Renormalization group approach to an extension of the law of iterated logarithms for one-dimensional (non-Markovian) stochastic chain

RIMS, 2003.9.10
Kumiko HATTORI (Shinshu U.)
Tetsuya HATTORI (Nagoya U.)

Tetsuya Hattori, *Random walks and renormalization groups — an introduction to mathematical physics,* Kyoritsu publishing, 2004.3, to appear (in Japanese).

1. Introduction
   - Expectation on 'Mathematics' of RG
2. Law of iterated logarithms (LIL)
   - Asymptotic behavior ('exponent') of paths
3. Main results
   - RG, construction of stochastic chains, generalized LIL
4. Displacement exponent for self-repelling walk

§1. Introduction.

- 'Mathematics' of RG (still a long way to go)

— a mathematical tool (calculus), and structure

- Return to a simplest model

scale change of the accuracy of observation

— Stochastic chains (probability measure on the set of paths) on $\mathbb{Z}$, with 1-dimensional RG (nearest neighbor jumps)

- 'A diet coke is good after chinese dishes.' (K.R.Ito)

We also have corresponding results on the following:

- Continuum limit continous processes
- Chains and processes on the Sierpinski gasket

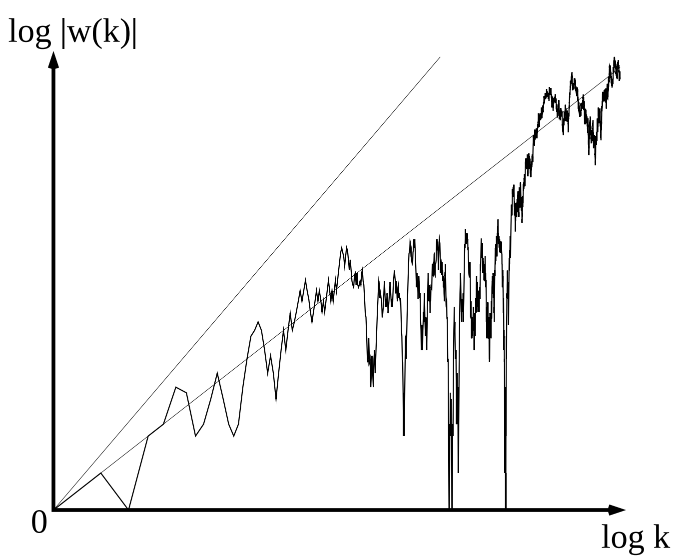
Hambly, K.Hattori, T.Hattori, PTRF 124 (2002)
K.Hattori, T.Hattori, preprint, 2003

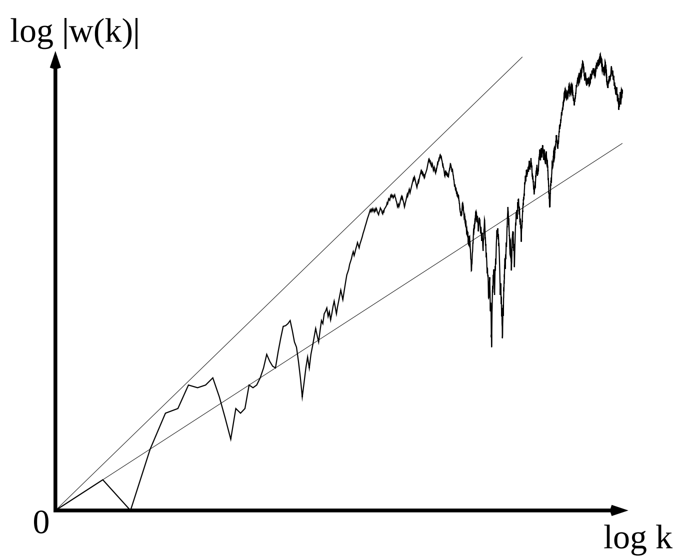
§2. Law of iterated logarithms (LIL).

