Renormalization group analysis of the self-avoiding paths on the d-dimensional Sierpiński gaskets

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Abstract

Notion of the renormalization group dynamical system, the self-avoiding fixed point and the critical trajectory are mathematically defined for the set of self-avoiding walks on the d-dimensional pre-Sierpiński gaskets (n-simplex lattices), such that their existence imply the asymptotic behaviors of the self-avoiding walks, such as the existence of the limit distributions of the scaled path lengths of 'canonical ensemble', the connectivity constant (exponential growth of path numbers with respect to the length), and the exponent for mean square displacement.

We apply the so defined framework to prove these asymptotic behaviors of the restricted self-avoiding walks on the 4-dimensional pre-Sierpiński gasket.

Key words. renormalization group, self-avoiding walk, fractals, Sierpinski gasket

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1 Introduction and main results.

1.1 Self-avoiding walk on the Sierpiński gasket.

Self-avoiding walks on hypercubic lattices $\mathbb{Z}^n$ have been studied mathematically for half a century, but compared to random walks (and diffusion processes, their continuum limits), amazingly little is known [11].

For random walks, nice properties such as Markov properties enabled deep and accurate studies, many of which are effective for spaces with any dimension n. On the other hand, self-avoiding walks seem to have little such strong general methods. In fact, their behaviors are expected to vary drastically with the dimension n for small n, so that effective methods possibly vary for different spaces.

Turning our attention to the 2- and 3-dimensional Sierpiński gaskets, there are works on the restricted self-avoiding walks (a subset of self-avoiding walks, to be defined in Section 3.1) in [1], and mathematically rigorous studies for the full self-avoiding walk (including a proof that the restriced self-avoiding walk of [1] are in the same universality classes with the full self-avoiding walk), with further precise asymptotic results, exist for both the 2-dimensional Sierpiński gasket [5, 4, 7] and the 3-dimensional Sierpiński gasket (4-simplex lattice) [6].

In the direction of generalization to d-dimensional Sierpiński gaskets, there is a work [9] on the restricted model for $d = 4, 5$, following the lines of [1] with a propsal of an approximation method for general d. (d-dimensional Sierpiński gasket is the $d+1$-simplex lattice in [9].) However, studies in the direction of extending the rigorous renormalization group analysis to d-dimensional cases have not appeared, to the authors' knowledge.

A main object of this paper to propose a general and mathematically rigorous renormalization group formulation of the self-avoiding walks on dSG for all d, from which one can derive asymptotic behaviors. As an application we prove asymptotic behaviors, such as the exponent for mean square displacement, of the restricted model of self-avoiding walks on 4SG. (The restricted model considers those self-avoiding walks which does not take 2 or more steps in row in each unit simplices (Section 3.1).)

We emphasize that a rigorous renormalization group analysis is non-trivial for the self-avoiding walks on dSG. Though it is easy to write down the renormalization group recursion equations for small d, it is of course another thing to analyze their trajectories rigorously. (Rigorous analysis of renormalization group trajectories and rigorous proofs of their implications on asymptotic behaviors of self-avoiding walks seem to have been ignored in the physics literature.)

It is not because the life is simple on gaskets that the gaskets are appealing, but because (as we will show in this paper) we can formulate and prove with mathematical rigor that an appropriate renormalization group formulation contains full imformation of asymptotic behaviors of self-avoiding walks. Since the renormalization group analysis contains full information on asymptotic behaviors, the authors think that it is too important not to analyze them with mathematical rigor and in generality (as we do in this paper).

1.2 Renormalization group approach.

General 'philosophy' of the renormalization group (RG) in physics (and the previous rigorous studies on 2SG and 3SG) suggest that a RG approach to the asymptotic behaviors of the self-avoiding paths on dSG starts with splitting the analysis into two parts:

(i) Formulate the RG, a dynamical system on a 'natural' parameter space, and then derive nice properties about the fixed points and the trajectories of the RG flows, such as uniqueness of certain fixed point and convergence of critical trajectories.

(ii) Derive asymptotic behaviors of the self-avoiding paths from the properties of RG flows.

The RG is a dynamical system determined by a recursion map $\vec{\Phi}$, which will be defined in (10), on a finite dimensional Eulidean space (the parameter space $\mathbb{R}^{\mathcal{I}_d}$ defined in (3)). For general case of physical interest, we should consider infinite dimensional parameter space, but the so called finite ramifiedness of dSG implies that the RG in the present study is finite dimensional. The RG map is a response in the parameter space to the 'scale transformation' (smoothing out or putting in finer structures to the paths) on the space of paths. (The transformation suitable for paths on dSG is a decimation, which will be implicit in the proof of Proposition 4.)

The quantities we need to extract from the RG map $\vec{\Phi}$ are the following.

(i) The largest eigenvalue λ of the differential map of $\vec{\Phi}$ at a self-avoiding fixed point $\vec{x}_c$.

(ii) The critical point β_c, which is the intersection point of the critical surface (the set of points from which the trajectories of RG converge to the self-avoiding fixed point $\vec{x}_c$) and the canonical curve (the curve defined by (17)).

We give the precise definitions of λ and β_c and also the assumptions on the RG map $\vec{\Phi}$ at (FP1) – (FP4) and $(CS1)$ in Section 3.1. (To state them rigorously, we need to prepare technically cumbersome definitions in Section 2 starting from the definition of dSG.)

In this paper we will prove the following. Fix $d \geqq 2$. For each $k \in \mathbb{Z}_+$, let $N(k)$ be the number of k step self-avoiding paths on dSG starting from the origin O, and let $E_k[\cdot]$ be the expectation with respect to the uniform distribution (averaging with equal weight) on such paths.

Theorem 1 (Theorem 10 and Theorem 11) *If there exists a critical point β_c then*

(i) $\displaystyle\lim_{k\to\infty} \frac{1}{k} \log N(k) = \beta_c\,.$

(ii) $\displaystyle\lim_{k\to\infty} \frac{1}{\log k} \log E_k[|w(k)|^{s\,d_w}] = s, \quad s \geqq 0,$ *where $|\cdot|$ denotes the Euclidean length and* $\displaystyle d_w = \frac{\log \lambda}{\log 2}\,.$

The first result says that the connectivity constant of the self-avoiding paths on dSG is e^{β_c}. The second result says that the exponent for mean square displacement is $1/d_w$, which indicates that a typical k step self-avoiding path w deviates from the starting point by $|w(k)| \sim k^{1/d_w}$. (Since Theorem 11 holds for all $s \geqq 0$, we have the exponent for all the moments as well as that for the mean square displacement, but we will keep the good old terminology in this paper.) We will prove an additional statement on the correction to the 'leading terms' $N(k) \sim e^{\beta_c k}$ and $|w(k)| \sim k^{1/d_w}$. See Theorem 10 and Theorem 11 for details.

Possibly the notions such as fixed points and critical points are not new from the view point of philosophy of RG. What is new here is that we propose a mathematically well-defined formulation (Section 3.1) which are sufficient (Section 3.2) to prove asymptotic behaviors of self-avoiding walks on dSG with all d, giving a mathematical evidence that the dynamics of RG contains information on the asymptotic behaviors of stochastic processes.

As an application of the formulation, we prove in Section 5 that the assumptions on the RG map in Section 3.1 are satisfied for the restricted model of self-avoiding paths on 4SG.

Theorem 2 (Theorem 31 and Theorem 33) *The self-avoiding fixed point $\vec{x}_c$ and the critical point $\beta_{c,res}$ of the restricted model on the 4 dimensional pre-Sierpiński gasket (4SG) exists.*

In particular, the number $N_{res}(k)$ of restricted self-avoiding paths of length k starting from 0 satisfies

$$\lim_{k\to\infty} \frac{1}{k} \log N_{res}(k) = \beta_{c,res}\,.$$

and the exponent for mean square displacement for the restricted model is $\displaystyle d_w = \frac{\log \lambda}{\log 2} = 1.6657696\cdots$, *in the sense that*

$$\lim_{k\to\infty} \frac{1}{\log k} \log E_{res,k}[|w(k)|^{s\,d_w}] = s, \quad s \geqq 0,$$

where $E_{res,k}$ is the expectation with respect to the probability measure with equal weight on length k restricted self-avoiding paths starting at O.

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2 Renormalization group.

2.1 Self-avoiding paths on the d-dimensional pre-Sierpiński gasket.

Let $d \geqq 2$ be an integer. We define a d-dimensional pre-Sierpiński gasket (pre-dSG) as follows. Consider a d-simplex of a unit side length embedded in $\mathbb{R}^d$, and let $G_0 = \{v_0, v_1, v_2, \cdots, v_d\}$ be the set of vertices of the d-simplex, where $v_0 = O = (0, 0, \cdots, 0)$ is the origin of $\mathbb{R}^d$. (We may occasionally also write $v_{0,i} = v_i$, $i = 1, \cdots, d$.) Let $B_0 = \{(v_i, v_j) \mid 0 \leqq i < j \leqq d\}$ be the set of non-ordered pairs of vertices, and we denote the pair (G_0, B_0) by F_0.

We define a sequence $F_n = (G_n, B_n)$, $n = 1, 2, 3, \cdots$, of finite pre-dSG inductively by

$$G_{n+1} = \bigcup_{i=0}^{d} (G_n + 2^n v_i), \ B_{n+1} = \bigcup_{i=0}^{d} (B_n + 2^n v_i), \quad n = 1, 2, 3, \cdots, \tag{1}$$

where we write $A + v = \{x + v \mid x \in A\}$ for a set A and a point v.

F_n is a d-simplex of side length 2^n, composed of $d+1$ copies of F_{n-1}, with $d+1$ outmost points being $v_{n,0} = O$ and $v_{n,i} = 2^n v_i$, $i = 1, 2, 3, \cdots, d$. G_n is a collection of vertices in the copies of G_{n-1}, and B_n is a collection of bonds in the copies of B_{n-1}.

We call

$$F = (G, B); \quad G = \bigcup_{n=0}^{\infty} G_n, \ B = \bigcup_{n=0}^{\infty} B_n, \tag{2}$$

the d-dimensional pre-Sierpiński gasket (pre-dSG). We identify $(v, v') \in B$ with line segments $\overline{vv'}$ whenever it would be natural to do so.

Denote the set of non-negative integers by $\mathbb{Z}_+$, and for $w : \mathbb{Z}_+ \to G$, denote by $L(w) \in \mathbb{Z}_+ \cup \{\infty\}$ ('the length of w') the smallest integer satisfying

$$w(i) = w(L(w)), \quad i \geqq L(w).$$

Define the set of self-avoiding paths W_0 to be the set of maps $w : \mathbb{Z}_+ \to G$, such that

$$\begin{array}{ll} w(i_1) \neq w(i_2), & 0 \leqq i_1 < i_2 \leqq L(w), \\ |w(i) - w(i+1)| = 1, & 0 \leqq i \leqq L(w) - 1, \\ \overline{w(i)w(i+1)} \in B, & 0 \leqq i \leqq L(w) - 1. \end{array}$$

2.2 Overview of technical definitions.

We need to prepare several basic definitions in Section 2.3, Section 2.4, and Section 2.5 before introducing the main notions in Section 3.1. Here we will briefly explain the basic definitions.

Section 2.3. We first classify how a self-avoiding path intersects a unit d-simplex. A path which enters a simplex moves within the simplex for at most d steps (because it may not hit the same vertex twice). If a path takes i_1 steps in the simplex and goes into an adjacent one, and never returns to the simplex, we label the intersection of the path and the simplex by the index $(i_1, 0, \cdots, 0)$. Alternatively, the path may return to the simplex a number of times, and for each return the intersection may be labelled by how many steps the path takes in the simplex. Thus if a path spends 3 steps for the first intersection and 1 step for the second intersection with a simplex, then we label the intersection by the index $(1, 3, 0, \cdots, 0)$. (For our purpose we may identify $(1, 3, 0, \cdots, 0)$ and $(3, 1, 0, \cdots, 0)$; we are free to rearrange a sequence in an index in the ascending order.) We denote the set of the indices by $\mathcal{I}_d$.

Each index corresponds to a component in the parameter space on which the RG map acts. Therefore for each index $I \in \mathcal{I}_d$, we need a set of self-avoiding paths $W_I^{(n)}$ on G_n labelled by I which has a similar structure as the intersection of a path and a unit simplex labelled by I. For an index with more than one non-zero entries, such as $I = (1, 3, 0, \cdots, 0)$, the set $W_I^{(n)}$ is defined to be a set of collection of self- and mutually-avoiding paths on G_n. For example, $W_{(1,3,0,\cdots,0)}^{(n)}$ is a set of disjoint pairs of self-avoiding paths on G_n, such that one path starts and ends at outmost vertices of G_n, but hits no other outmost vertices, while the other path hits two outmost vertices other than the endpoints.

Section 2.4. The RG in our study is the recursion map in n of the joint generating functions $\vec{X}_n = (X_{n,I}(\vec{x}),\ I \in \mathcal{I}_d)$ of $s_J,\ J \in \mathcal{I}_d$, for $W_I^{(n)}$, where s_J is the number of unit simplices whose intersection with the path is of type J.

