a \leq b$ if $a_k \leq b_k$, $k = 0, 1, 2, \cdots$. Obviously we have, for a non-negative sequence a,

$$(2.5) \ \ M_r(a) \geq M_{r'}(a), \ \text{if } 0 < r \leq r'.$$

Define $\alpha(r) = \{\alpha(r)_k \mid k = 0, 1, 2, \cdots\}$ by

$$(2.6) \ \ M_r(\alpha(r)) = \alpha(r).$$

Then $\alpha = \{\alpha_k \mid k = 0, 1, 2, \cdots\}$ in (1.8) is $\alpha = \alpha(\rho)$.

Next we assume all the assumptions in Theorem 1.1. In particular, we assume $a_{n,0} = q_n \leq q_{n+1} = a_{n+1,0}$. Then, by induction in n using (2.2) we see that

$$(2.7) \ \ a_{n,k} \leq a_{n,0}^{k+1}, \ k = 0, 1, 2, \cdots , \ n = 0, 1, 2, \cdots .$$

In fact, we assumed this for $n = 0$ in Theorem 1.1. Assume that (2.7) holds for some n. Then using (2.3)

$$a_{n+1,k} \leq a_{n,0}^{k+1} \leq a_{n+1,0}^{k+1} \, , \ k = 1, 2, 3, \cdots .$$

Hence (2.7) holds also for $n + 1$. Put

$$\bar{\phi}_n(z) = \sum_{k=0}^{\infty} (-1)^k \bar{\alpha}_{n,k} z^k, \ z \in \mathbf{C}, \ n = 0, 1, 2, \cdots .$$

Then (1.7) implies

$$(2.8) \ \ \bar{\alpha}_{n,k} = q_n^{-k-1} a_{n,k}, \ k = 0, 1, 2, \cdots ,$$

which, with (2.3) and (2.4), satisfies

(2.9) $\bar{\alpha}_{n+1} = M_{q_{n+1}/q_n}(\bar{\alpha}_n),\ n = 0, 1, 2, \cdots,$

where we put $\bar{\alpha}_n = \{\bar{\alpha}_{n,k} \mid k = 0, 1, 2, \cdots\}$.

We now prove the following (2.10) and (2.11):

(2.10) $\{\bar{\phi}_n \mid n = 0, 1, 2, \cdots\}$ is a set of uniformly bounded and equicontinuous functions on $\{z \in \mathbf{C} \mid |z| \le 1/2\}$.

(2.11) $\lim_{n\to\infty} \bar{\alpha}_{n,k} = \alpha_k,\ k = 0, 1, 2, \cdots,$

Applying a standard argument using Ascoli–Arzelà Theorem, we see that (2.10) and (2.11) imply that $\bar{\phi}_n(z)$ converges uniformly to $\bar{\phi}(z)$ on $|z| \le 1/2$. Note that (1.3) implies

(2.12) $$\bar{\phi}_{n+1}(z) = \frac{q_n}{q_{n+1}} \int_{q_n z/q_{n+1}}^{\infty} \bar{\phi}_n(z')^2\, dz'\ ,\ z \in \mathbf{C},\quad n = 0, 1, 2, \cdots.$$

This with (1.6) then imply that $\bar{\phi}_n(z)$ actually converges uniformly to $\bar{\phi}(z)$ on any compact sets as $n \to \infty$, hence $\bar{\phi}$ is entire, and our proof will be complete.

We are left with proving (2.10) and (2.11). To prove (2.10), note that (2.7) and (2.8) (with $a_{n,0} = q_n$) imply $(0 \le)\ \bar{\alpha}_{n,k} \le 1,\ k, n = 0, 1, 2, \cdots$. Hence, if $|z| \le 1/2$ then

$$|\bar{\phi}_n(z)| \le \sum_{k=0}^{\infty} \bar{\alpha}_{n,k} |z|^k \le 2,\ n = 0, 1, 2, \cdots,$$

implying uniform boundedness. Let $\epsilon > 0$. If $|z_i| \le 1/2$, $i = 1, 2$, and $|z_1 - z_2| < \epsilon/4$, then

$$|\bar{\phi}_n(z_1) - \bar{\phi}_n(z_2)| \le \sum_{k=0}^{\infty} \bar{\alpha}_{n,k} |z_1^k - z_2^k| \le |z_1 - z_2| \sum_{k=0}^{\infty} 2^{-k+1} k < \epsilon,$$

which implies equicontinuity. Thus (2.10) is proved.

