Renormalization group approach to an extension of the law of iterated logarithms for one-dimensional (non-Markovian) stochastic chain

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§0. Excuses. I remember a post-doc. physicist from Germany 20 years ago, making a keen observation that Japanese seminar talks start with 'excuses'.

Now, I first declined the organizer K.R.Ito-san's invitation, writing 'A talk like law of iterated logarithms for 1-dim chain will receive negative response from the audience equipped with deep physics and hard analysis', to which his answer was 'A diet coke is good after chinese dishes'. I was interested in this seemingly contradictory responce, that Ito-san invites a talk which he thinks has no calorie in it. Hoping to solve the contradiction at the meeting!

§1. Definitions and main results. For $L \in \mathbb{Z}_+ \cup \{\infty\}$ we say that $w: \{0,1,\cdots,L\} \to \mathbb{Z}$ is a path of length L (on $\mathbb{Z}$ starting from the origin), if it satisfies $w(0) = 0$ and $|w(i) - w(i-1)| = 1$ $(i = 1,2,\cdots,L)$. Let $\tilde{W}_n$ be a set of paths of finite length, ending at 2^n, and which do not hit -2^n. Let

$$\Phi_1(z) = \sum_{w \in \tilde{W}_1} b_1(w) z^{L(w)} =: \sum_{k=0}^{\infty} c_k z^k$$

be a generating function (with weight b_1) of L for $\tilde{W}_1$, and we shall call the dynamical system generated by Φ_1, the renormalization group. Φ_n, defined inductively by $\Phi_{n+1} = \Phi_1 \circ \Phi_n$, $n = 1,2,3,\cdots$, has a canonical form $\Phi_n(z) = \sum_{w \in \tilde{W}_n} b_n(w) z^{L(w)}$.

The main aim of the talk is [1, 3]: (i) to state the existence proof of stochastic chain compatible with the renormalization group, and (ii) to give explicitly the 'exponent' for the asymptotic properties of the chain (a generalization of the law of iterated logarithms) in terms of the differential at the fixed point of the renormalization group.

Our (mild) assumptions on Φ_1 is the following: $c_k \geqq 0$ $(k = 2,3,4,\cdots)$, radius of convergence $r > 0$, $c_2 > 0$, $\exists k \geqq 3$; $c_k > 0$. (Note $c_0 = c_1 = 0$ is automatic.)

Proposition. There exists a unique fixed point x_c of Φ_1, satisfying $0 < x_c < r$, and it holds that $\lambda := \Phi'(x_c) > 2$. $\diamond$

Define a probability measure on $\tilde{W}_n$ by $\mathrm{P}_n[\ \{w\}\ ] = b_n(w) x_c^{L(w)-1}$.

Theorem. A sequence of scaled length distributions defined by the generating function $x_c^{-1}\Phi_n(e^{-\lambda^{-n}s} x_c) = \sum_{w \in \tilde{W}_n} e^{-s\lambda^{-n}L(w)} \mathrm{P}_n[\ \{w\}\ ]$ converges weakly as $n \to \infty$. $\diamond$

Additional results on rate of convergence and on detailed properties of the limitting distributions also follow from the renormalization group and Tauberian theorems.

Let $\tilde{W}_n^r$ be a set of paths consisting of paths in $\tilde{W}_n$ reflected at the origin $(w \mapsto -w)$. Define a probability measure $\mathrm{P}_{r,n}$ on $\tilde{W}_n^r$ by $\mathrm{P}_{r,n}[\ \{w\}\ ] = \mathrm{P}_n[\ \{-w\}\ ]$. The next theorem states the existence of a stochastic chain compatible with the renormalization group.

Theorem. There exists a stochastic chain $\{W_k\}$ such that for each path w of length k,

$$\begin{aligned}\mathrm{P}[\ W_j = w(j),\ 0 \leqq j \leqq k\ ] = &\frac{1}{2}\mathrm{P}_n[\ \{w' \in \tilde{W}_n \mid w'(j) = w(j)\,,\ 0 \leqq j \leqq k\}\ ] \\ &+ \frac{1}{2}\mathrm{P}_{r,n}[\ \{w' \in \tilde{W}_n^r \mid w'(j) = w(j)\,,\ 0 \leqq j \leqq k\}\ ]\end{aligned}$$

holds for any $n \in \mathbb{N}$ satisfying $2^n > \max_{0 \leqq j < k} |w(j)|$. ◇

The stochastic chain constructed by this theorem satisfies the following generalized law of iterated logarithms.

Theorem. There exist positive constants $C_\pm$ such that $C_- \leqq \overline{\lim_{k\to\infty}} \frac{|W_k|}{k^\nu (\log\log k)^{1-\nu}} \leqq C_+$, *a.s.*, where $\nu = \frac{\log 2}{\log \lambda}$. ◇

§2. Motivation. Renormalization group (RG) has attracted so many people as a program to analyze complex (non-differentiable, random) geometrical objects such as fields and configurations. In physics, some people even went as far as to say that RG is finished long time ago. However, the progress of RG as a mathematical analysis is far from satisfactory. That half a century has past without final results for critical exponents for 3 dimensional Ising model and continuum limits of 4 dimensional QCD, may suggest that we might be more humble and return to the simplest models.

RG is the dynamical system on a parameter space, representing the response of a system to a scale change of the accuracy of observation. As a simplest model with non-trivial action of RG, we consider a one-dimensional stochastic chain (probability measure on the set of paths on $\mathbb{Z}$). This talk employs a further simplification of restricting to the 1-dimensional (1–parameter) RG.

The law of iterated logarithms is originally a statement for 1 dimensional simple random walk W_k, stating that each sample has asymptotically a maximal range of $\sqrt{k \log\log k}$ at k-th step. More precisely, it states $\overline{\lim_{k\to\infty}} \frac{W_k}{\sqrt{k \log\log k}} = 1$, *a.s.* The theorem is long known for the simple random walk, and there are results on other stochastic processes too, but there were no generalizations to the case with the 'exponent' ν other than 1/2, to the authors' knowledge.

As a differential equation determines a trajectory, and as a partial differential equation (PDE) or a stochastic differential equation (SDE) determines a time evolution of a classical field or a stochastic process, we expect that RG, when mathematically completed, proves existence and determines propeties of a stochastic process (and a quantum field, eventually). In contrast to the processes determined by PDE (through transition probabilities) or SDE, which assumes martingale or Markov properties (hence the 'exponent' must essentially be 1/2), RG is expected to push the horizon of analysis to the non-Markovian processes. The present talk is a mathematical evidence on the simplest toy model for such expectations.

In this talk, we consider stochastic chains for simplicity. As RG is compatible with continuum limits, we have rigorous results also for the continuous stochastic processes [2].

References

[1] Tetsuya HATTORI, *Random walks and renormalization groups — an introduction to mathematical physics —,* Kyoritsu publishing, 2004.3, to appear (in Japanese).

[2] B. Hambly, K. Hattori, T. Hattori, *Self-repelling Walk on the Sierpiński Gasket,* Probability Theory and Related Fields **124** (2002) 1–25.

[3] K. Hattori, T. Hattori, *Displacement exponents of self-repelling walks on the pre-Sierpiński gasket and* $\mathbb{Z}$*,* preprint, 2003.