A similarity of finite gaskets G_n among different ns implies a recursion relation to hold for all n, and this is our RG. In this way we arrive at a mathematically well-defined notion of 'a response in the parameter space of the scale transformation in the path space'.

Section 2.5. A study in 3SG shows [6] that in general there are more than one non-trivial fixed points of the RG. Therefore we have to know which fixed point is relevant for the asymptotic behavior of the self-avoiding paths. It turns out that the condition that the fixed point is in a certain invariant set of the RG ensures our proof to work. To formulate the condition (see (FP4)), we introduce the invariant set Ξ_d.

We note that it would also be useful for intuitive understanding to look at the case of 3SG, which is explicitly given in [6],

2.3 Classification of self-avoiding paths.

Denote by $\mathcal{T}_b$ the family of all the translations of B_0 that are subsets of B. Namely, $\mathcal{T}_b$ contains all the unit d-simplices which compose the pre-dSG. (with each simplex regarded as a collection of bonds). Put

$$\mathcal{I}_d = \{(i_1, i_2, \cdots, i_k) \in \mathbb{Z}_+^k \mid k = 1, 2, 3, \cdots,\ 0 < i_1 \leqq i_2 \leqq \cdots \leqq i_k\,,\ i_1 + i_2 + \cdots + i_k + k \leqq d + 1\}, \tag{3}$$

and denote the number of elements of $\mathcal{I}_d$ by $f_d = \sharp \mathcal{I}_d$.

Proposition 3 *Let $w \in W_0$ and $\Delta \in \mathcal{T}_b$, and consider the set of bonds*

$$A = \{\overline{w(i)w(i+1)} \in \Delta \mid i = 0, 1, 2, \cdots, L(w)\}.$$

If A is not empty, then there exists $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_d$ such that A is congruent to

$$\Delta_I = \{\overline{Ov_1v_2\cdots v_{i_1-1}v_{i_1}}, \overline{v_{i_1+1}\cdots v_{i_1+i_2}}, \cdots, \overline{v_{i_1+\cdots i_{k-1}+1}\cdots v_{i_1+\cdots i_k}}\}, \tag{4}$$

where we used an abbreviation such as

$$\overline{Ov_1v_2\cdots v_{i_1-1}v_{i_1}} = \overline{Ov_1}, \overline{v_1v_2}, \overline{v_2v_3}, \cdots, \overline{v_{i_1-1}v_{i_1}}\,.$$

Example.

- $\mathcal{I}_2 = \{(1), (2)\}$: $A \neq \emptyset$ is congruent to either $\{\overline{Ov_1}\}$ or $\{\overline{Ov_1v_2}\}$.
- $\mathcal{I}_3 = \{(1), (2), (3), (1, 1)\}$: There is a possibility that a path enters a unit tetrahedron twice, as $\{\overline{Ov_1}, \overline{v_2v_3}\}$.
- $\mathcal{I}_4 = \{(1), (2), (3), (4), (1, 1), (1, 2)\}$.
  Correspondingly, $f_2 = 2$, $f_3 = 4$, $f_4 = 6$. ◇

Proof of Proposition 3. If $A \neq \emptyset$, namely, if the path w enters the unit d-simplex specified by Δ, then A is composed of one or more connected clusters. That is, w may pass through Δ and may come back and reenter Δ. Since w is self-avoiding, the second passage does not intersect with the first one. Thus we can classify A by the size of the connected segments. One may rearrange the segments in an increasing order of size, hence each class is determined by an increasing finite sequence of positive integers, $i_1 \leqq i_2 \leqq i_3 \leqq \cdots \leqq i_k$ for some $k \geqq 1$. The meaning of the conditions in the definition of $\mathcal{I}_d$ should now be obvious. Since Δ is a translation of B_0 which is the set of bonds in the unit d-simplex $Ov_1v_2\cdots v_d$, the statement follows. □

In analogy with Proposition 3 we can classify the set of self-avoiding paths on F_n by $\mathcal{I}_d$, and also generalize to two or more self-avoiding paths.

For $n \in \mathbb{Z}_+$ and $u, v \in G_n$, define $W^{(n,u,v)}$ by

$$W^{(n,u,v)} = \{w \in W_0 \mid w(0) = u,\ w(L(w)) = v, \quad w(i) \in G_n\,,\ i \in \mathbb{Z}_+\}.$$

For $n \in \mathbb{Z}_+$ and $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_d$, define $W_I^{(n)}$ by

$$\begin{aligned} W_I^{(n)} = \{&(w_1, w_2, \cdots, w_k) \in W^{(n,O,v_{n,i_1})} \times W^{(n,v_{n,i_1+1},v_{n,i_1+i_2+1})} \times W^{(n,v_{n,i_1+i_2+2},v_{n,i_1+i_2+i_3+2})} \\ &\times \cdots \times W^{(n,v_{n,i_1+i_2+\cdots+i_{k-1}+k-1},v_{n,i_1+i_2+\cdots+i_k+k-1})} \mid \\ &\text{if } i \neq j \text{ then } w_i \text{ and } w_j \text{ do not hit common points, and for each } j \\ &w_j \text{ hits points } v_{n,i_1+i_2+\cdots+i_{j-1}+j-1},\ v_{n,i_1+i_2+\cdots+i_{j-1}+j},\ v_{n,i_1+i_2+\cdots+i_{j-1}+j+1}, \\ &\cdots\ v_{n,i_1+i_2+\cdots+i_{j-1}+i_j+j-1},\ \text{in this order}, \\ &\text{but hits no other points in } \{v_{n,\ell} \mid \ell = 0, 1, 2, \cdots, d\}\}. \end{aligned} \tag{5}$$

Obviously, k is equal to the number of path segments that form an element in $W_I^{(n)}$.

Example. For $d=4$, there are $f_4=6$ types of sets $W_I^{(n)}$, which are

$\{(1)\}$: Set of paths from O to $v_{n,1}$ which do not hit $v_{n,2}$, $v_{n,3}$, $v_{n,4}$.

$\{(2)\}$: Set of paths from O to $v_{n,2}$ passing through $v_{n,1}$ which do not hit $v_{n,3}$, $v_{n,4}$.

$\{(3)\}$: Set of paths from O to $v_{n,3}$ passing through $v_{n,1}$ and $v_{n,2}$ in this order and avoiding $v_{n,4}$.

$\{(4)\}$: Set of paths from O to $v_{n,4}$ passing through $v_{n,1}$, $v_{n,2}$, and $v_{n,3}$ in this order.

$\{(1,1)\}$: Set of pair of (self- and mutually-avoiding) paths, one from O to $v_{n,1}$ and the other from $v_{n,2}$ to $v_{n,3}$ neither hitting $v_{n,4}$.

$\{(1,2)\}$: Set of pair of paths, one from O to $v_{n,1}$ and the other from $v_{n,2}$ to $v_{n,4}$ via $v_{n,3}$.

◇

For $w\in\bigcup_{n\in\mathbb{Z}_+}\bigcup_{I\in\mathcal{I}_d}W_I^{(n)}$ denote by $\hat{w}$ the set of bonds which w passes. Namely, for $n\in\mathbb{Z}_+$ and $I=(i_1,i_2,\cdots,i_k)\in\mathcal{I}_d$, and for $w=(w_1,w_2,\cdots,w_k)\in W_I^{(n)}$,

$$\hat{w}=\{\overline{w_j(i)w_j(i+1)}\in B \mid i=0,1,2,\cdots,L(w_j)-1,\ j=1,2,\cdots,k\}.$$

Also define $S_I(w)$, $I\in\mathcal{I}_d$, by,

$$S_I(w)=\{\Delta\in\mathcal{T}_b \mid \hat{w}\cap\Delta \text{ is congruent to } \Delta_I \text{ of (4)}\}, \tag{6}$$

and denote by $s_I(w)=\sharp S_I(w)$, the cardinality of $S_I(w)$. $s_I(w)$ is the number of unit d-simplices in F such that the trajectory of the path (or the paths) w is congruent to Δ_I. It is a generalized notion of the length of the path in the sense that

$$\sum_{i=1}^{k}L(w_i)=\sum_{J\in\mathcal{I}_d}|J|s_J(w),\quad w=(w_1,w_2,\cdots,w_k)\in W_I^{(n)},\ I=(i_1,\cdots,i_k)\in\mathcal{I}_d,\ n\in\mathbb{Z}_+, \tag{7}$$

where, for $J\in\mathcal{I}_d$ we define $|J|$, the length of J, by

$$|J|=j_1+\cdots+j_\ell,\quad \text{if}\quad J=(j_1,\cdots,j_\ell). \tag{8}$$

2.4 Parameter space and the renormalization group.

Assumptions of the main results are stated in terms of the flows of the associated renormalization group (RG), which is a map (discrete-time dynamical system) in a parameter space of variables in the generating function of generalized path length $(s_J,\ J\in\mathcal{I}_d)$. The dynamical system is derived as the response in the parameter space to the change in n. A graphical property of dSG called finite ramifideness implies that the RG is a finite dimensional dynamical system.

Define the generating function

$$\vec{X}_n=(X_{n,I},\ I\in\mathcal{I}_d):\ \mathbb{C}^{\mathcal{I}_d}\to\mathbb{C}^{\mathcal{I}_d}$$

of $(s_J,\ J\in\mathcal{I}_d)$ for a family of paths sets $(W_I^{(n)},\ I\in\mathcal{I}_d)$, by,

$$X_{n,I}(\vec{x})=\sum_{w\in W_I^{(n)}}\prod_{J\in\mathcal{I}_d}x_J^{s_J(w)},\quad \vec{x}=(x_J,\ J\in\mathcal{I}_d)\in\mathbb{C}^{\mathcal{I}_d},\ n=0,1,2,\cdots. \tag{9}$$

The right hand side is a finite summation, so $X_{n,I}$ is defined on $\mathbb{C}^{\mathcal{I}_d}$.

The starting point of our analysis is the following.

Proposition 4 $\vec{X}_n=(X_{n,I},\ I\in\mathcal{I}_d)$, $n=0,1,2,\cdots$, *satisfy the following recursion relations.*

$$\vec{X}_0(\vec{x})=\vec{x},\ \vec{x}\in\mathbb{C}^{\mathcal{I}_d},$$

and

$$\vec{X}_{n+1} = \vec{\Phi} \circ \vec{X}_n \,, \tag{10}$$

where

$$\vec{\Phi} = (\Phi_I,\ I \in \mathcal{I}_d) = \vec{X}_1 \,,$$

is a f_d dimensional vector valued function whose components are polynomials in f_d variables with positive integer coefficients. In particular, $\mathbb{R}_+^{\mathcal{I}_d}$ is an invariant set of $\vec{\Phi}$.

The degree of each term in the polynomials are no less than 2 and no greater than $d+1$, and $\Phi_{(1)}$ contains terms ${x_{(1)}}^2$ and ${x_{(1)}}^{d+1}$.

Proof. Let $n \in \mathbb{Z}_+$. F_1 is composed of $d+1$ d-simplices congruent to F_0. Similarly, F_{n+1} is composed of $d+1$ d-simplices F_n. The similarity of the two compositions leads to a natural map

$$\pi : \ W_I^{(n+1)} \to W_I^{(1)} \,, \quad I \in \mathcal{I}_d.$$

For each $X_{n+1,I}$, classify the summation in the right hand side of (9) (with $n+1$ in place of n) by $\pi(w) \in W_I^{(1)}$ to find

$$X_{n+1,I}(\vec{x}) = \sum_{w \in W_I^{(n+1)}} \prod_{J \in \mathcal{I}_d} x_J^{s_J(w)} = \sum_{w' \in W_I^{(1)}} \sum_{w \in W_I^{(n+1)};\ \pi(w)=w'} \prod_{J \in \mathcal{I}_d} x_J^{s_J(w)}$$

$$= \sum_{w' \in W_I^{(1)}} \prod_{I' \in \mathcal{I}_d} \left(\sum_{w'' \in W_{I'}^{(n)}} \prod_{J \in \mathcal{I}_d} x_J^{s_J(w'')} \right)^{s_{I'}(w')} = \sum_{w' \in W_I^{(1)}} \prod_{I' \in \mathcal{I}_d} (X_{n,I'}(\vec{x}))^{s_{I'}(w')} = X_{1,I}(\vec{X}_n(\vec{x})).$$

By definition (9), each term in $\Phi_I = X_{1,I}$ has a form $\prod_{J \in \mathcal{I}_d} x_J^{s_J(w)}$, hence its degree $\sum_{J \in \mathcal{I}_d} s_J(w)$ is, by definition (6), the number of unit simplices in F_1 that a path w passes through. This is bounded from above by the total number of unit simplices in F_1, which is $d+1$, and from below by 2, because any two extreme (outmost) vertices of F_1 is apart by length 2.