Theorem (Khintchine, 1924). Let W_k, $k \in \mathbb{Z}_+$, be SRW on $\mathbb{Z}$ with $W_0 = 0$. Then

$$\mathrm{P}\Big[\ \overline{\lim_{k\to\infty}} \frac{W_k}{\sqrt{k \log\log k}} = 1\ \Big] = 1. \qquad \diamond$$

• $W_k \sim k^{1/2}$ (as for CLT and displacement exponent). $\log\log$ correction is automatic from RG — The exponent $1/2$ is a consequence of fluctuations, hence path fluctuates around the average $k^{1/2}$


log |w(k)|
0
log k


log |w(k)|
0
log k

What is new in our work?

• Previous works — exponent $\nu = 1/2$

We generalize to all ν — existence proof of a chain with exponent ν.

• Decimation for SRW known. (F.B.Knight, 1962)

We do not use Markov properties. (Note $\nu \neq 1/2$ suggests non-Markov.)

RG as a new math. to analyze non-Markov proc.

Remark.

Why LIL and not displacement exponent?

$\mathrm{E}[\ w(k)^s\ ] \sim k^{\nu s}$

For Markov chains displacement exponents are easier because of independence of increments $W_{k+1} - W_k$, but we are working on non-Markovian chains!

• Displacement exponent for self-repelling walks (K.Hattori, T.Hattori, 2003)

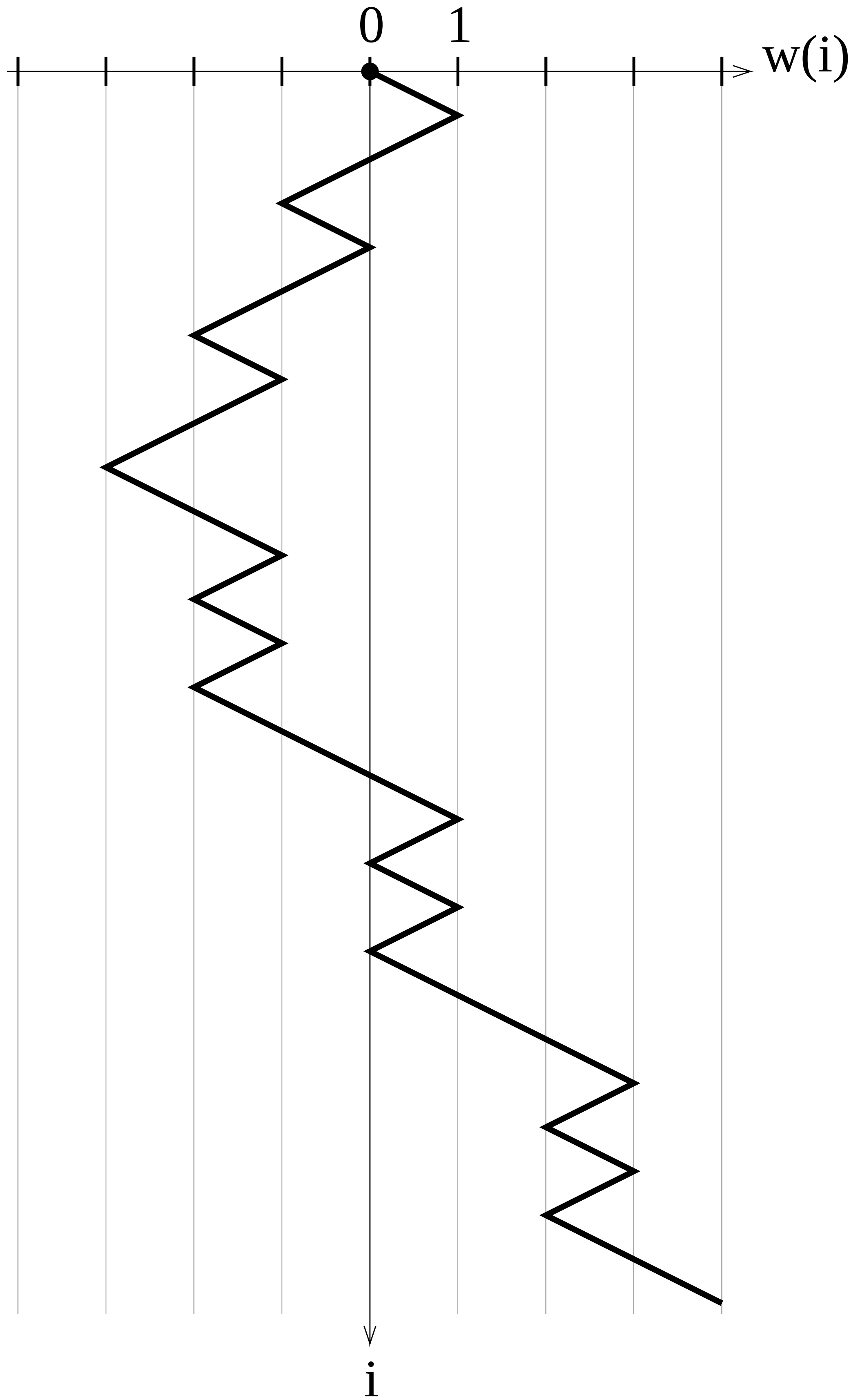
§3. Main results.