Finally, we prove (2.11). Let $\gamma = \{\gamma_k \mid k = 0, 1, 2, \cdots\}$, be a non-negative sequence satisfying $\gamma_0 = 1$, and for $r > 0$ define $\tilde{a}(r)_n = \{\tilde{a}(r)_{n,k} \mid k = 0, 1, 2, \cdots\}$, $n = 0, 1, 2, \cdots$, by

(2.13) $\tilde{a}(r)_n = M_r^n(\gamma),\ n = 0, 1, 2, \cdots.$

By (2.5), we see that $\tilde{a}(r)_n$ is decreasing in r.

Lemma 2.1. *For $r > 0$ and for any γ in (2.13), $\tilde{a}(r)_{n,k} = \alpha(r)_k$ if $n \ge k \ge 0$, where $\alpha(r)$ is as in (2.6).*

Proof. By definition, $\tilde{a}(r)_{n,0} = \alpha(r)_0 = 1$, $n = 0, 1, 2, \cdots$, hence in particular the claim holds for $n = 0$. Assume that the claim holds for some n and for all k satisfying $0 \le k \le n$. Then for $1 \le k \le n + 1$ we have

$$\tilde{a}(r)_{n+1,k} = \frac{1}{kr^{k+1}} \sum_{j=1}^{k} \tilde{a}(r)_{n,k-j} \tilde{a}(r)_{n,j-1} = \frac{1}{kr^{k+1}} \sum_{j=1}^{k} \alpha(r)_{k-j} \alpha(r)_{j-1} = \alpha(r)_k,$$

hence the claim holds for $n + 1$. $\square$

Let us proceed with the proof of (2.11), and let $0 < \epsilon < 1$. The assumption (1.6) implies that there exists n_0 such that

$$(1-\epsilon)\rho \le \frac{q_{n+1}}{q_n} \le (1+\epsilon)\rho, \ n \ge n_0,$$

which further implies, by induction, with (2.9) and (2.5),

$$(2.14) M^n_{(1+\epsilon)\rho}(\bar{\alpha}_{n_0}) \le \bar{\alpha}_{n_0+n} = M^n_\rho(\bar{\alpha}_{n_0}) \le M^n_{(1-\epsilon)\rho}(\bar{\alpha}_{n_0}), \ n = 0, 1, 2, \cdots.$$

Put $\gamma = \bar{\alpha}_{n_0}$ in (2.13). Comparing (2.14) with (2.13) we have,

$$\tilde{a}(\rho(1+\epsilon))_n = M^n_{\rho(1+\epsilon)}(\bar{\alpha}_{n_0}) \le \bar{\alpha}_{n+n_0} \le M^n_{\rho(1-\epsilon)}(\bar{\alpha}_{n_0}) = \tilde{a}(\rho(1-\epsilon))_n, \ n = 0, 1, 2, \cdots.$$

With Lemma 2.1 we further have,

$$\alpha(\rho(1+\epsilon))_k \le \bar{\alpha}_{n+n_0,k} \le \alpha(\rho(1-\epsilon))_k, \ n \ge k \ge 0,$$

Hence

$$\alpha(\rho(1+\epsilon))_k \le \liminf_{n\to\infty} \bar{\alpha}_{n,k} \le \limsup_{n\to\infty} \bar{\alpha}_{n,k} \le \alpha(\rho(1-\epsilon))_k, \ k = 0, 1, 2, \cdots.$$

Noting that $0 < \epsilon < 1$ is arbitrary and $\alpha(r)_k$ is a polynomial in r^{-1}, we have (2.11). □

3 Monotonicity arguments.

Here we prove Theorem 1.2.

Put

$$\Omega = \{f : [0,\infty) \to [0,\infty) \mid \text{decreasing, right continuous, } f(0) = 1, \ \lim_{t\to\infty} f(t) = 0\}.$$

If $\phi \in \mathcal{O}_a$, then the corresponding F appearing in the expression (1.9) satisfies $1 - F \in \Omega$.

We shall first rewrite the recursion equation (1.3) in terms of corresponding F's.

For $\ell > 0$ and $f \in \Omega$ define $R_\ell(f) : [0,\infty) \to [0,\infty)$ by $R_\ell(f)(0) = 1$ and

$$(3.1) \ R_\ell(f)(t) = \frac{1}{\ell t}\int_0^{\ell t} f(s) f(\ell t - s)\, ds, \ t > 0.$$

Lemma 3.1. *(i)* $R_\ell(\Omega) \subset \Omega$.

(ii) Let $\ell > 0$ *and* $f \in \Omega$ *and* $g \in \Omega$. *If* $f(t) \le g(t)$, $t > 0$, *then* $R_\ell(f)(t) \le R_\ell(g)(t)$, $t > 0$.