Positivity of coefficients of $X_{1,I}$ are obvious. Existence of terms $x_{(1)}^2$ and $x_{(1)}^{d+1}$ in $\Phi_{(1)} = X_{1,(1)}$ follows from the paths $\overline{Ov_{0,1}v_{1,1}}$ and $\overline{Ov_{0,d}(v_{0,d}+v_{0,d-1})(v_{0,d-1}+v_{0,d-2})\cdots(v_{0,2}+v_{0,1})v_{1,1}}$ in $W_{(1)}^{(1)}$. □

Large n means that the endpoints of the paths are far apart, hence it corresponds to large path length L. Intuitively speaking Proposition 4 therefore gives a response to the change in the length scale of the system in consideration, the sets of self-avoiding paths, in terms of the parameter space of variables in the generating functions of s_I, the generalized path length. Global properties of the trajectories of the map $\vec{\Phi}$ therefore is expected to give (and we will show that it does) large length asymptotic behaviors of self-avoiding paths on dSG.

In analogy to the (mathematically misleading) terminology in physics literature, we call the discrete-time dynamical system on $\mathbb{R}_+^{\mathcal{I}_d}$ defined by the map $\vec{\Phi}$, the renormalization group (RG).

2.5 Invariant sets.

If there is a subset of $\mathbb{R}_+^{\mathcal{I}_d}$ which is an invariant set of the RG map $\vec{\Phi}$, then the recursion (10) is naturally regarded as a recursion equation on the subset.

For $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_\ell)$ in $\mathcal{I}_d$, denote by $I \oplus J$ the rearrangement of $i_1, \cdots, i_k, j_1, \cdots, j_\ell$ in non-decreasing order, and define $\Xi_d \subset \mathbb{R}_+^{\mathcal{I}_d}$ by

$$\Xi_d = \{\vec{x} \in \mathbb{R}_+^{\mathcal{I}_d} \mid x_{I \oplus J} \leqq x_I x_J \ \text{ for all } \ I, J \in \mathcal{I}_d \ \text{ such that } \ I \oplus J \in \mathcal{I}_d\}. \tag{11}$$

Example. $\Xi_3 = \{\vec{x} \in \mathbb{R}_+^{\mathcal{I}_3} \mid x_{(11)} \leqq x_{(1)}^2\}$, $\Xi_4 = \{\vec{x} \in \mathbb{R}_+^{\mathcal{I}_4} \mid x_{(11)} \leqq x_{(1)}^2,\ x_{(12)} \leqq x_{(1)} x_{(2)}\}$. ◇

Proposition 5 *Ξ_d is an invariant set of $\vec{\Phi}$.*

Proof. Let $I, J, I\oplus J \in \mathcal{I}_d$. Note that there is a natural one-to-one into map $W^{(1)}_{I\oplus J} \to W^{(1)}_I \times W^{(1)}_J$. For $w\in W^{(1)}_{I\oplus J}$, let $(w_1,w_2)\in W^{(1)}_I \times W^{(1)}_J$ be the corresponding pair. Then, for each $\Delta\in\mathcal{T}_b$, $\hat{w}\cap\Delta$ may be regarded as a composition of $\hat{w}_1\cap\Delta$ and $\hat{w}_2\cap\Delta$, hence if $\hat{w}\cap\Delta$ is congruent to Δ_K of (4) for some $K\in\mathcal{I}_d$, then there exists $K_1,K_2\in\mathcal{I}_d$ (allowing an emptyset) such that $K=K_1\oplus K_2$ and such that $\hat{w}_i\cap\Delta$, $i=1,2$, is congruent to Δ_{K_i}, $i=1,2$, respectively. Note also that $\vec{x}\in\Xi_d$ implies $x_{K_1\oplus K_2}\leqq x_{K_1}x_{K_2}$.

Therefore by definition (Proposition 4 and (9)),

$$\Phi_{I\oplus J}(\vec{x})=\sum_{w\in W^{(1)}_{I\oplus J}}\prod_{K\in\mathcal{I}_d}{x_K}^{s_K(w)}\leqq\sum_{w_1\in W^{(1)}_I}\sum_{w_2\in W^{(1)}_J}\prod_{K_1\in\mathcal{I}_d}{x_{K_1}}^{s_{K_1}(w_1)}\prod_{K_2\in\mathcal{I}_d}{x_{K_2}}^{s_{K_2}(w_2)}=\Phi_I(\vec{x})\Phi_J(\vec{x}).$$

□

In the following, for $\mathcal{K}\subset\mathcal{I}_d$, we use a (somewhat irregular) notation

$$\mathbb{R}_+^{\mathcal{K}}=\{\vec{x}\in\mathbb{R}_+^{\mathcal{I}_d}\mid x_J=0,\ J\notin\mathcal{K}\}\ \subset\ \mathbb{R}_+^{\mathcal{I}_d}. \tag{12}$$

We also write $\mathbb{C}^{\mathcal{K}}\subset\mathbb{C}^{\mathcal{I}_d}$, $\mathbb{Z}_+^{\mathcal{K}}\subset\mathbb{Z}_+^{\mathcal{I}_d}$, etc.

Define

$$\mathcal{K}_{res}=\{(1),(11),\cdots,(1\cdots1)\}. \tag{13}$$

The indices in $\mathcal{K}_{res}$ correspond to those paths which go out of a simplex after single step passage each time they enter the simplex.

Proposition 6 $\mathbb{R}_+^{\mathcal{K}_{res}}$ *is an invariant subset of* $\vec{\Phi}$.

Proof. This is proved by generalizing the arguments in the proof of [6, Proposition 2.1 (4)(5)]. □

3 Main results.

3.1 Fixed point and critical trajectory.

Based on experiences with dSG for $d=2,3,4$, we define notions which are relevant for asymptotic behaviors of self-avoiding paths on dSG.

Denote the Jacobi matrix of $\vec{\Phi}$ in Proposition 4 by $\mathcal{J}=(\mathcal{J}_{IJ})$:

$$\mathcal{J}_{IJ}(\vec{x})=\frac{\partial\Phi_I}{\partial x_J}(\vec{x}),\quad I,J\in\mathcal{I}_d\,,\ \vec{x}\in\mathbb{C}^{\mathcal{I}_d}. \tag{14}$$

We say that $\vec{x}_c\in\mathbb{R}_+^{\mathcal{I}_d}$ is a *self-avoiding fixed point*, if the following hold.

(FP1) $\vec{\Phi}(\vec{x}_c)=\vec{x}_c$.

(FP2) $\mathcal{J}(\vec{x}_c)$ in (14) is diagonalizable by an invertible matrix. The eigenvalue λ which is largest in absolute value satisfies $\lambda>1$ with multiplicity 1, and all the other eigenvalues have absolute values strictly less than 1 .

Denote by $\vec{v}_L=(v_{L,I},\ I\in\mathcal{I}_d)$ a left eigenvector of $\mathcal{J}(\vec{x}_c)$ corresponding to λ;

$$\sum_{I\in\mathcal{I}_d}v_{L,I}\mathcal{J}_{IJ}(\vec{x}_c)=\lambda v_{L,J}\ ,\quad J\in\mathcal{I}_d\,,$$

which we chose to have non-negative components (possible, thanks to Frobenius' theorem). Then $v_{L,J}>0$, $J\in\mathcal{I}_d$.

Similarly, denote by $\vec{v}_R$ a right eigenvector corresponding to λ with non-negative components;

$$\sum_{J\in\mathcal{I}_d}\mathcal{J}_{IJ}(\vec{x}_c)v_{R,J}=\lambda v_{R,I}\ ,\quad I\in\mathcal{I}_d\,.$$

Then $v_{R,(1)}>0$.

(FP3) For all $I \in \mathcal{I}_d$ such that $x_{c,I} \neq 0$, there exists $m \in \mathbb{Z}_+^{\mathcal{I}_d}$, satisfying $m_{(1)} > 0$ and $m_J = 0$ if $x_{c,J} = 0$, such that there is a term $\prod_{J \in \mathcal{I}_d} {x_J}^{m_J}$ in Φ_I .

(FP4) $\vec{x}_c \in \Xi_d \setminus \{\vec{0}\}$.

Assume that there exists a self-avoiding fixed point $\vec{x}_c$. We say that $\vec{x} \in \mathbb{R}_+^{\mathcal{I}_d}$ is *in the domain of attraction of* $\vec{x}_c$, if the following hold.

(DA1) $\lim_{n \to \infty} \vec{X}_n(\vec{x}) = \vec{x}_c$.

(DA2) If $x_{c,I} \neq 0$ then $x_I \neq 0$.

We denote by $\mathcal{D}om(\vec{x}_c)$ the set of $\vec{x} \in \mathbb{R}_+^{\mathcal{I}_d}$ which are in the domain of attraction of $\vec{x}_c$.

Example. If $\vec{x}_c$ is a self-avoiding fixed point, then $\vec{x}_c \in \mathcal{D}om(\vec{x}_c)$, i.e., a self-avoiding fixed point satisfies (DA1) – (DA2).

◇

Let $\mathcal{K} \subset \mathcal{I}_d$. Instead of (5), we may consider a set of walks $W_{\mathcal{K},I}^{(n)}$ by restricting to those paths in $W_I^{(n)}$ which satisfy $s_J(w) = 0$ if $J \notin \mathcal{K}$:

$$W_{\mathcal{K},I}^{(n)} = \{w \in W_I^{(n)} \mid s_J(w) = 0,\ J \notin \mathcal{K}\}. \tag{15}$$

If $\mathcal{K} = \mathcal{I}_d$, then we are dealing with the original (full) model; $W_{\mathcal{I}_d,I}^{(n)} = W_I^{(n)}$. We define the corresponding generating functions by

$$X_{\mathcal{K},n,I}(\vec{x}) = \sum_{w \in W_{\mathcal{K},I}^{(n)}} \prod_{J \in \mathcal{K}} x_J^{s_J(w)}, \quad \vec{x} = (x_J,\ J \in \mathcal{K}) \in \mathbb{C}^{\mathcal{K}},\ n = 0, 1, 2, \cdots,\ I \in \mathcal{I}_d . \tag{16}$$

If $\mathbb{R}_+^{\mathcal{K}}$ is an invariant subset of $\mathbb{R}_+^{\mathcal{I}_d}$, then $\vec{X}_{\mathcal{K},n}$ satisfy (10), with a convention that the components corresponding to $J \notin \mathcal{K}$ are 0 .

For $\mathcal{K} \subset \mathcal{I}_d$ and $\beta \in \mathbb{R}$, define $\vec{x}_{can,\mathcal{K}}(\beta) = (x_{can,\mathcal{K},I}(\beta),\ I \in \mathcal{I}_d)$ by

$$x_{can,\mathcal{K},I}(\beta) = \begin{cases} e^{-\beta|I|}, & I \in \mathcal{K}, \\ 0, & I \notin \mathcal{K}, \end{cases} \tag{17}$$

where $|I|$ is defined in (8). Following the notions in statistical mechanics, the partition function for a set of self-avoiding paths specified by $\mathcal{K}$ is defined by $\vec{Z}_{\mathcal{K},n} = (Z_{\mathcal{K},n,I}\ ,\ I \in \mathcal{I}_d)$, with

$$Z_{\mathcal{K},n,I}(\beta) = \sum_{w \in W_{\mathcal{K},I}^{(n)}} e^{-\beta L(w)}, \quad \beta \in \mathbb{R},\ n = 0, 1, 2, \cdots. \tag{18}$$

With (7), we see that

$$Z_{\mathcal{K},n,I}(\beta) = X_{\mathcal{K},n,I}(\vec{x}_{can,\mathcal{K}}(\beta)). \tag{19}$$

In view of this relation, we will occasionally refer to the curve in the parameter space $\mathbb{R}^{\mathcal{I}_d}$ defined by (17) as the 'canonical curve'.

If $\mathcal{K} = \mathcal{I}_d$ we also use an abbreviation $\vec{x}_{can}(\beta) = \vec{x}_{can,\mathcal{I}_d}(\beta)$ and $\vec{Z}_n(\beta) = \vec{Z}_{\mathcal{I}_d\,,n}(\beta)$. Hence

$$Z_{n,I}(\beta) = X_{n,I}(\vec{x}_{can}(\beta)) = \sum_{w \in W_I^{(n)}} e^{-\beta L(w)}, \quad \beta \in \mathbb{R},\ n = 0, 1, 2, \cdots. \tag{20}$$

In the following, the set of paths in (15) with $\mathcal{K} = \mathcal{K}_{res}$ will be called the restricted self-avoiding paths, the corresponding generating function (16), the generating function for the restricted model, and so on.

We need the following additional definitions for Theorem 10.

(CS1) We say that $\beta_c \in \mathbb{R}$ is a critical point of the full model if $\vec{x}_{can}(\beta_c) \in \mathcal{D}om(\vec{x}_c)$ for a self-avoiding fixed point $\vec{x}_c$.

(CS2) We say that $\beta_{c,res} \in \mathbb{R}$ is a critical point of the restricted model if $\vec{x}_{can,\mathcal{K}_{res}}(\beta_{c,res}) \in \mathcal{D}om(\vec{x}_c)$ for a self-avoiding fixed point $\vec{x}_c$.