Path on $\mathbb{Z}$.

$L \in \mathbb{Z}$ or $L = \infty$ (length).

$w: \ \{0, 1, \cdots, L\} \to \mathbb{Z}$;

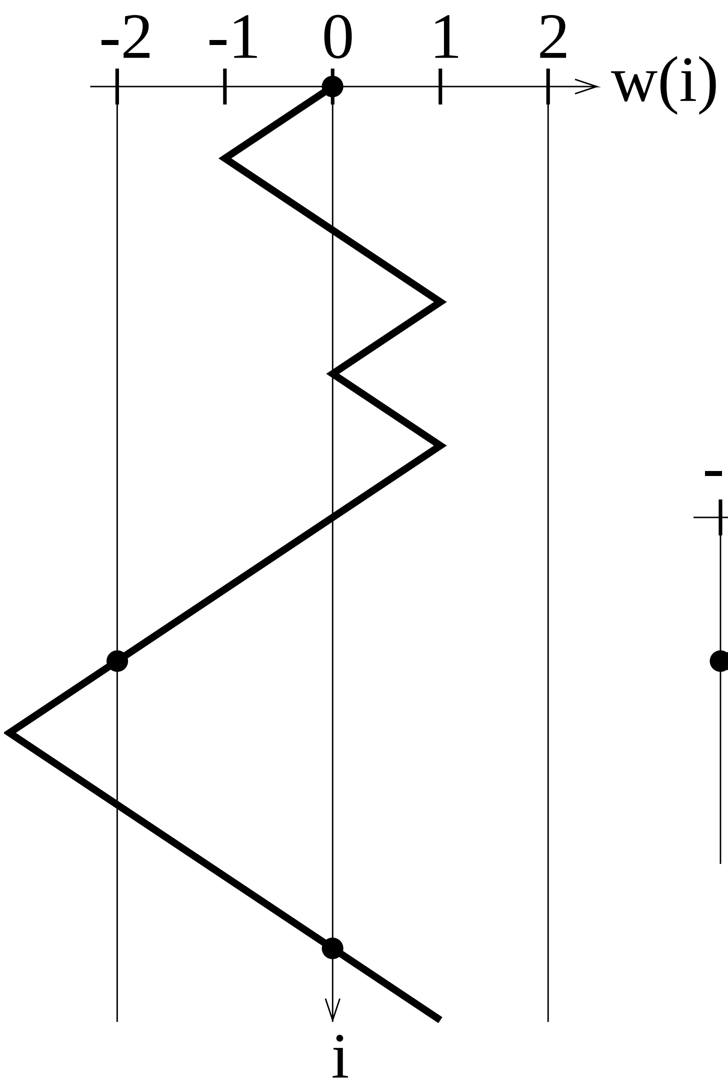
$w(0) = 0$, $|w(i) - w(i-1)| = 1$, $i = 1, \cdots, L$