Proof. Note that

$$(3.2) \ R_\ell(f)(t) = \int_0^1 f(\ell t \frac{1-s}{2}) f(\ell t \frac{1+s}{2})\, ds, \ t > 0.$$

Continuity and non-negativity of $R_\ell(f)$ follows from (3.2) and $f \in \Omega$. Then the decreasing property and $R_\ell(f)(t) \le 1$, $t > 0$, follows. $\lim_{t\to\infty} R_\ell(f)(t) = 0$ then follows from (3.2) and the dominated convergence theorem. Hence (i) is proved. Using (3.2) and non-negativity of f and g, (ii) also follows easily. □

Lemma 3.2. *Let $a > 1$ and $\phi_0 \in \mathcal{O}_a$, and let ϕ_n, $n = 0, 1, 2, \cdots$, be a sequence defined recursively by (1.3) on $[0, \infty)$. Then $\phi_n \in \mathcal{O}_a$, $n = 0, 1, 2, \cdots$, and each ϕ_n has an expression*

(3.3) $\phi_n(x) = \int_0^\infty e^{-xt}(1 - F_n(t))\, dt\,,\ x \ge 0,$

with $1 - F_n \in \Omega$, and F_n, $n = 0, 1, 2, \cdots$, satisfies

(3.4) $1 - F_{n+1} = R_1(1 - F_n),\ n = 0, 1, 2, \cdots.$

Proof. By assumption there exists $F_0 \in \Omega$ such that

(3.5) $\phi_0(x) = \int_0^\infty e^{-xt}(1 - F_0(t))\, dt\,,\ x \ge 0\,.$

Define $F_n :\ [0, \infty) \to [0, \infty)$, $n = 0, 1, 2, \cdots$, recursively by (3.4), and put

(3.6) $\tilde{\phi}_n(x) = \int_0^\infty e^{-xt}(1 - F_n(t))\, dt\,,\ x \ge 0,\ n = 0, 1, 2, \cdots.$

Lemma 3.1(i) implies

(3.7) $1 - F_n \in \Omega,\ n = 1, 2, 3, \cdots.$

By definition, $\phi_0 \in \mathcal{O}_a$ implies $F_0(t) \le Ct^a$, $t > 0$. On the other hand, if, for some n, there exists $C_n > 0$ (independent of t) such that $F_n(t) \le C_n t^a$, $t > 0$, then (3.2) implies that there exists $C_{n+1} > 0$ such that

$$F_{n+1}(t) \le C_{n+1} t^{2a} \le C_{n+1} t^a,\ 0 < t \le 1\,.$$

We have already seen that $1 - F_{n+1} \in \Omega$, hence F_{n+1} is bounded. Therefore there exists $C' > 0$ such that $F_{n+1}(t) \le C' t^a$, $t > 0$. By induction, similar estimates hold for all $n = 0, 1, 2, \cdots$. This with (3.7) and (3.6) implies that $\tilde{\phi}_n \in \mathcal{O}_a$, $n = 0, 1, 2, \cdots$. Therefore, if we can prove that $\tilde{\phi}_n = \phi_n$ for all n, the proof of Lemma 3.2 is complete. Since this holds by definition for $n = 0$, it suffices to prove that $\tilde{\phi}_n$ satisfies the same recursion relation (1.3) as ϕ_n. Using (3.1) in (3.4), we have, from (3.6),

$$\begin{aligned}
&\tilde{\phi}_{n+1}(x) = \int_0^\infty dt\, e^{-xt} \frac{1}{t} \int_0^t (1 - F_n(s))(1 - F_n(t - s))\, ds \\
&= \int_0^\infty du \int_0^\infty dv\, \frac{1}{u + v} e^{-x(u+v)} (1 - F_n(u))(1 - F_n(v)) \\
&= \int_x^\infty dy \int_0^\infty du \int_0^\infty dv\, e^{-y(u+v)} (1 - F_n(u))(1 - F_n(v)) \\
&= \int_x^\infty (\tilde{\phi}_n(y))^2\, dy,\ x \ge 0,
\end{aligned}$$

which proves that $\tilde{\phi}_n$ satisfies (1.3). □

Using Lemma 3.1(ii), we can now state a sufficient condition for q_n, $n = 0, 1, 2, \cdots$, to be increasing.