Note that by definition (11),

$$\vec{x}_{can}(\beta_c) \in \Xi_d\,, \qquad \text{and} \qquad \vec{x}_{can,\mathcal{K}_{res}}(\beta_{c,res}) \in \Xi_d\,. \tag{21}$$

Remark. (i) The boundary ∂D of the set $D \subset \Xi_d$ defined in (31) is a bounded closed non-empty $\vec{\Phi}$–invariant subset of $\mathbb{R}^{\mathcal{I}_d}$ (which are easy consequences of Theorem 15). Hence, the fixed point theorem implies that there exists a fixed point of $\vec{\Phi}$ which satisfies (FP1) and (FP4).

The other conditions on the self-avoiding fixed point (FP2) and (FP3) depend more on the details of the self-avoiding paths on dSG. However, these conditions deal with conditions of Perron-Frobenius type and irreducibility, which are 'soft' conditions, hence we expect them to hold. (These conditions are used in the proofs of Proposition 12, Proposition 14, Proposition 17, Proposition 20, and Lemma 22.)

(ii) What may be more difficult is the condition (CS1), which states existence of a trajectory converging to a fixed point. This essentially suggests that a bounded trajectory necessary converges to a fixed pooint (at least in the domain Ξ_d), that the renormalization group dynamical system is free of limit cycles much less any chaotic behaviors. There are of course many discrete dynamical systems, even on one-dimensional space, which exhibit chaotic behaviors, hence this condition is far from trivial.

On the other hand, it is proved in [5] and [6] that for $d = 2$ and $d = 3$, all the conditions (FP1) – (FP4) and (CS1) are satisfied. We also prove in Section 5 that (FP1) – (FP4) and (CS2) are satisfied for the restricted model on 4SG. Based on these results, we conjecture that these conditions are satisfied (hence the results about the asymptotic behaviors of the self-avoiding walks hold) for all d. ◇

3.2 Asymptotic behaviors.

Here we will state main consequences of assumptions on RG formulated in Section 3.1.

First we note the following characterization of a critical point β_c.

Theorem 7 *If $\beta_c \in \mathbb{R}$ is a critical point of the full model and $\vec{x}_c$ the corresponding fixed point (implicit in the definition (CS1)), then for $I \in \mathcal{I}_d$,*

$$\lim_{n\to\infty} Z_{n,I}(\beta) = \begin{cases} 0, & \beta > \beta_c\,, \\ x_{c,I}\,, & \beta = \beta_c\,. \end{cases}$$

Moreover,

$$\lim_{n\to\infty} \sum_{i=1}^{d} Z_{n,(i)}(\beta) = \infty, \quad \beta < \beta_c\,.$$

In particular, critical point (if exists) is unique.
Similar result holds also for the restricted model (CS2).

Since the critical point (if exists) is unique, there is one and only one self-avoiding fixed point that is related by (CS1) to the critical point.

Though it is not trivial to prove the uniqueness of self-avoiding fixed point, we therefore can (and we will, in the proof of Theorem 10) talk about the unique self-avoiding fixed point that is related to the critical point, under the asumption that the critical point exists.

To state the next Theorem, we note a relation between 0 components of a fixed point and an invariant subset of $\vec{\Phi}$. In the known case of $d = 2$ and $d = 3$, the self-avoiding fixed points have 0 components. We will write

$$\mathcal{K}_{\vec{x}_c} = \{I \in \mathcal{I}_d \mid x_{c,I} \neq 0\}, \tag{22}$$

and, as in (12),

$$\mathbb{R}_+^{\mathcal{K}_{\vec{x}_c}} = \{\vec{x} \in \mathbb{R}_+^{\mathcal{I}_d} \mid x_J = 0,\ J \notin \mathcal{K}_{\vec{x}_c}\}\,.$$

Proposition 8 $\mathbb{R}_+^{\mathcal{K}_{\vec{x}_c}}$ *is an invariant subset of* $\vec{\Phi}$.

Proof. If $x_{c,I} = 0$ then $\Phi_I(\vec{x}_c) = x_{c,I} = 0$. On the other hand Φ_I is a polynomial with positive coefficients. Therefore, each term in $\Phi_I(\vec{x})$ contains one of x_J's such that $x_{c,J} = 0$. In other words, each term in $\Phi_I(\vec{x})$ contains x_J such that $J \notin \mathcal{K}_{\vec{x}_c}$.

Therefore, if $\vec{x} \in \mathbb{R}_+^{\mathcal{K}_{\vec{x}_c}}$, then $\Phi_I(\vec{x}) = 0$ for those I satisfying $x_{c,I} = 0$, or equivalently, $I \notin \mathcal{K}_{\vec{x}_c}$. □

Remark. For $d = 2$ and $d = 3$, the results in [5] and [6] respectively proves (by explicit calculations) that the self-avoiding fixed point $\vec{x}_c$ is unique and that $\mathcal{K}_{res} = \mathcal{K}_{\vec{x}_c}$. ◇

For $I \in \mathcal{I}_d$, $n \in \mathbb{Z}_+$, and $\vec{x} \in \mathbb{R}_+^{\mathcal{I}_d}$, define a probability measure $\mu_{\vec{x},n,I}$ on the finite set $W_I^{(n)}$ by

$$\mu_{\vec{x},n,I}[\{w\}] = \frac{1}{X_{n,I}(\vec{x})} \prod_{J\in\mathcal{I}_d} x_J^{s_J(w)}, \quad w \in W_I^{(n)}, \tag{23}$$

whenever $X_{n,I}(\vec{x}) \neq 0$.

Note that if $x_{c,I} \neq 0$ and $\vec{x} \in \mathcal{D}om(\vec{x}_c)$, then (DA1) implies that $X_{n,I}(\vec{x}) > 0$ for sufficiently large n, hence if $\vec{x} \in \mathcal{D}om(\vec{x}_c)$ then $\mu_{\vec{x},n,I}$ is well defined.

Theorem 9 *Let $\vec{x}_c$ be a self-avoiding fixed point and $\vec{x} \in \mathcal{D}om(\vec{x}_c)$. Then the following hold.*

(i) There exists $f_d \times f_d$ matrix $\Lambda(\vec{x})$ whose elements are non-negative such that

$$\Lambda(\vec{x}) = \lim_{n\to\infty} \lambda^{-n} \mathcal{J}_n(\vec{x}), \tag{24}$$

where $\mathcal{J}_n$ is as in (14).

(ii) For $I \in \mathcal{K}_{\vec{x}_c}$, the joint distribution of scaled generalized path lengths $(\lambda^{-n} s_J,\ J \in \mathcal{K}_{\vec{x}_c})$ under $\mu_{\vec{x},n,I}$ converges weakly to a Borel probability measure $p^*_{\vec{x},I}$ on $\mathbb{R}^{\mathcal{I}_d}$ as $n \to \infty$. Here, λ is as in (FP2). $p^*_{\vec{x},I}$ is supported on $\mathbb{R}_+^{\mathcal{I}_d}$.

The generating function $\varphi^*_I = \varphi^*_{\vec{x},I}$, as a function of $(t_J,\ J \in \mathcal{K}_{\vec{x}_c})$, defined by

$$\varphi^*_{\vec{x},I}(\vec{t}) = \int_0^\infty e^{\vec{t}\cdot\vec{\xi}} p^*_{\vec{x},I}[d\vec{\xi}], \quad \vec{t} \in \mathbb{C}^{\mathcal{K}_{\vec{x}_c}},$$

is an entire function in $\vec{t}$.

(iii) The set of functions $\varphi^*_I = \varphi^*_{\vec{x},I}$, $I \in \mathcal{K}_{\vec{x}_c}$, are uniquely determined by

$$\begin{aligned} &x_{c,I} \frac{\partial \varphi^*_I}{\partial t_J}(\vec{0}) = \Lambda_{IJ}(\vec{x})\, x_J, \quad \text{if } I,\, J \in \mathcal{K}_{\vec{x}_c}, \\ &x_{c,I}\varphi^*_I(\lambda\vec{t}) = \Phi_I(\vec{x}_c\,\vec{\varphi}^*(\vec{t})), \quad \vec{t} \in \mathbb{C}^{\mathcal{K}_{\vec{x}_c}}, \quad \text{if } I \in \mathcal{K}_{\vec{x}_c}, \end{aligned} \tag{25}$$

where we define $\varphi^*_J = 0$ for $J \notin \mathcal{K}_{\vec{x}_c}$, and in the variable for $\vec{\Phi}$ we used an (irregular) notation

$$(\vec{x}\,\vec{\varphi}^*(\vec{t}))_J = x_J\, \varphi^*_J(\vec{t}), \quad J \in \mathcal{I}_d.$$

(iv) If $\vec{x} \in \mathcal{D}om(\vec{x}_c) \cap \Xi_d$ and $I \in \mathcal{K}_{\vec{x}_c}$, then the distribution of $\lambda^{-n} L(w)$, the scaled length of $w \in W_I^{(n)}$, under $\mu_{\vec{x},n,I}$ converges weakly to a Borel probability measure $\bar{p}^*_{\vec{x},I}$, which has a C^∞ density $\bar{\rho}^*_{\vec{x},I}$.

In particular, $\bar{\rho}^*_{\vec{x},(1)}(\xi) > 0$, $\xi > 0$.

Remark. (i) If $x_{c,I} = 0$, then the right hand sides in (25) are 0 because of Proposition 8 and (24), hence the equations in (25) are trivially correct for if $I \notin \mathcal{K}_{\vec{x}_c}$.

(ii) Existence and positivity of density is used in the proof of Theorem 10.

◇

We move on to the results on paths with step numbers fixed, instead of paths with endpoints fixed.

We denote the self-avoiding paths starting from origin O by $W^{(0)}$: $W^{(0)} = \{w \in W_0 \mid w(0) = O\}$. Also, we define, in analogy with (15), $W_{\mathcal{K}}^{(0)} = \{w \in W^{(0)} \mid s_J(w) = 0,\ J \notin \mathcal{K}\}$, for $\mathcal{K} \subset \mathcal{I}_d$.

For each $k \in \mathbb{Z}_+$, let

$$N(k) = \sharp\{w \in W^{(0)} \mid L(w) = k\}$$

be the number of self-avoiding paths of length k starting from O, and for $\mathcal{K} \subset \mathcal{I}_d$,

$$N_{\mathcal{K}}(k) = \sharp\{w \in W_{\mathcal{K}}^{(0)} \mid L(w) = k\}.$$

Theorem 10 *If there exists a critical point $\beta_c \in \mathbb{R}$ of the full model, then there exist positive constants C_i, $i = 1, 2$, and real constants C_i, $i = 3, 4$, such that*

$$C_1 k^{C_3} e^{\beta_c k} \leqq N(k) \leqq C_2 k^{C_4} e^{\beta_c k}, \quad k = 1, 2, 3, \cdots.$$

Similarly, if there exists a critical point $\beta_{c,res}$ of the restricted model, then there exist positive constants C_i', $i = 1, 2$, and real constants C_i', $i = 3, 4$, such that

$$C_1' k^{C_3'} e^{\beta_{c,res} k} \leqq N(k) \leqq C_2' k^{C_4'} e^{\beta_{c,res} k}, \quad k = 1, 2, 3, \cdots.$$

For each positive integer k, let $\tilde{P}_k$ be a distribution on $W^{(0)}$, defined by

$$\tilde{P}_k[A] = \frac{1}{N(k)} \sharp\{w \in A \mid L(w) = k\}, \quad A \subset W^{(0)}.$$

The next result shows the existence of the exponent for mean square displacement, which indicates (in a log ratio sense) that a typical self-avoiding path w of length $L(w) = k$ deviates from the starting point by $|w(k)| \asymp k^{1/d_w}$, where

$$d_w = \frac{\log \lambda}{\log 2}. \tag{26}$$

Theorem 11 *If there exists a critical point $\beta_c \in \mathbb{R}$ of the full model, then there exist constants α, k_0, C, and C' such that*

$$s \log k - s\alpha \log\log k + C \leqq \log E_k[|w(k)|^{s\, d_w}] \leqq s \log k + s\alpha \log\log k + C', \quad k \geqq k_0,\ s \geqq 0,$$

where E_k denotes expectation with respect to $\tilde{P}_k$, and $|\cdot|$ denotes the (Euclidean) length in $\mathbb{R}^d$.

A similar result holds for the restricted model.

Remark. (i) The intuitive meaning of (26) is as follows. λ is the asymptotic rate of increase of the number of steps as n is increased. Since the size (scale) is increased by a factor 2 as n is increased by 1, the log ratio of the number of steps to the distance scale is equal to the log ratio of λ and 2. Though this is a standard idea in the renormalization group approaches to asymptotic behaviors, our emphasis here is on the precise mathematical statements and rigorous proofs that fit to such intuitive pictures.

(ii) As may be seen from the fact that λ is defined in (FP2) as the largest eigenvalue of the differential map of $\vec{\Phi}$ at $\vec{x}_c$ while β_c is defined in (CS1) as the intersection of the canonical curve and the critical surface, these two quantities have no direct relations. In fact, in the common wisdom of the renormalization group ideas, the exponent for mean square displacement is considered to be universal, i.e., independent of details of the system, (in fact the full model and the restricted model have the same λ), while the connective constant depends on the details of the system (the full model and the restricted model have different values of β_c). This is related to the fact that λ is a solution to an algebraic equation and can be calculated explicitly to arbitrary precision, while β_c has no such simple algebraically closed formula and is difficult to calculate explicitly.