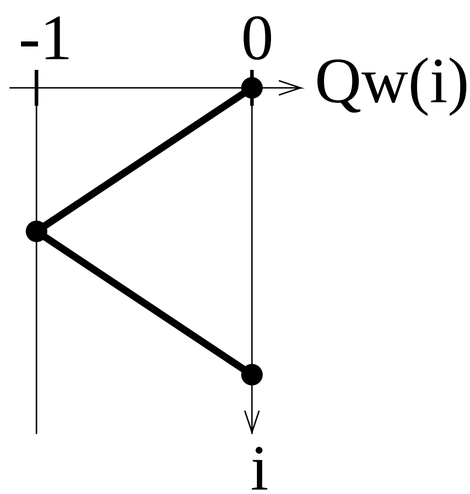


Stochastic chain = Prob. measure on the set of $L = \infty$ paths

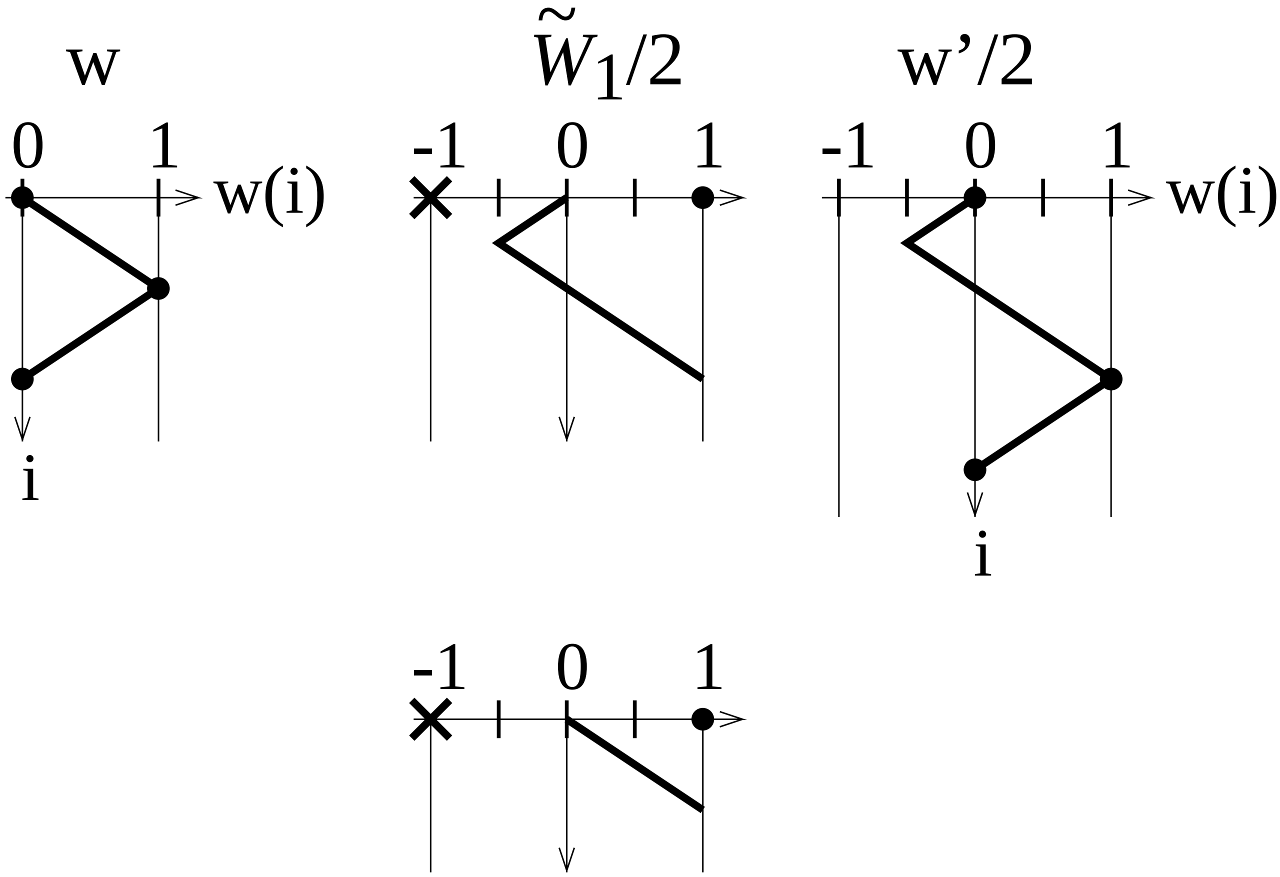
1. Decimation
2. Analysis of RG
3. Construction of chains consistent with RG
4. Asymptotics from RG (generalized LIL)