Proposition 3.3. *Let $a > 1$, $\phi_0 \in \mathcal{O}_a$, and F_0 be as in (3.5), and let ϕ_n, $n = 0, 1, 2, \cdots$, be a sequence defined recursively by (1.3) on $[0, \infty)$. If $(1 - F_0)(t) \le R_1(1 - F_0)(t)$, $t > 0$, then $q_n = \phi_n(0)$, $n = 0, 1, 2, \cdots$, is increasing.*

Proof. Lemma 3.2 implies that there exists F_n, $n = 0, 1, 2, \cdots$, such that

$$1 - F_n = R_1^n(1 - F_0) \in \Omega,\ \phi_n(x) = \int_0^\infty e^{-xt}(1 - F_n(t))\, dt\,,\ x \ge 0, \quad n = 0, 1, 2, \cdots.$$

By assumption, $1 - F_0(t) \le 1 - F_1(t)$, $t > 0$. Applying R_1 on both sides and using Lemma 3.1(ii), we inductively obtain

$$(0 \le)\ 1 - F_0(t) \le 1 - F_1(t) \le 1 - F_2(t) \le \cdots,\ t > 0\,.$$

Integrating by $t > 0$, we see that $q_n = \phi_n(0)$ is increasing. □

We next turn to a proof of Theorem 1.2.

Lemma 3.4. *Assume that* $F_0 : [0, \infty) \to [0, \infty)$ *satisfies* $1 - F_0 \in \Omega$ *and* $F_0(t) \le Ct^a$, $t > 0$, *for some positive constants* C *and* a. *Then* F_n, $n = 0, 1, 2, \cdots$, *defined by (3.4) satisfies* $\lim_{n\to\infty} F_n(\ell^n t) = 0$, $t > 0$, *if* $0 < \ell < \ell_s(a)$, *where* $\ell_s(a) = \sup_{0<a'<a} \ell(a')$ *and* $\ell(a) = \left(\dfrac{a+1}{2}\right)^{1/a}$.

Remark. By elementary calculus, we see that $\ell_s(a)$ in this Lemma is equal to $\ell_s(a)$ of (1.10) in Theorem 1.2.

Proof of Lemma 3.4. For $a > 0$ define $\tilde f_a \in \Omega$ by

(3.8) $\tilde f_a(t) = \max\{(1 - t^a), 0\}, \quad t > 0\,.$

Let $a > 0$, $0 < t \le 1$, and $0 < \ell \le \ell(a)$, and put

$$A = \{s \in (0, 1] \mid \tilde f_a(\ell t(1 + s)/2) \ne 0\}\,.$$

We have

$$(0, 1] \setminus A = \{s \in (0, 1] \mid \ell^a t^a (1 + s)^a 2^{-a} \ge 1\}\,.$$

Since $\tilde f_a$ is decreasing, we have,

$$\{s \in (0, 1] \mid \tilde f_a(\ell t(1 - s)/2) = 0\} \subset \{s \in (0, 1] \mid \tilde f_a(\ell t(1 + s)/2) = 0\}\,.$$

Therefore

$$\begin{aligned}
R_\ell(\tilde f_a)(t) &= \int_0^1 \tilde f_a(\ell t\frac{1-s}{2})\tilde f_a(\ell t\frac{1+s}{2})\, ds \\
&= \int_A (1 - \ell^a t^a (1 - s)^a 2^{-a})(1 - \ell^a t^a (1 + s)^a 2^{-a})\, ds \\
&= \int_0^1 [1 - \ell^a t^a (1 - s)^a 2^{-a} - \ell^a t^a (1 + s)^a 2^{-a}]\, ds \\
&\quad + \int_{(0,1]\setminus A} [\ell^a t^a (1 - s)^a 2^{-a} - 1 + \ell^a t^a (1 + s)^a 2^{-a}]\, ds \\
&\quad + \int_A \ell^{2a} t^{2a} (1 - s^2)^a 2^{-2a}\, ds\,.
\end{aligned}$$

Performing the first integration in the right hand side we have

$$R_\ell(\tilde{f}_a)(t) \geq 1 - \ell^a \frac{2}{a+1} t^a \geq 1 - \ell(a)^a \frac{2}{a+1} t^a = 1 - t^a = \tilde{f}_a(t),$$

if $0 < t \leq 1$ and $0 < \ell \leq \ell(a)$. For $t > 1$ we have $R_\ell(\tilde{f}_a)(t) \geq 0 = \tilde{f}_a(t)$. Hence

(3.9) $R_\ell(\tilde{f}_a)(t) \geq \tilde{f}_a(t),\ t > 0,\ 0 < \ell \leq \ell(a)$.

For $a > 0$ define $T_a :\ \Omega \to \Omega$ by $T_a(f)(t) = f(at)$, $t > 0$. Then

(3.10) $R_\ell = R_1 \circ T_\ell$,

and

(3.11) $R_\ell \circ T_a = T_a \circ R_\ell$.