(iii) It may be worthwhile to note that the self-avoiding walks on hypercubic lattice $\mathbb{Z}^n$ in high dimensions ($n > 4$) are proved to be in the same universality class as the random walks — i.e., they have similar asymptotic behaviors — by the lace expansion methods [2, 3]. In a sense, the self-avoiding walks in high dimensional spaces may be seen as (non-trivial) perturbations to the random walks.

However, it is also believed (and trivially true for $n = 1$!), that for $n < 4$ the asymptotic behaviors are very different, hence the problem remains in lower dimensional spaces. We note that dSG are, from the renormalization group point of view, spaces 'between $\mathbb{Z}^1$ and $\mathbb{Z}^2$'. We also point out that the lace expansion method heavily uses translational invariance of $\mathbb{Z}^n$, while fractals lack the invariance. In fact, the random walks and the self-avoiding walks are known to be in different universality classes on 2SG and 3SG; the values of exponents for mean square displacement of the self-avoiding walks and the random walks have no explicit simple relations [8].

◇

4 Proofs.

4.1 Phase structure.

Here we will prove Theorem 7.

We first note some simple consequences of the definitions.

Proposition 12 *Let $\vec{x}_c$ be a self-avoiding fixed point and λ be as in (FP2). Then $2 < \lambda < d+1$.*

Proof. Proposition 4 and (14) imply

$$2\Phi_I(\vec{x}) < \sum_{J\in\mathcal{I}_d} \mathcal{J}_{IJ}(\vec{x})x_J < (d+1)\Phi_I(\vec{x}), \quad \vec{x}\in\mathbb{R}_+^{\mathcal{I}_d};\ x_{(1)} > 0\,.$$

This, with (FP1) and (FP2), further implies

$$2\sum_{I\in\mathcal{I}_d} v_{L,I}x_{c,I} < \lambda\sum_{J\in\mathcal{I}_d} v_{L,J}x_{c,J} < (d+1)\sum_{I\in\mathcal{I}_d} v_{L,I}x_{c,I}\,,$$

which implies the statement.

□

Note that if $\vec{x}_c$ is a self-avoiding fixed point then Proposition 4 and (FP1) imply

$$\vec{X}_n(\vec{x}_c) = \vec{x}_c\,, \quad n\in\mathbb{Z}_+, \tag{27}$$

For $n\in\mathbb{Z}_+$, denote the Jacobi matrices of $\vec{X}_n$ by $\mathcal{J}_n$:

$$\mathcal{J}_{n,IJ}(\vec{x}) = \frac{\partial X_{n,I}}{\partial x_J}(\vec{x}), \quad I,J\in\mathcal{I}_d\,,\ \vec{x}\in\mathbb{C}^{\mathcal{I}_d}. \tag{28}$$

In particular, we have, from (14), $\mathcal{J}_1 = \mathcal{J}$. Proposition 4 implies, with an aid of the chain rule,

$$\mathcal{J}_n(\vec{x}) = \mathcal{J}(\vec{X}_{n-1}(\vec{x}))\cdot\mathcal{J}(\vec{X}_{n-2}(\vec{x}))\cdot\cdots\mathcal{J}(\vec{X}_1(\vec{x}))\cdot\mathcal{J}(\vec{x}), \quad n\in\mathbb{Z}_+, \tag{29}$$

which further implies with (27),

$$\mathcal{J}_n(\vec{x}_c) = \mathcal{J}(\vec{x}_c)^n, \quad n\in\mathbb{Z}_+. \tag{30}$$

Proposition 13 *$\Phi_{(1)}(\vec{x})$ contains terms ${x_{(i)}}^2$, $1\leqq i\leqq d$,*

Proof. This is proved by a similar graphical consideration as that in the proof of Proposition 4 that $\Phi_{(1)}(\vec{x})$ contains a term ${x_{(1)}}^2$.

□

Proposition 14 *If $\vec{x}_c$ is a self-avoiding fixed point, then*

(i) $x_{c,(1)} > 0$*. Namely,* $(1)\in\mathcal{K}_{\vec{x}_c}$ *.*

(ii) $v_{R,I} > 0$*,* $I\in\mathcal{K}_{\vec{x}_c}$ *, where* $\vec{v}_R$ *is as in (FP2).*

Proof. (i) $\vec{x}_c \neq \vec{0}$, by (FP4). Hence there exists $I\in\mathcal{I}_d$ such that $x_{c,I} > 0$. With (FP4), we further see that there exists $1\leqq i\leqq d$ such that $x_{c,(i)} > 0$. Proposition 13 then implies that $x_{c,(1)} = \Phi_{(1)}(\vec{x}_c) \geqq x_{c,(i)}^2 > 0$.

(ii) (FP3) and $I \in \mathcal{K}_{\vec{x}_c}$ imply $\mathcal{J}(\vec{x}_c)_{I,(1)} > 0$. With (FP2), this further implies $v_{R,I} \geqq \dfrac{1}{\lambda} J(\vec{x}_c)_{I,(1)}\, v_{R,(1)} > 0$.

□

Let

$$D = \{\vec{x} \in \Xi_d \mid \sup_{n \in \mathbb{Z}_+} \max_{I \in \mathcal{I}_d} X_{n,I}(\vec{x}) < \infty\}, \tag{31}$$

and denote its exterior, boundary, interior in Ξ_d by D^c, ∂D, and D^o, respectively. (Namely, $D^c = \Xi_d \setminus \overline{D}$, $\partial D = \overline{D} \cap \overline{D^c}$, $D^o = D \setminus \partial D$.) Let also

$$\tilde{D} = \{\vec{x} \in \Xi_d \mid \lim_{n\to\infty} \max_{I \in \mathcal{I}_d} X_{n,I}(\vec{x}) = 0\}.$$

Theorem 15 *(i) It holds that*

$$D = \{\vec{x} \in \Xi_d \mid \sup_{n \in \mathbb{Z}_+} \max_{I \in \mathcal{I}_d} X_{n,I}(\vec{x}) \leqq 1\}. \tag{32}$$

In particular, D is a closed subset of Ξ_d.

(ii) Let $\vec{x} \in D$ and $\vec{x}' \in \Xi_d$. If, for each $I \in \mathcal{I}_d$ either $x'_I < x_I$ or $x'_I = x_I = 0$ holds, then $\vec{x}' \in \tilde{D}$.

(iii) It holds that

$$D^o = \tilde{D} \tag{33}$$

(iv) D^c, ∂D, and D^o are non-empty invariant sets of $\vec{\Phi}$.

Remark. Note that this theorem holds independently of the notion of self-avoiding fixed points and critical points. ◇

Proof. (i) Denote the right hand side of (32) by D'. Obviously $D' \subset D$. To prove $D'^c \subset D^c$ (in Ξ_d), let $\vec{x} \in D'^c \subset \Xi_d$. Then the definition of D' implies that there exists $n \in \mathbb{Z}_+$ and $I \in \mathcal{I}_d$ such that $X_{n,I}(\vec{x}) > 1$, and the definition of Ξ_d therefore implies $X_{n,(i)}(\vec{x}) > 1$ for some $1 \leqq i \leqq d$. Then Proposition 13 implies $X_{n+1,(1)}(\vec{x}) > 1$, which, with Proposition 4, further implies

$$\lim_{n\to\infty} X_{n,(1)}(\vec{x}) = \infty. \tag{34}$$

Hence $D'^c \subset D^c$.

(ii) The case $\vec{x}' = \vec{0}$ is trivial. For $\vec{x}$ and $\vec{x}'$ in $\mathbb{R}_+^{\mathcal{I}_d} \setminus \{0\}$, let $r = \max_{I \in \mathcal{I}_d;\, x_I \neq 0} \dfrac{x'_I}{x_I}$. Since Φ_I, $I \in \mathcal{I}_d$, are polynomials with positive coefficients and each term of degree no less than 2 (Proposition 4), we have $\max_{I \in \mathcal{I}_d} \Phi_I(\vec{x}') \leqq r^2 \max_{I \in \mathcal{I}_d} \Phi_I(\vec{x})$ if $0 \leqq r \leqq 1$. By induction, $\max_{I \in \mathcal{I}_d} X_{n,I}(\vec{x}') \leqq r^{2^n} \max_{I \in \mathcal{I}_d} X_{n,I}(\vec{x})$, $n \in \mathbb{Z}_+$. If $\vec{x} \in D \setminus \{\vec{0}\}$ and $\vec{x}' \in \Xi_d$, and if for each $I \in \mathcal{I}_d$ either $x'_I < x_I$ or $x'_I = x_I = 0$ holds, then $0 \leqq r < 1$. Since $\{X_{n,I}(\vec{x})\}$ is bounded, we have $\vec{x}' \in \tilde{D}$.

(iii) For $\epsilon > 0$ let $D_\epsilon = \{\vec{x} \in \Xi_d \mid \sum_{I \in \mathcal{I}_d} x_I < \epsilon\}$. Since Φ_I is a polynomial with each term of order no less than 2 (Proposition 4), there exists a constant $M > 0$ such that

$$\sum_{I \in \mathcal{I}_d} \Phi_I(\vec{x}) \leqq M\epsilon \sum_{I \in \mathcal{I}_d} x_I\,, \quad \vec{x} \in D_\epsilon,\ 0 < \epsilon < 1\,.$$

Let $\epsilon = \frac{1}{2M} \wedge \frac{1}{2}$. Then by induction in n with

$$\sum_{I \in \mathcal{I}_d} X_{n+1,I}(\vec{x}) = \sum_{I \in \mathcal{I}_d} \Phi_I(\vec{X}_n(\vec{x})),$$

we have, for $\vec{x} \in D_\epsilon$, $\vec{X}_n(\vec{x}) \in D_\epsilon$, $n \in \mathbb{Z}_+$. Then we further have

$$\sum_{I \in \mathcal{I}_d} X_{n+1,I}(\vec{x}) \leqq M\epsilon \sum_{I \in \mathcal{I}_d} X_{n,I}(\vec{x}) \leqq \frac{1}{2} \sum_{I \in \mathcal{I}_d} X_{n,I}(\vec{x}), \quad n \in \mathbb{Z}_+,$$

which implies $\vec{x} \in \tilde{D}$. Hence $D_\epsilon \subset \tilde{D}$ for $\epsilon = \frac{1}{2M} \wedge \frac{1}{2}$. On the other hand, the definition of $\tilde{D}$ implies that if $\vec{x} \in \tilde{D}$ then $\vec{X}_n(\vec{x}) \in D_\epsilon$ for sufficiently large n. $\vec{X}_n$ is a continuous function and D_ϵ is an open set, hence there exists an open neighborhood U of $\vec{x}$ such that $\vec{X}_n(U) \subset D_\epsilon$. Hence $U \subset \tilde{D}$. This implies that $\tilde{D}$ is an open subset of D. Since D^o is the largest open set of D, this implies $\tilde{D} \subset D^o$.

To prove $\tilde{D} \supset D^o$, assume $\vec{x}' \in D^o$. Since D^o is an open set, there exists $\vec{x} \in D^o$ such that $x'_I < x_I$, $I \in \mathcal{I}_d$. Then, the above claim imples $\vec{x}' \in \tilde{D}$.

(iv) By (33), (32), and (34), we see that D, D^o, and D^c are invariant sets of $\vec{\Phi}$. Since $D = \overline{D}$, ∂D also is invariant. Obviously, $\vec{x}$ with small enough components is in D^o and that with sufficiently large components is in D^c, hence the boundary ∂D is also non-empty.

□

Proof of Theorem 7. The case $\beta = \beta_c$ holds by the definition (CS1).

By the definitions (CS1) and (FP4), $\lim_{n\to\infty} \vec{X}_n(\vec{x}_{can}(\beta_c)) = \vec{x}_c \neq \vec{0}$, which, with (31) and (33), implies $\vec{x}_{can}(\beta_c) \in \partial D$. Monotonicity property in Theorem 15 then implies $\vec{x}_{can}(\beta) \in \tilde{D}$ if $\beta > \beta_c$, hence, in particular, $\lim_{n\to\infty} \vec{Z}_n(\beta) = 0$.

Finally, if $\beta < \beta_c$ and $\vec{x}_{can}(\beta) \in D = \overline{D}$, then the monotonicity property in Theorem 15 implies $\vec{x}_{can}(\beta_c) \in \tilde{D} = D^o$, which contradicts $\vec{x}_{can}(\beta_c) \in \partial D$. Hence $\vec{x}_{can}(\beta) \in D^c$ and in particular, with the same argument as that led to (34), we have $\lim_{n\to\infty} \sum_{i=1}^{d} Z_{n,(i)}(\beta) = \infty$.

The case of restricted model is similarly proved, if we note (CS2) in place of (CS1).

□

We will use the following in the proof of Theorem 10.