1. Decimation. Scale change of the accuracy of observation
$Q: \ w \mapsto Qw$; $(Qw)(i) = \frac{1}{2} w(T_i(w))$; $T_0(w) = 0$,
$T_{i+1}(w) = \inf\{j > T_i(w) \mid w(j) \in 2\mathbb{Z} \backslash \{w(T_i(w))\}\}$





‘Fine structures’ lost by Q (added by Q^{-1}):
$\tilde{W}_1$: set of paths; $L < \infty$, ending at 2, which do not hit -2.



$$\boxed{\Phi_1(z) = \sum_{w \in \tilde{W}_1} b_1(w) z^{L(w)} =: \sum_{k=0}^{\infty} c_k z^k.}$$

(Note $c_0 = c_1 = 0$.)

Assumptions on b_1 (or c_k):

(i) $b_1(w) \geqq 0$

(ii) radius of convergence $r > 0$

(iii) $c_2 > 0$ and $\exists k \geqq 3$; $c_k > 0$

Proposition.

(i) $\exists! x_c$; $\Phi_1(x_c) = x_c$, $0 < x_c < r$

(ii) $\lambda := \Phi_1'(x_c) > 2$ $\diamond$

(SRW: $\Phi_1(z) = \frac{x^2}{1-2x^2}$, $x_c = 1/2$, $\lambda = 4$)

RG: the dynamical system determined by Φ_1

$\Phi_{n+1} = \Phi_1 \circ \Phi_n$, $n = 1, 2, 3, \cdots$

Note $\Phi_n(z) = \sum_{w \in \tilde{W}_n} b_n(w) z^{L(w)}$; $\tilde{W}_n$: $0 \to 2^n$, $\not\longrightarrow 2^n$

— Representation in the parameter space of the scale change (addition of fine structure $\tilde{\mathcal{W}}_1$)

2. Analysis of RG.

$\mathrm{P}_n[\ \{w\}\ ] := b_n(w) {x_c}^{L(w)-1}$ defines prob. meas. on $\tilde{\mathcal{W}}_n$

$\tilde{\mathrm{P}}_n$: scaled length distribution on $\tilde{W}_n$;

$$\int e^{-s\xi} \tilde{\mathrm{P}}_n[\ d\xi\ ] = x_c^{-1} \Phi_n(e^{-\lambda^{-n} s} x_c)$$

$(= \sum_{w \in \tilde{W}_n} e^{-s\lambda^{-n} L(w)} \mathrm{P}_n[\ \{w\}\ ])$

Theorem. $\exists \tilde{\mathrm{P}}_*$; $\tilde{\mathrm{P}}_n \to \mathrm{P}_*$. Additional estimates on rate of convergence and limitting distributions such as:

(i) $\exists \rho(\xi) d\xi = \tilde{\mathrm{P}}_*[\ d\xi\ ]$, C^∞, positive.

(ii) $\nu = \log 2 / \log \lambda$,

$-C \leqq \varliminf_{x \to 0} x^{\nu/(1-\nu)} \log \tilde{\mathrm{P}}_*[\ [0, x]\ ] \leqq \varlimsup \leqq -C'$, $x > 0$ $\diamond$

Note $\exists \rho(\xi)$ implies non-deterministic (non-trivial).