The assumptions on F_0 implies $1 - F_0(t) \geq \tilde{f}_a(C^{1/a}t)$, $t > 0$. Since F_0 is bounded, this implies that for any $0 < a' \leq a$ there exists a positive constant C' such that $1 - F_0(t) \geq \tilde{f}_{a'}(C't) = T_{C'}(\tilde{f}_{a'})(t)$, $t > 0$. Lemma 3.1 and (3.9) with (3.10) and (3.11) then imply

$$\begin{aligned} &1 - F_n(t) = R_{\tilde{\ell}}{}^n(1 - F_0)(\tilde{\ell}^{-n}t) \geq T_{C'} R_{\tilde{\ell}}{}^n(\tilde{f}_{a'})(\tilde{\ell}^{-n}t) \geq T_{C'}(\tilde{f}_{a'})(\tilde{\ell}^{-n}t) = \tilde{f}_{a'}(C'\tilde{\ell}^{-n}t), \\ &\quad t > 0,\ 0 < \tilde{\ell} \leq \ell(a'),\ 0 < a' \leq a. \end{aligned}$$

Therefore if $0 < \ell < \ell_s(a)$, choose $\tilde{\ell} > \ell$ and $a' \leq a$ such that $\tilde{\ell} < \ell(a')$ to find

$$0 \leq \lim_{n\to\infty} F_n(\ell^n t) \leq 1 - \lim_{n\to\infty} \tilde{f}_{a'}(C'\ell^n \tilde{\ell}^{-n} t) = 1 - \tilde{f}_{a'}(+0) = 0,\ t > 0.$$

□

Lemma 3.5. *Assume that* a, b, ℓ *satisfy* $1 < a < c - 1$, $a < b < 2a$, $\ell_s(a) \leq \ell < \ell_s(b)$, *and define* $f_{a,+} :\ [0, \infty) \to [0, 1]$ *by*

$$f_{a,+}(t) = \min\{(1 - t^a + Ct^b), 1\}, \quad t \geq 0,$$

where C *is a constant satisfying*

$$C \geq \max\{1, C_1, C_2\}, \quad C_1 = \left(\frac{\sqrt{\pi}\Gamma(1+a)\,(\ell/2)^{2a}}{2\Gamma(3a/2)\,(1 - (\ell/\ell_s(b))^b)} \right)^{(b-a)/a}, \quad C_2 = \frac{a}{b} \left(\frac{b-a}{b} \right)^{(b-a)/a}.$$

Then

$$R_\ell(f_{a,+}(t)) \leq f_{a,+}(t),\ t \geq 0.$$

Remark. If $1 < a < c - 1$, ℓ satisfying the assumptions exists, but if $a \geq c - 1$ such ℓ does not exist, because then $\ell_s(b) = \ell_s(a) = e^{1/c}$.

Proof. Note that $\ell_s(b) < 2$ for all $b > 1$, as is obvious from (1.10). Note also that $C \geq C_2$ implies $1 - t^a + Ct^b \geq 0$, $t \geq 0$, hence $f_{a,+}$ is non-negative on $[0, \infty)$.

Define $t_0 > 0$ by $Ct_0^{b-a} = 1$. $C \geq 1$ and $b > a$ imply $t_0 \leq 1$. Also it is easy to see that $f_{a,+}(t) = 1$, $t \geq t_0$. By definition, $0 \leq f_{a,+}(t) \leq 1$, $t \geq 0$, hence $0 \leq R_\ell(f_{a,+})(t) \leq 1$, $t \geq 0$. Therefore the statement holds for $t \geq t_0$. In the following we assume $0 < t < t_0$.

Using $f_{a,+}(t) \leq 1 - t^a + Ct^b$ and (3.2), we see that

$$
\begin{aligned}
&R_\ell(f_{a,+})(t) \leq 1 - \left(\frac{\ell}{\ell_s(a)}\right)^a t^a + I_2 + I_3 \,, \\
&I_2 = Ct^b \left(\frac{1}{C} \left(\frac{\ell}{2}\right)^{2a} \int_0^1 (1-u^2)^a \, du \, t^{2a-b} + \left(\frac{\ell}{\ell_s(b)}\right)^b \right), \\
&I_3 = C \left(\frac{\ell t}{2}\right)^{a+b} \int_0^1 (1-u^2)^a \left(-(1-u)^{b-a} - (1+u)^{b-a} + C \left(\frac{\ell t}{2}\right)^{b-a} (1-u^2)^{b-a} \right) du.
\end{aligned}
$$

Using $t < t_0 = C^{-1/(b-a)}$ and $C \geq C_1$ (noting $a < b < 2a$), we find $I_2 \leq Ct^b$. Using $Ct_0^{b-a} = 1$ and $\ell < \ell_s(b) < 2$, we have $C \left(\frac{\ell t}{2}\right)^{b-a} \leq 1$, which further implies, for $0 \leq u \leq 1$,

$$
\begin{aligned}
&-(1-u)^{b-a} - (1+u)^{b-a} + C\left(\frac{\ell t}{2}\right)^{b-a} (1-u^2)^{b-a} \\
&\leq -(1-u)^{b-a} - (1+u)^{b-a} + (1-u^2)^{b-a} \\
&\leq -2\sqrt{(1-u^2)^{b-a}} + (1-u^2)^{b-a} \leq \sqrt{(1-u^2)^{b-a}}\,(-2+1) \leq 0,
\end{aligned}
$$

hence $I_3 \leq 0$.