Proposition 16 *If $\vec{x} \in D^o$ then* $\overline{\lim_{n\to\infty}} \max_{I \in \mathcal{I}_d} 2^{-n} \log X_{n,I}(\vec{x}) < 0$.

Proof. Write $Y_n = \max_{I \in \mathcal{I}_d} \log X_{n,I}$, $n \in \mathbb{Z}_+$. Since Φ_I are polynomials of positive coefficients with each term of degree no less than 2 (Proposition 4), there exists a polynomial P_1 with positive coefficients such that

$$Y_{n+1}(\vec{x}) \leqq 2Y_n(\vec{x}) + \log P_1(\exp(Y_n(\vec{x}))), \quad n \in \mathbb{Z}_+, \ \vec{x} \in \mathbb{R}_+^{\mathcal{I}_d}.$$

If $\vec{x} \in D^o$, Theorem 15 implies that $\vec{X}_n(\vec{x}) \in D^o$, $n \in \mathbb{Z}_+$. Hence, $\{\vec{X}_n(\vec{x}) \mid n \in \mathbb{Z}_+\}$ is bounded, and there exists $M \in \mathbb{R}$, a constant in n, such that

$$Y_{n+1}(\vec{x}) \leqq 2Y_n(\vec{x}) + M, \quad n \in \mathbb{Z}_+. \tag{35}$$

In particular, $2^{-n}(Y_n(\vec{x}) + M)$ is decreasing, hence

$$\overline{\lim_{n\to\infty}} 2^{-n}(Y_n(\vec{x}) + M) \leqq 2^{-n_0}(Y_{n_0}(\vec{x}) + M), \quad n_0 \in \mathbb{Z}_+.$$

Also $\vec{x} \in D^o$ implies

$$\lim_{n\to\infty} Y_n(\vec{x}) = -\infty.$$

Hence

$$\overline{\lim_{n\to\infty}} 2^{-n} Y_n(\vec{x}) = \overline{\lim_{n\to\infty}} 2^{-n}(Y_n(\vec{x}) + M) < 0,$$

which implies the statement.

□

4.2 Distribution of path length.

Here we will prove Theorem 9.

Proposition 17 *Let $\vec{x}_c$ be a self-avoiding fixed point and $\vec{x} \in \mathcal{D}om(\vec{x}_c)$ then there exists $f_d \times f_d$ matrix $\Lambda(\vec{x})$ such that (24) holds. The elements of the matrix are non-negative, and*

$$\Lambda(\vec{x})_{I,(1)} > 0, \quad I \in \mathcal{K}_{\vec{x}_c}\,. \tag{36}$$

Proof. First note that (FP2) and (30) imply an existence of the limit in (24) for $\vec{x} = \vec{x}_c$.

Next recall the following two results.

Lemma 18 ([6, Lemma 4.2], [5, Lemma (3.1)]) *Let N be a positive integer, and A and A_n, $n \in \mathbb{N}$, be $N \times N$ matrices. Assume that there is an invertible $N \times N$ matrix Q such that $Q^{-1}AQ$ is a diagonal matrix whose eigenvalue λ that is largest in absolute value, is positive. Assume further that $\sum_{n=1}^{\infty} \|A_n - A\| < \infty$. Then there exists a matrix R such that*

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \left\|\lambda^{-n} A_{n+m} A_{n+m-1} \cdots A_{m+1} \,-\, R\right\| = 0\,.$$

$Q^{-1}RQ$ is a diagonal matrix whose diagonal elements satisfy $(Q^{-1}RQ)_{ii} = 1$ if $(Q^{-1}AQ)_{ii} = \lambda$, and $(Q^{-1}RQ)_{ii} = 0$ otherwise. Moreover $\lim_{n\to\infty} \lambda^{-n} A_n A_{n-1} \cdots A_1$ exists.

($\|A\|$ is the standard operator norm defined as the square root of largest eigenvalue of A^*A.)

Lemma 19 ([6, Proposition 3.8]) *If $\vec{x}_c$ is a self-avoiding fixed point and $\vec{x} \in \mathcal{D}om(\vec{x}_c)$, then there exist positive constants ρ and C, satisfying $\rho < 1$, such that*

$$|X_{n,I}(\vec{x}) - x_{c,I}| \leqq C\rho^n, \quad n \in \mathbb{Z}_+, \; I \in \mathcal{I}_d\,.$$

Remark. (i) The proof in [6, Proposition 3.8] is for $d = 3$, but the proof is directly applicable to general cases. In fact, the proof is an application of a general theory for diffeomorphisms [10].

(ii) Though we do not use this, the proof in [6, Proposition 3.8] implies that Lemma 19 holds for any ρ satisfying

$$\max\{\lambda^{-1}, |\lambda_2|\} < \rho < 1,$$

where λ_2 is the eigenvalue of $\mathcal{J}(\vec{x}_c)$ which is second largest in absolute value.

◇

Let us proceed with the proof of existence of the limit in Proposition 17. By the mean-value theorem,

$$\mathcal{J}_{IJ}(\vec{x}) - \mathcal{J}_{IJ}(\vec{x}_c) = \sum_{K\in\mathcal{I}_d} \frac{\partial \mathcal{J}_{IJ}}{\partial x_K}(\vec{u})\,(x_K - x_{c,K}), \quad I, J \in \mathcal{I}_d\,. \tag{37}$$

where $\vec{u} = \vec{x}_c + (\vec{x} - \vec{x}_c)\theta$ for some $\theta \in (0, 1)$.

Since $\vec{x} \in \mathcal{D}om(\vec{x}_c)$, $\vec{X}_m(\vec{x})$ is in a neghborhood of $\vec{x}_c$ for sufficiently large m. Therefore $\vec{X}_m(\vec{x})$ is in a bounded domain. In particular, there exists a positive constant M such that

$$\left|\frac{\partial \mathcal{J}_{IJ}}{\partial x_K}(\vec{u})\right| \leqq M, \quad I, J, K \in \mathcal{I}_d\,, \; \vec{u} = \vec{x}_c + (\vec{X}_m(\vec{x}) - \vec{x}_c)\theta, \; 0 < \theta < 1\,, \; m \in \mathbb{N}.$$

This with (37) and Lemma 19 implies

$$\left\|\mathcal{J}(\vec{X}_m(\vec{x})) - \mathcal{J}(\vec{x}_c)\right\| \leqq C'\rho^m, \; m \in \mathbb{Z}_+,$$

for some $C' > 0$ and $0 < \rho < 1$. Therefore we see that the assumptions in Lemma 18 are satisfied with

$$A_n = \mathcal{J}(\vec{X}_n(\vec{x})) \tag{38}$$

and $A = \mathcal{J}(\vec{x}_c)$. Hence, the limit (24) exists.

To prove (36), let i be an index satisfying $(Q^{-1}AQ)_{ii} = \lambda$. Then the column i of the matrix Q in Lemma 18, applied to the present case, can be chosen to be proportional to the right eigenvector $\vec{v}_R$ of $A = \mathcal{J}(\vec{x}_c)$ in (FP2). Similarly, the row i of matrix Q^{-1}, applied to the present case, can be chosen to be the left eigenvector $\vec{v}_L$ in (FP2). Therefore Lemma 18 implies that if $v_{R,I} > 0$ then for sufficiently large m, $\lim_{n\to\infty} \lambda^{-n} A_{n+m}A_{n+m-1}\cdots A_{m+1}$ for A_n given by (38) has positive (I,J) element for all $J \in \mathcal{I}_d$, in particular, for $J = (1)$. Also, since $\Phi_{(1)}(\vec{x})$ contains a term ${x_{(1)}}^2$ (Proposition 4), A_n has positive $((1),(1))$ element, which further implies that $A_mA_{m-1}\cdots A_1$ has positive $((1),(1))$ element. Finally, we know from Proposition 14 that $v_{R,I} > 0$ for $I \in \mathcal{K}_{\vec{x}_c}$. Hence (36) holds.

□

For $\vec{x} = (x_I,\ I \in \mathcal{I}_d) \in \mathbb{C}^{\mathcal{I}_d}$ and $\vec{t} = (t_I,\ I \in \mathcal{I}_d) \in \mathbb{C}^{\mathcal{I}_d}$, we use an (irregular) notation

$$\vec{x}(\vec{t}) = (x_I \exp(\lambda^{-n} t_I),\ I \in \mathcal{I}_d). \tag{39}$$

Proposition 20 *If $\vec{x} \in \mathcal{D}om(\vec{x}_c)$ then $\vec{X}_n(\vec{x}(\vec{t}))$ converges uniformly in $\vec{t} \in \mathbb{C}^{\mathcal{I}_d}$ for any compact subset of $\mathbb{C}^{\mathcal{I}_d}$. Furthermore,*

(i) If $I \in \mathcal{I}_d$ satisfies $x_{c,I} = 0$ (i.e., if $I \notin \mathcal{K}_{\vec{x}_c}$), then

$$\lim_{n\to\infty} X_{n,I}(\vec{x}(\vec{t})) = 0, \quad \vec{t} \in \mathbb{C}^{\mathcal{I}_d}.$$

(ii) Define $\vec{\varphi}^* = \vec{\varphi}^*_{\vec{x}} = (\varphi^*_{\vec{x},I}\,,\ I \in \mathcal{I}_d)$ by

$$\varphi^*_{\vec{x},I}(\vec{t}) = \begin{cases} \dfrac{1}{x_{c,I}} \lim_{n\to\infty} X_{n,I}(\vec{x}(\vec{t})), \quad \vec{t} \in \mathbb{C}^{\mathcal{I}_d}, & \text{if } I \in \mathcal{K}_{\vec{x}_c}\,, \\ 0, & \text{if } I \notin \mathcal{K}_{\vec{x}_c}\,. \end{cases}$$

Then $\vec{\varphi}^*$ is entire and satisfies

$$x_{c,I}\varphi^*_I(\lambda\vec{t}) = \Phi_I(\vec{x}_c\vec{\varphi}^*(\vec{t})), \quad \vec{t} \in \mathbb{C}^{\mathcal{I}_d},\ I \in \mathcal{I}_d\,.$$

where, the notation of the variable for Φ_I is as in Theorem 9. It also holds that

$$x_{c,I}\,\frac{\partial \varphi^*_I}{\partial t_J}(\vec{0}) = x_J\,\Lambda_{IJ}(\vec{x}), \quad I, J \in \mathcal{I}_d\,,$$

where Λ is defined in (24).

Proof. For $\vec{z} \in \mathbb{C}^{\mathcal{I}_d}$, define

$$|\vec{z}|_* = \sum_{I\in\mathcal{I}_d} v_{L,I}\,|z_I|,$$

where $v_{L,I}$ is as in (FP2). It is easy to see that (FP2) implies that $|\cdot|_*$ is a norm. (Note that $v_{L,I} > 0$, $I \in \mathcal{I}_d$, implies uniqueness, i.e., that $|a|_* = 0$ implies $a = 0$.)

Using non-negativity of elements of $\mathcal{J}$, we have

$$|(\mathcal{J}(\vec{x}_c)\,\vec{z})_I| \leqq \sum_{J\in\mathcal{I}_d} \mathcal{J}(\vec{x}_c)_{IJ}\,|z_J|, \quad I \in \mathcal{I}_d\,.$$

Hence

$$|\mathcal{J}(\vec{x}_c)\,\vec{z}|_* \leqq \lambda|\vec{z}|_*\,, \quad \vec{z} \in \mathbb{C}^{\mathcal{I}_d}. \tag{40}$$

We now see that Proposition 20 is a consequence of the following, which is proved in the proof of [6, Proposition 4.4].

Lemma 21 *Let $p \in \mathbb{N}$, and let $H^{(n)}$: $\mathbb{C}^p \to \mathbb{C}^p$, $n \in \mathbb{N}$, be a sequence of polynomials with positive coefficients, and each term with degree 2 or larger, which satisfies a recursion relation*

$$H^{(n+1)} = H^{(1)} \circ H^{(n)}, \quad n \in \mathbb{N}. \tag{41}$$

Let $\partial H^{(1)}$ be the Jacobi matrix of $H^{(1)}$:

$$\partial H^{(1)}_{ij}(z) = \frac{\partial H^{(1)}_i}{\partial z_j}(z), \quad i, j \in 1, \cdots, p,\ z \in \mathbb{C}^p.$$

Assume that $a \in \mathbb{R}_+^p$ is a fixed point of $H^{(1)}$, and that there exists a norm $|\cdot|_*$ on $\mathbb{C}^p$ and $\lambda > 1$ such that

$$|\partial H^{(1)}(a)z|_* \leqq \lambda |z|_* , \quad z \in \mathbb{C}^p, \tag{42}$$

and that $x \in \mathbb{R}_+^p$ is a point for which there exist positive constants ρ and C, satisfying $\rho < 1$, such that

$$|H_i^{(n)}(x) - a_i| \leqq C\rho^n, \quad i = 1, \cdots, p, \; n \in \mathbb{Z}_+, \tag{43}$$

and for which the limit

$$\Lambda(x) = \lim_{n\to\infty} \lambda^{-n} \partial H^{(n)}(x) \tag{44}$$

exists.