$k \sim \lambda^n \ \Leftrightarrow \ x = 2^n \quad \Rightarrow \quad x \sim k^\nu$

3. Chains consistent with RG.

$(\tilde{\mathcal{W}}_n, \mathrm{P}_n)$: paths with fixed endpoints. $\leftrightarrow$

Chain: meas. on infinite length path (LIL considers limit for each path) with pos. at fixed length W_k measurable

$\tilde{W}_n^r$: $\tilde{W}_n$ with $w \mapsto -w$

Prob. meas. $\mathrm{P}_{r,n}$ on $\tilde{W}_n^r$; $\mathrm{P}_{r,n}[\ \{w\}\ ] = \mathrm{P}_n[\ \{-w\}\ ]$

Theorem (Hattori—Hattori, 2003).

$\exists \{W_k\};\ (\forall w;\ L(w) = k)(\forall n;\ 2^n > \max\limits_{0 \leqq j < k} |w(j)|)$

$$\mathrm{P}[\ W_j = w(j),\ 0 \leqq j \leqq k\ ]$$
$$= \tfrac{1}{2}\mathrm{P}_n[\ \{w' \in \tilde{W}_n \mid w'(j) = w(j)\,,\ 0 \leqq j \leqq k\}\ ]$$
$$+ \tfrac{1}{2}\mathrm{P}_{r,n}[\ \{w' \in \tilde{W}_n^r \mid w'(j) = w(j)\,,\ 0 \leqq j \leqq k\}\ ]$$

$\diamond$

RG serves as consistency condition!

4. Asymptotics from RG (generalized LIL).

Theorem (Hattori—Hattori, 2003).
Let $\nu = \dfrac{\log 2}{\log \lambda}$; $\lambda = \Phi'(x_c)$. Then $\exists C_\pm > 0$;

$$\mathrm{P}[\ C_- \leqq \varlimsup_{k\to\infty} \frac{|W_k|}{k^\nu (\log\log k)^{1-\nu}} \leqq C_+\ ] = 1 \qquad \diamond$$

Idea of proof.

- RG estimates on hitting time of $2^n \to \mathrm{P}[\ W_k < C2^n\ ]$.
- Prob. 1 statement from (modified) Borel-Cantelli Th. for scale parameter n.

Lower bd: BC2 (independence among scales). cf. Previous results on SRW: BC2 for step number $k \leftarrow$ requires Markov property.

$\{A_k\}\ \perp$, $\sum_{k=1}^\infty \mathrm{P}[\ A_k\ ] = \infty \to \mathrm{P}[\ \varlimsup_{k\to\infty} A_k\ ] = 1$