We therefore have

$$
R_\ell(f_{a,+})(t) \leq 1 - \left(\frac{\ell}{\ell_s(a)}\right)^a t^a + Ct^b \leq 1 - t^a + Ct^b \leq f_{a,+}(t).
$$

$\square$

Proof of Theorem 1.2. Let $0 < \ell < \ell_s(a)$. By assumptions, ϕ_n has an expression

$$
\phi_n(x) = \int_0^\infty e^{-xt}(1 - F_n(t))\, dt \,, \ x \geq 0,
$$

where F_n is as in (3.3). Then Lemma 3.4 and dominated convergence theorem imply

$$
\lim_{n\to\infty} \ell^{-n}\phi_n(\ell^{-n}x) = \lim_{n\to\infty} \int_0^\infty e^{-xs}(1 - F_n(\ell^n s))\, ds = \int_0^\infty e^{-xs}\, ds = x^{-1}, \ x > 0\,.
$$

For $x = 0$, Lemma 3.4 and Fatou's Lemma imply

$$
\liminf_{n\to\infty} \int_0^\infty (1 - F_n(\ell^n s))\, ds \geq \int_0^\infty \liminf_{n\to\infty} (1 - F_n(\ell^n s))\, ds = \infty,
$$

which implies,

$$
\lim_{n\to\infty} \ell^{-n}\phi_n(0) = \infty.
$$

The second part of the Theorem is proved similarly using Lemma 3.5 in place of Lemma 3.4.

$\square$

4 Rod bisection.

Here we prove Theorem 1.4.

By explicit calculation, we see that

$$\frac{1}{x}(1-e^{-x}) = \int_0^\infty e^{-xt}(1-F_0(t))\,dt$$

with

$$(4.1)\quad F_0(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x \ge 1\,. \end{cases}$$

Define $F_n:\ [0,\infty) \to [0,1]$, $n = 0,1,2,\cdots$, recursively by (3.4). Then Lemma 3.2 implies that ϕ_n in Theorem 1.4 is given by the formula (3.3).

Lemma 4.1. *$q_n = \phi_n(0)$ is increasing in n.*

Proof. By explicit calculation, we have

$$F_1(x) = \begin{cases} 0, & 0 < x < 1, \\ 2 - \dfrac{2}{x}, & 1 \le x < 2, \\ 1, & x \ge 2\,. \end{cases}$$

In particular, we have $1 - F_0(x) \le 1 - F_1(x)$, $x > 0$, hence Proposition 3.3 implies that q_n is increasing. □

Note that (4.1) and (3.4) inductively imply

$$(4.2)\quad F_n(x) = 1,\ x > 2^n,\ n = 0,1,2,\cdots.$$

Lemma 4.2. *$\lim_{n\to\infty} q_n^{1/n}$ exists and satisfies $\lim_{n\to\infty} q_n^{1/n} \ge e^{1/c}$.*

Proof. It is noted in eq. (5.1) of [7] that $1 - F_n(x)$ (for the choice (4.1)) is the probability that at the nth stage of random sequential bisection of a rod of length x, all the pieces have length shorter than 1. Namely, one starts with a rod of length x and breaks it into two pieces randomly with uniform distribution. Then one breaks each of the resulting two pieces randomly (with no correlation between the pieces), and so on, and see, after n steps, whether all the 2^n pieces are shorter than unit length.

By performing partial integration on

$$(4.3)\quad q_n = \phi_n(0) = \int_0^\infty (1 - F_n(t))\,dt,$$

and noting (4.2) to deal with the boundary terms, we have, for $n > 0$,

$$(4.4)\quad q_n = \int_0^\infty tF_n'(t)\,dt = \int_0^\infty \frac{1}{t}\sigma_n(t)\,dt\,,$$

where, in the second equality, we made a change of variable $x \to 1/x$ and put

$$\sigma_n(t) = -\frac{dF_n(1/t)}{dt} = \frac{1}{t^2}{F_n}'(\frac{1}{t})\,.$$

$1 - F_n(1/t)$ is the probability that all the pieces have length shorter than 1 at the nth stage of random sequential bisection of a rod of length $1/t$. In other words, it is the probability that starting from a rod of unit length, X_n, the longest piece at nth stage, is shorter than t. Therefore σ_n is the probability density of X_n. Then (4.4) implies that q_n is the expectation value of $1/X_n$. If we denote the expectations by E[$\cdot$], then $q_n = \mathrm{E}[\,1/X_n\,]$.