Then there exists an entire function $H^* = (H_1^*, \cdots, H_p^*) : \mathbb{C}^p \to \mathbb{C}^p$ such that

$$\lim_{n\to\infty} H^{(n)}(x_1 \exp(\lambda^{-n} t_1), \cdots, x_p \exp(\lambda^{-n} t_p)) = H^*(t),$$

uniformly in $t = (t_1, \cdots, t_p)$ on any compact set of $\mathbb{C}^p$.

H^* satisfies

$$H^*(\lambda t) = H^{(1)}(H^*(t)), \quad t \in \mathbb{C}^p,$$

and

$$\frac{\partial H_i^*}{\partial t_j}(0) = x_j \Lambda_{ij}(x), \quad i, j \in \{1, 2, \cdots, p\}.$$

Furthermore, if $a_i = 0$ for some $i \in \{1, \cdots, p\}$, then $H_i^*(t) = 0$.

Remark. The proof of [6, Proposition 4.4] is for 3SG, but the proof is valid for the general case of Lemma 21. In the proof of [6, Proposition 4.4], [6, Theorem 3.5] is referred to, which should be replaced, in the proof of Lemma 21, by $\lim_{n\to\infty} H^{(n)}(x) = a$, which is a consequence of the assumption (43). Similarly, Proposition 3.8, Proposition 4.3, and (4.4) in [6] are replaced by (43), (44), and (41), respectively.

◇

To apply Lemma 21 to the proof of Proposition 20, we only need to note that Proposition 4, (40), Lemma 19, and Proposition 17, correspond to (41), (42), (43), and (44), respectively.

□

Proof of Theorem 9. Assume that $x_{c,I} \neq 0$ and $\vec{x} \in \mathcal{D}om(\vec{x}_c)$.

Denote by $p_{\vec{x},n,I}$, the joint distribution of $(\lambda^{-n} s_J, \; J \in \mathcal{I}_d)$ under $\mu_{\vec{x},n,I}$. Its generating function is expressed, with the definitions (9) and (23), as

$$\int_0^\infty e^{\vec{t}\cdot\vec{\xi}} p_{\vec{x},n,I}[d\vec{\xi}] = \frac{X_{n,I}(\vec{x}(\vec{t}))}{X_{n,I}(\vec{x})}, \quad \vec{t} \in \mathbb{C}^{\mathcal{I}_d}, \tag{45}$$

where the notation $\vec{x}(\vec{t})$ is as in (39). Since the probability measure in question is a measure on a finite set, the integration over $\vec{\xi}$ in (45) is actually a finite summation, hence the quantity is defined for all complex $\vec{t}$.

Using Proposition 20 and (DA1) in (45), we see that the right hand side of (45) converges to $\varphi_I^*(\vec{t}) = \varphi_{\vec{x},I}^*(\vec{t})$ of Proposition 20, uniformly in $\vec{t}$ on compact sets. Hence $p_{\vec{x},n,I}$ converges weakly to a probability measure $p_{\vec{x},I}^*$ whose generating function is $\varphi_I^*(\vec{t})$.

We therefore see that the claims in Theorem 9, except those on the existence and positivity of the density of $\bar{p}_{\vec{x},(1)}^*$, are direct consequences of Proposition 17 and Proposition 20. That (25) determines $\vec{\varphi}^*$ follows from the fact that its Taylor coefficients are determined recursively by (25). (Note that Φ_I is a polynomial with each term having degree 2 or larger. Hence when comparing the coefficients of t^N in both hand sides of (25), the right hand side contains coefficients for lower order than N, hence the coefficients of $\vec{\varphi}^*$ is determined inductively in N.)

To prove that $\bar{p}_{\vec{x},I}^*$ has a C^∞ density, we prepare the following.

Denote by $\bar{p}_{\vec{x},n,I}$, the distribution of $\lambda^{-n}L$ under $\mu_{\vec{x},n,I}$. Its generating function is, from (7) and (45),

$$\int_0^\infty e^{t\xi}\bar{p}_{\vec{x},n,I}[d\xi] = \frac{X_{n,I}(\vec{x}(t\vec{e}))}{X_{n,I}(\vec{x})}\,, \quad t \in \mathbb{C}, \tag{46}$$

where $\vec{e} = (e_J,\ J \in \mathcal{I}_d)$ is given by

$$e_J = |J|. \tag{47}$$

According to what has been proved below (45), (46) converges to an entire function

$$g_I(t) = g_{\vec{x},I}(t) = \varphi_I^*(t\vec{e}) \tag{48}$$

uniformly in t on compact sets of $\mathbb{R}$, and $\bar{p}_{\vec{x},n,I}$ converges weakly to a probability measure $\bar{p}^*_{\vec{x},I}$ whose generating function is g_I .

Proposition 20 also implies that

$$x_{c,I}\frac{dg_I}{dt}(0) = x_{c,I}\int_0^\infty \xi\bar{p}^*_{\vec{x},I}[d\xi] = \sum_{J\in\mathcal{I}_d} e_J\, x_J\, \Lambda_{IJ}(\vec{x}), \quad I \in \mathcal{I}_d\,, \tag{49}$$

and

$$x_{c,I}g_I(\lambda t) = \Phi_I(\vec{x}_c\vec{g}(t)), \quad I \in \mathcal{I}_d,\ t \in \mathbb{C}, \tag{50}$$

where, the notation of the variable for Φ_I is as in Theorem 9, and we also defined

$$g_J = 0, \quad \text{if } x_{c,J} = 0\,.$$

Lemma 22 *(i)* $M_I = M_I(\vec{x}) := \int_0^\infty \xi\bar{p}^*_{\vec{x},I}[d\xi] > 0, \quad I \in \mathcal{K}_{\vec{x}_c}\,,\ \vec{x} \in \Xi_d \cap \mathcal{D}om(\vec{x}_c).$

(ii) If $\vec{x} \in \Xi_d \cap \mathcal{D}om(\vec{x}_c)$ and $I \in \mathcal{K}_{\vec{x}_c}$ then $\bar{p}^*_{\vec{x},I}$ is not concentrated on a single point (i.e., has non-zero variance). Namely,

$$V_I = V_I(\vec{x}) := \int_0^\infty (\xi - M_I(\vec{x}))^2\bar{p}^*_{\vec{x},I}[d\xi] > 0, \quad I \in \mathcal{K}_{\vec{x}_c}\,,\ \vec{x} \in \Xi_d \cap \mathcal{D}om(\vec{x}_c).$$

(iii) If $\vec{x} \in \Xi_d \cap \mathcal{D}om(\vec{x}_c)$ then there exist positive constants C_1 and C_2 such that

$$\max_{I\in\mathcal{K}_{\vec{x}_c}} |g_I(\sqrt{-1}t)| \leqq C_2\exp(-C_1|t|^{1/d_w}), \quad t \in \mathbb{R},$$

where d_w is as in (26).

Proof. (i) Using (49) we have

$$\int_0^\infty \xi\bar{p}^*_{\vec{x},I}[d\xi] \geqq e_{(1)}\frac{x_{(1)}}{x_{c,I}}\,\Lambda_{I\,(1)}(\vec{x}) > 0,$$

where we used (47) for $e_{(1)} > 0$, $\vec{x} \in \Xi_d \cap \mathcal{D}om(\vec{x}_c)$ for $x_{(1)} > 0$, and (36) for $\Lambda_{I\,(1)}(\vec{x}) > 0$.

(ii) By definition (Proposition 4 and (9)), we have

$$\Phi_I(\vec{x}) = \sum_{m\in\mathbb{Z}_+^{\mathcal{I}_d}} b_{I,m}\prod_{J\in\mathcal{I}_d} {x_J}^{m_J}, \quad \vec{x} \in \mathbb{C}^{\mathcal{I}_d},\ I \in \mathcal{I}_d\,,$$

where

$$b_{I,m} = \sharp\{w \in W_I^{(1)} \mid \vec{s}(w) = m\}, \quad I \in \mathcal{I}_d\,,\ m \in \mathbb{Z}_+^{\mathcal{I}_d}.$$

Then (50) implies

$$x_{c,I}g_I(\lambda t) = \Phi_I(\vec{x}_c\,\vec{g}(t)) = \sum_{m\in\mathbb{Z}_+^{\mathcal{I}_d}} C_{I,m}\prod_{J\in\mathcal{I}_d} g_J(t)^{m_J}, \quad t \in \mathbb{C}, \tag{51}$$

where we wrote

$$C_{I,m} = b_{I,m}\prod_{J\in\mathcal{I}_d} x_{c,J}^{m_J}\,.$$

Now assume that $\bar{p}_{\vec{x},I}$ is concentrated on a single point for some $I \in \mathcal{K}_{\vec{x}_c}$. Then its generating function is

$$g_I(t) = e^{M_I t}. \tag{52}$$

Differentiating (51) by t up to twice, putting $t = 0$, and using (52) on the left hand side, we obtain equations, which eventually lead to

$$(0 \leqq) \quad \sum_m C_{I,m}(\lambda M_I - \sum_J m_J M_J)^2 = -\sum_m C_{I,m} \sum_J m_J V_J \,.$$

Since $C_{I,m}$, m_J, and V_J are non-negative, it follows that $V_J = 0$ for all $J \in \mathcal{I}_d$ such that there exists $m \in \mathbb{Z}_+^{\mathcal{I}_d}$ satisfying $m_J > 0$ and $m_K = 0$ if $K \notin \mathcal{K}_{\vec{x}_c}$. In other words, $V_J = 0$ for all $J \in \mathcal{I}_d$ such that $\Phi_I(\vec{x})$ has a term which contains a positive power of x_J and composed of those x_K with $K \in \mathcal{K}_{\vec{x}_c}$. With (FP3) we see that $J = (1)$ has such property for all $I \in \mathcal{K}_{\vec{x}_c}$. Hence $V_{(1)} = 0$, and $\bar{p}_{\vec{x},(1)}$ is concentrated on a single point.

The arguments after (52) can now be repeated with $I = (1)$. Using also (FP3) again, we eventually see that (51) implies

$$x_{c,(1)} \exp(\lambda\, M_{(1)}\, t) = \sum_{m \in \mathbb{Z}_+^{\mathcal{K}_{\vec{x}_c}}} C_{(1),m} \exp(\sum_{J \in \mathcal{K}_{\vec{x}_c}} M_J\, m_J\, t), \quad t \in \mathbb{C}.$$

But $\Phi_{(1)}$ is a polynomial, hence the right hand side is actually a finite summation with positive coefficients, so this can hold only if

$$\lambda\, M_{(1)} = \sum_{J \in \mathcal{K}_{\vec{x}_c}} M_J\, m_J,$$

for all m such that $C_{(1),m} > 0$. Proposition 4 implies that $C_{(1),m} > 0$ for m such that;

(a) $m_{(1)} = 2$ and $m_J = 0,\ J \neq (1)$,

(b) $m_{(1)} = d+1$ and $m_J = 0,\ J \neq (1)$.

Therefore $2M_{(1)} = (d+1)M_{(1)}$, hence $M_{(1)} = 0$ for $d \geqq 2$, which is a contradiction, because Proposition 14 implies $(1) \in \mathcal{K}_{\vec{x}_c}$, which, with the preceding result, implies $M_{(1)} > 0$.

This proves that if $I \in \mathcal{K}_{\vec{x}_c}$ and $\vec{x} \in \mathcal{D}om(\vec{x}_c) \cap \Xi_d$, then $\bar{p}_{\vec{x},I}$ is not concentrated on a single point.

(iii) The preceeding result implies, by a standard argument relating positivity of variances and the absolute values of characteristic functions, that there exists $a > 0$ such that

$$|g_I(\sqrt{-1}t)| < 1, \quad -a < t < a,\ t \neq 0,\ I \in \mathcal{K}_{\vec{x}_c}\,.$$

Since characteristic functions are continuous, this further implies that there exist $a' > 0$ and $0 < C < 1$ such that

$$|g_I(\sqrt{-1}t)| \leqq C, \quad \frac{a'}{\lambda} \leqq |t| \leqq a',\ t \in \mathbb{R},\ I \in \mathcal{K}_{\vec{x}_c}\,. \tag{53}$$

Also (51) and (FP1) and the fact that each term in Φ_I has degree 2 or more (Proposition 4) imply (together with the fact that absolute values of characteristic functions are no greater than 1),

$$\max_{I \in \mathcal{K}_{\vec{x}_c}} |g_I(\sqrt{-1}\lambda t)| \leqq \max_{I \in \mathcal{K}_{\vec{x}_c}} |g_I(\sqrt{-1}t)|^2, \quad t \in \mathbb{R}.$$

This and (53) imply

$$|g_I(\sqrt{-1}t)| \leqq C_2 \exp(-C_1 |t|^{1/d_w}), \quad t \in \mathbb{R},\ I \in \mathcal{K}_{\vec{x}_c}\,,$$

for some positive constants C_1 and C_2.

This completes a proof of Lemma 22.