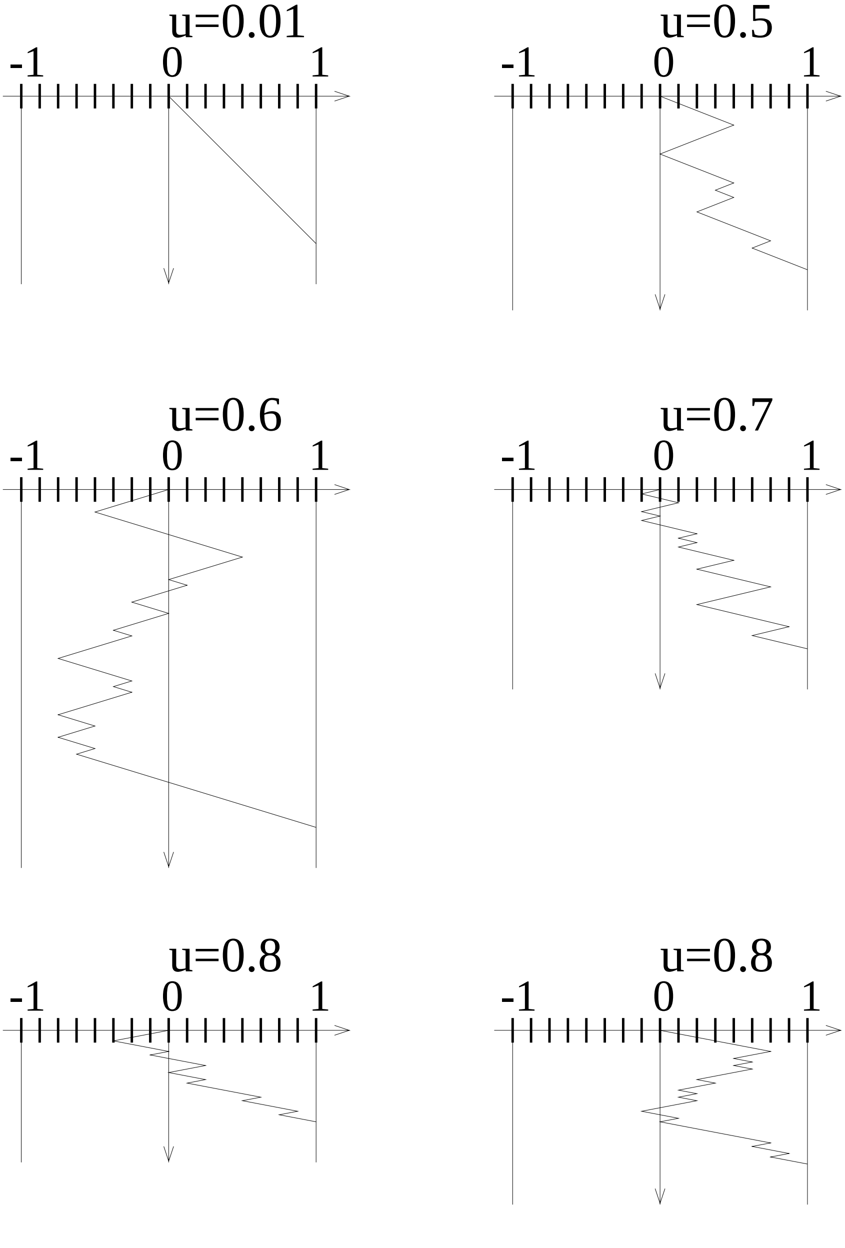
§4. Displacement exponent for self-repelling walk.

SRW — Markov, $\nu = 1/2$; $|W_k| \sim k^\nu$
Self-avoiding path — non-Markov extreme, $\nu = 1$ on $\mathbb{Z}$
• continous interpolation?

Theorem (Hattori—Hattori, 2003). $\exists$ measures on $L = \infty$ path P_u, $u \in [0,1]$;

1. $u = 1$: SRW on $\mathbb{Z}$ (or Sierpiński gasket)
2. $u = 0$: SAP
3. Displacement exponent
$$\lim_{k\to\infty} \frac{1}{\log k} \log \mathrm{E}_u[\, |W_k|^s \,] = s\nu_u,\ s \geqq 0\,,$$
is conti. in u

Construction for measures on $\mathbb{Z}$:
Generating function of L for SAP $\Phi_{0,1}(z) = z^2$
Generating function for SRW
$$\Phi_{1,1}(z) = \Phi_1(z) = \frac{z^2}{1 - 2z^2}$$
Interpolation! $\Phi_{u,1}(z) = \dfrac{z^2}{1 - 2u^2 z^2}$
$\nu_u = \dfrac{\log 2}{\log \lambda_u}$, $\lambda_u = \Phi'_{u,1}(x_{c,u})$, $x_{c,u} = \Phi_{u,1}(x_{c,u})$
displacement exponent $\leftarrow$ reflection principle $\leftarrow$ explicit form of weights


u=0.01
-1
0
1
u=0.5
-1
0
1
u=0.6
-1
0
1
u=0.7
-1
0
1
u=0.8
-1
0
1
u=0.8
-1
0
1