Now consider the longest piece $\tilde{X}_{n,m}$ at $n+m$th stage among descendants from the longest piece X_n at nth stage. Clearly $\tilde{X}_{n+m} \le X_{n+m}$. Note also that $\tilde{X}_{n,m}/X_n$ and X_n are independent and the former is equal in distribution to X_m. Therefore

$$q_{n+m} = \mathrm{E}[\,\frac{1}{X_{n+m}}\,] \le \mathrm{E}[\,\frac{1}{\tilde{X}_{n+m}}\,] = \mathrm{E}[\,\frac{X_n}{\tilde{X}_{n,m}}\,]\mathrm{E}[\,\frac{1}{X_n}\,] = \mathrm{E}[\,\frac{1}{X_m}\,]\mathrm{E}[\,\frac{1}{X_n}\,] = q_m q_n\,.$$

Hence

(4.5) $q_{n+m} \le q_n q_m\,,\ n, m = 1, 2, 3, \cdots.$

Put $p_n = \log q_n\,,\ n = 1, 2, 3, \cdots$. Then $p_{n+m} \le p_n + p_m,\ n = 1, 2, 3, \cdots$. Using standard arguments on subadditivity, we deduce that

$$\lim_{n\to\infty} \frac{1}{n} p_n = \inf_{n\ge 1} \frac{1}{n} p_n\,.$$

Therefore, the limit

$$\lim_{n\to\infty} \exp(p_n/n) = \lim_{n\to\infty} {q_n}^{1/n} = \inf_{n\ge 1} {q_n}^{1/n}$$

exists.

Note that Theorem 1.2 is applicable to the present choice (4.1) of F_0 with any $a > 1$. Choose $a \ge c-1$. Suppose that $e^{1/c} = \ell_s(a) > \lim_{n\to\infty} q_n^{1/n}$ and choose ℓ so that $e^{1/c} > \ell > \lim_{n\to\infty} q_n^{1/n}$. Then Theorem 1.2 for $x = 0$ implies

$$\lim_{n\to\infty} (\ell^{-1} q_n^{1/n})^n = \lim_{n\to\infty} \ell^{-n}\phi_n(0) = \infty,$$

which further implies $\lim_{n\to\infty} q_n^{1/n} \ge \ell$, which contradicts the choice of ℓ. Therefore $e^{1/c} \le \lim_{n\to\infty} q_n^{1/n}$. □

Lemma 4.3. *If q_n satisfies (1.6), then* $\lim_{n\to\infty} q_n^{1/n} \le e^{1/c}$.

Proof. In [3] it is essentially proved that

(4.6) $\lim_{n\to\infty} F_n(\ell^n x) = 1,\ \text{ if } \ell > e^{1/c},\ x > 0\,.$

(In the reference, (4.6) may not be explicit, but this property of F_n is essentially used there to prove that $H_k/\log k \to c$, in probability, as $k \to \infty$. Here H_k is the height of a binary search tree with k nodes constructed by standard insertions from a random permutation of k positive integers. In fact, it is easy to see that (4.6) and Lemma 2.1 in [3] imply $H_k/\log k \to c$ in probability. One should note the correspondence $F_n(t) = \mathrm{Prob}[\,Z_n \ge t^{-1}\,]$ between our notation and the notation in the reference.) Hence dominated convergence implies, for each $x > 0$,

(4.7) $\lim_{n\to\infty} \ell^{-n}\phi_n(\ell^{-n}x) = 0,\ \ell > e^{1/c}\,.$

As noted in the Introduction, ϕ_0 in Theorem 1.4 satisfies all the assumptions in Theorem 1.1 except perhaps (1.6). Hence if (1.6) also holds, the consequences of Theorem 1.1 hold. In particular, the uniform convergence of $\bar{\phi}_n(z) = q_n^{-1}\phi_n(q_n^{-1}z)$ (in a neighborhood of $z = 0$) and $\bar{\phi}_n(0) = 1$, $n \in \mathbf{Z}_+$, imply, with (3.3),

$$(\forall\epsilon > 0)\exists\delta > 0,\ \exists n_0 \in \mathbf{N};\ (\forall n \geq n_0)\ (\forall 0 \leq x \leq \delta)\ \ |\int_0^\infty e^{-xt}(1 - F_n(q_n t))dt - 1| < \epsilon.$$

If $q = \lim\limits_{n\to\infty} q_n^{1/n} > \ell$, then $\ell^n \leq q_n$ for sufficiently large n, Since $F_n(t)$ is increasing in t, this implies, for sufficiently large n,

$$\ell^{-n}\phi_n(\ell^{-n}x) \geq 1 - \epsilon, \quad 0 \leq x \leq \delta.$$

Comparing with (4.7), we see that $\ell \leq e^{1/c}$. This holds for all $\ell < q$, hence $q \leq e^{1/c}$. □

Lemma 4.1, Lemma 4.2, and Lemma 4.3 prove Theorem 1.4.

Remark. The arguments in [3] use, in particular, the Biggins-Kingman-Hammersley theorem, which in turn is based on arguments of Chernov type inequalities together with the law of large numbers for superconvolutive sequences [1, 2, 5, 6]. The arguments are strong enough to control ${q_n}^{1/n}$, but unfortunately cannot control q_n to prove the existence of $\lim\limits_{n\to\infty} q_{n+1}/q_n$.

A Appendix.

We have no explicit example for which all the assumptions in Theorem 1.1 are proven to hold. Even for the most promising case (1.11), we lack a proof of (1.6). We give some of our numerical results for the ratio q_{n+1}/q_n in Table 1.

Table 1: Numerical results for q_n/q_{n-1}. The number of sample points is $N = 3200$. The digits shown are stable between $N = 1600$ and $N = 3200$ results.

n	0	1	2	3	4	5	6	7	8	9	10	20	30	40
q_{n+1}/q_n	1.3863	1.3666	1.3520	1.3408	1.3319	1.3247	1.3188	1.3138	1.3095	1.3059	1.3027	1.2851	1.2777	1.2737

Numerical values are obtained by discretizing $1 - F_n$ (i.e., represent the function by its values at a finite number, say N, of points), and performing numerical integration (i.e., approximating by a discrete sum of N terms) of (3.4), starting from (4.1). The results suggest that q_{n+1}/q_n is decreasing in n, hence (1.6) does hold. Our data is also consitent with $\lim\limits_{n\to\infty} q_{n+1}/q_n = e^{1/c} = 1.261\cdots$.

In spite of the promising numerical results, a proof of (1.6) seems not very easy. Numerical results suggest that q_{n+1}/q_n is decreasing in n. In fact $q_2/q_1 \leq q_1/q_0$, or equivalently, $q_2 \leq q_1^2$ does hold, by substituting $n = m = 1$ in (4.5). However, $q_3/q_2 \leq q_2/q_1$, or $q_3 q_1 \leq q_2^2$, already seems rather hard. For example, recall the proof of Lemma 4.2 and for $n = 1, 2, 3, \cdots$ and $k = 0, 1, 2, \cdots, 2^{n-1} - 1$, let $Z_{n,k}$ be the length ratio of left piece of rod at n-th bisection stage, to the k-th piece at $n-1$-st stage. Start with two rods and perform two (independent) bisection stages $Z_{n,k}$ and $Z'_{n',k'}$. Denote by X_n and X'_n the length of the longest pieces, respectively, at

n-th stage. For example, $X_1 = \max\{Z_{10}, 1 - Z_{10}\}$. Then $q_n q_m = \mathrm{E}[\ X_n X'_m\ ]$. Now consider conditional expectations

$$Q_{31}(Z_{10}, Z_{20}, Z_{21}, Z'_{10}) = \mathrm{E}[\ X_3 X'_1 \mid Z_{10}, Z_{20}, Z_{21}, Z'_{10}\ ]$$

and

$$Q_{22}(Z_{10}, Z_{20}, Z_{21}, Z'_{10}) = \mathrm{E}[\ X_2 X'_2 \mid Z_{10}, Z_{20}, Z_{21}, Z'_{10}\ ].$$

Our expectation is $\mathrm{E}[\ Q_{31}\ ] = q_3 q_1 \le q_2^2 = \mathrm{E}[\ Q_{22}\ ]$. A sufficient condition for this to hold is

(A.1) $Q_{31} \le Q_{22},\ a.s.$ (?)

However, it turns out that (taking an obvious contiuous versions of conditional expectations)

$$x\, Q_{22}(1/2, x, 1/2, 1/2) = 16(1 - \log 2) = 4.9\cdots,\ \frac{1}{2} \le x \le 1,$$

while

$$\lim_{x \uparrow 1} x\, Q_{31}(1/2, x, 1/2, 1/2) = 8 \log 2 = 5.5\cdots,$$

and

$$\frac{1}{2}\, Q_{31}(1/2, 1/2, 1/2, 1/2) = \frac{80}{3} - 32 \log 2 = 4.4\cdots,$$

so that (A.1) does not hold. This shows some difficulties encountered in an attempt to prove monotonicity of q_{n+1}/q_n.

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