□

The last statement in Lemma 22 implies (through the inversion formula and the convergence of $\int_{-\infty}^{\infty} t^n g_I(\sqrt{-1}t)\,dt$, $t = 1, 2, 3, \cdots$,) the existence of C^∞ density $\bar{\rho}^*_{\vec{x},(1)}$ of $\bar{p}^*_{\vec{x},(1)}$.

To prove the positivity of $\bar{\rho}^*_{\vec{x},(1)}$, substitute $g_{(1)}(t) = \int_0^\infty e^{t\xi} \bar{\rho}^*_{\vec{x},(1)}(\xi)\,d\xi$ in (50), to find

$$\lambda^{-1}\bar{\rho}^*_{\vec{x},(1)}(\lambda^{-1}\xi) = x_{c,(1)}(\bar{\rho}^*_{\vec{x},(1)} * \bar{\rho}^*_{\vec{x},(1)})(\xi) + {x_{c,(1)}}^d(\bar{\rho}^*_{\vec{x},(1)} * \bar{\rho}^*_{\vec{x},(1)} * \cdots * \bar{\rho}^*_{\vec{x},(1)})(\xi) + \text{ non-negative terms}, \quad \xi \geqq 0,$$

where the operation $*$ denotes convolution, and we also used Proposition 4. Note first that since $\bar{\rho}^*_{\vec{x},(1)}$ is continuous and $M_{(1)} > 0$, there is a $\xi_0 > 0$ for which $\bar{\rho}^*_{\vec{x},(1)}(\xi) > 0$ in a neighborhood of ξ_0. Furthermore, the above relation implies that if $\bar{\rho}^*_{\vec{x},(1)}$ is positive in a neighborhood of $\xi_1, \xi_2 \geqq 0$, then it also is in a neighborhood of $\lambda^{-1}(\xi_1 + \xi_2)$. With Proposition 12 and the continuity of $\bar{\rho}^*_{\vec{x},(1)}$, we therefore see that $\bar{\rho}^*_{\vec{x},(1)}$ is positive in a neighborhood of 0. The above relation also implies that if $\bar{\rho}^*_{\vec{x},(1)}$ is positive in a neighborhood of $\xi_1, \xi_2, \cdots, \xi_{d+1} \geqq 0$, then it also is in a neighborhood of $\lambda^{-1}(\xi_1 + \xi_2 + \cdots + \xi_{d+1})$. With Proposition 12, we inductively see that $\bar{\rho}^*_{\vec{x},(1)}$ is positive on $(0, \infty)$.

This completes a proof of Theorem 9.

□

4.3 Exponent for mean square displacement.

Here we prove Theorem 10 and Theorem 11. Since the proofs are similar for the full model and the restricted model, we will concentrate on the full model.

We introduce the following notation which we use throughout this section.

Define $\nu : W^{(0)} \to \mathbb{Z}_+ \cup \{\infty\}$ by

$$\nu(w) = \min\{n \in \mathbb{Z}_+ \cup \{\infty\} \mid w(k) \in G_n\,,\ k = 0, 1, 2, \cdots, L(w)\}, \quad w \in W^{(0)}. \tag{54}$$

Note the obvious relation

$$2^{\nu(w)-1} < L(w) \leqq d(d+1)^{\nu(w)}, \quad w \in W^{(0)}. \tag{55}$$

(The second inequality is because there are $(d+1)^n$ unit simplices in F_n, and within each unit simplex a self-avoiding walk can spend at most d steps.)

For $w \in W^{(0)}$, let $\|w\| = \max\{|w(k)| \mid k = 0, 1, 2, \cdots, L(w)\}$, where $|\cdot|$ denotes the (Euclidean) length in $\mathbb{R}^d \supset F$. By definition,

$$2^{\nu(w)-1} < \|w\| \leqq 2^{\nu(w)}. \tag{56}$$

(To see this, note that by definition of $\nu(w)$, w is contained in the ball of radius $2^{\nu(w)}$ centered at O, but not in the ball of radius $2^{\nu(w)-1}$.)

4.3.1 Tauberian type estimates and number of paths.

Here we prove Theorem 10 for the full model.

Let λ be as before, β_c be a critical point of the full model, and $\vec{x}_{can}(\beta_c) = \vec{x}_{can,\mathcal{I}_d}(\beta_c)$ be as in (17).

Proposition 23 *Define, for $b > 0$, $n \in \mathbb{Z}_+$, $\xi \in \mathbb{R}$, $c_n = b\lambda^{-n}\sqrt{n}$ and $h_n(\xi) = \dfrac{1}{\sqrt{2\pi}c_n} e^{-\xi^2/(2c_n^2)}$. Then for sufficiently large b it holds that*

$$\lim_{n\to\infty} (\bar{p}_{\vec{x}_{can}(\beta_c),n,I} * h_n)(\xi) = \bar{\rho}^*_{\vec{x}_{can}(\beta_c),I}(\xi), \quad I \in \mathcal{K}_{\vec{x}_c},$$

uniformly in $\xi \in \mathbb{R}$, where

$$(\bar{p}_{\vec{x},n,I} * h_n)(\xi) = \int_{\mathbb{R}} h_n(\xi - \eta)\, \bar{p}_{\vec{x},n,I}(d\eta)$$

is a convolution.

Proof. For $n \in \mathbb{Z}_+$ and $I \in \mathcal{I}_d$, denote the characteristic function of $(\bar{p}_{\vec{x}_{can}(\beta_c),n,I} * h_n)(\xi)\,d\xi$ by $\phi_{n,I}$. Then

$$\phi_{n,I}(t) = \int e^{\sqrt{-1}\xi t}(\bar{p}_{\vec{x}_{can}(\beta_c),n,I} * h_n)(\xi)\,d\xi = \bar{\varphi}_{\vec{x}_{can}(\beta_c),n,I}(t)e^{-c_n^2 t^2/2}, \tag{57}$$

where, using (17), (19), (47), (39) in (46), we find

$$\bar{\varphi}_{\vec{x}_{can}(\beta_c),n,I}(t) = \frac{Z_{n,I}(\beta_c - \sqrt{-1}\lambda^{-n}t)}{Z_{n,I}(\beta_c)}, \quad t \in \mathbb{C}.$$

According to what is proved for (48), we have

$$\lim_{n\to\infty} \bar{\varphi}_{\vec{x}_{can}(\beta_c),n,I}(t) = g_{\vec{x}_{can}(\beta_c),I}(t), \tag{58}$$

uniformly in t on compact sets. Let $A = \{t \in \mathbb{C} \mid \mathbf{Im}t \geqq 0,\ \lambda^{-1} \leqq |t| \leqq 1\}$. Then Lemma 22 and (CS1) implies $\sup_{t\in A} |g_{\vec{x}_{can}(\beta_c),I}(\sqrt{-1}t)| < 1$ for $I \in \mathcal{K}_{\vec{x}_c}$. With (CS1) we therefore see that for sufficiently small positive $\epsilon > 0$ there exists an integer n_1 such that

$$|Z_{n,I}(\beta_c - \sqrt{-1}\lambda^{-n}t)| = Z_{n,I}(\beta_c)|\bar{\varphi}_{\vec{x}_{can}(\beta_c),n,I}(t)| < x_{c,I} - \epsilon, \quad I \in \mathcal{K}_{\vec{x}_c},\ n = n_1, n_1+1, \cdots,\ I \in \mathcal{K}_{\vec{x}_c},\ t \in A. \tag{59}$$

Define $\vec{x}' = (x'_I;\ I \in \mathcal{I}_d)$ by

$$x'_I = \begin{cases} x_{c,I} - \epsilon, & I \in \mathcal{K}_{\vec{x}_c}, \\ 0 & I \notin \mathcal{K}_{\vec{x}_c}. \end{cases}$$

Then $\vec{x}' \in D^o \setminus \{0\}$ by Theorem 15. Since D^o is an open set in Ξ_d, there exists $\delta > 0$ such that if we define $\vec{x}'' = (x''_I;\ I \in \mathcal{I}_d)$ by

$$x''_I = \begin{cases} x_{c,I} - \epsilon, & I \in \mathcal{K}_{\vec{x}_c}, \\ \delta & I \notin \mathcal{K}_{\vec{x}_c}, \end{cases}$$

then $\vec{x}'' \in D^o$. Also, by the definition of $\mathcal{K}_{\vec{x}_c}$ and (CS1), there exists $n_0 \geqq n_1$ such that

$$|Z_{n,I}(\beta_c - \sqrt{-1}\lambda^{-n}t)| \leqq Z_{n,I}(\beta_c) < \delta, \quad I \notin \mathcal{K}_{\vec{x}_c},\ n = n_0, n_0+1, \cdots,\ t \in \mathbb{R}. \tag{60}$$

Proposition 16, (59), and (60) then imply, with $\vec{x}'' \in D^o$, that there exists positive constants C_1 and C_2 such that

$$\begin{aligned} &|Z_{m+n,I}(\beta_c - \sqrt{-1}\lambda^{-n}t)| = |X_{m,I}(\vec{Z}_n(\beta_c - \sqrt{-1}\lambda^{-n}t)| \leqq X_{m,I}(\vec{x}'') \leqq C_1 e^{-C_2 2^m}, \\ &m \in \mathbb{Z}_+,\ n = n_0, n_0+1, \cdots,\ I \in \mathcal{I}_d,\ t \in A. \end{aligned} \tag{61}$$

Let n be an integer satisfying $n > n_0$, and t be a real satisfying $|t| \in [1, \lambda^{n-n_0-1})$, and $m = \left[\dfrac{\log|t|}{\log\lambda}\right] + 1$, where $[x]$ is the largest integer not exceeding x. Then $n - m \geqq n_0$ and $\lambda^{-m}t \in A$. Hence (61) and (57) imply

$$\begin{aligned} &|\phi_{n,I}(t)| \leqq \frac{|Z_{m+n-m,I}(\beta_c - \sqrt{-1}\lambda^{-(n-m)}(\lambda^{-m}t))|}{Z_{n,I}(\beta_c)} \leqq \frac{C_1}{Z_{n,I}(\beta_c)}\, e^{-C_2 2^m} \leqq \frac{C_1}{Z_{n,I}(\beta_c)}\, e^{-C_2|t|^{1/d_w}}, \\ &n > n_0,\ I \in \mathcal{I}_d,\ |t| \in [1, \lambda^{n-n_0-1}). \end{aligned}$$

On the other hand, we have $\lim_{n\to\infty} Z_{n,I}(\beta_c) = x_{c,I}$, and $\lim_{n\to\infty} \phi_{n,I}(t) = g_{\vec{x}_{can}(\beta_c),I}(t)$, $t \in \mathbb{R}$, by (58). Hence the dominated convergence theorem implies

$$\lim_{n\to\infty} \int_{\mathbb{R}} |\chi_{[-\lambda^{n-n_0-1},\lambda^{n-n_0-1}]}(t)\phi_{n,I}(t) - g_{\vec{x}_{can}(\beta_c),I}(t)|\,dt = 0, I \in \mathcal{K}_{\vec{x}_c},$$

where χ_A denotes the indicator function of a set A.

On the other hand,

$$\begin{aligned} &\int_{\mathbb{R}} |\chi_{[-\lambda^{n-n_0-1},\lambda^{n-n_0-1}]}(t)\phi_{n,I}(t) - \phi_{n,I}(t)|\,dt \leqq 2\int_{\lambda^{n-n_0-1}}^{\infty} e^{-c_n^2 t^2/2}\,dt \\ &\leqq 2\int_{\lambda^{n-n_0-1}}^{\infty} t e^{-c_n^2 t^2/2}\,dt = \frac{2}{b^2 n} e^{-n(b^2\lambda^{-2n_0-2}/2 - 2\log\lambda)} \ \to\ 0, \quad n \to \infty, \end{aligned}$$

for sufficiently large b. Therefore

$$\lim_{n\to\infty}\int_{\mathbb{R}} |\phi_{n,I}(t) - g_{\vec{x}_{can}(\beta_c),I}(t)|\, dt = 0\,.$$

Using the inversion formula we therefore have

$$\sup_{\xi\in\mathbb{R}} |(\bar{p}_{\vec{x}_{can}(\beta_c),n,I} * h_n)(\xi) - \bar{\rho}^*_{\vec{x}_{can}(\beta_c),I}(\xi)| \leqq \int_{\mathbb{R}} |e^{-\sqrt{-1}\xi t}\phi_{n,I}(t) - \bar{\varphi}_{\vec{x}_{can}(\beta_c),I}(t)|\, dt \ \to\ 0, \quad n\to\infty.$$

□

We will use the following in Section 4.3.2.

Corollary 24 *(i) There exists positive constants C_1 and C_2 such that*

$$Z_{m+n,I}(\beta_c+\lambda^{-n}) \leqq C_1 e^{-C_2 2^m}, \quad m,n\in\mathbb{Z}_+,\ I\in\mathcal{I}_d\,.$$

(ii) There exists a positive constant C such that

$$Z_{n,I}(\beta_c-\lambda^{-n}) \leqq C, \quad n\in\mathbb{Z}_+,\ I\in\mathcal{I}_d\,.$$

Proof. (i) Since $\vec{x}_{can}(\beta_c+\lambda^{-n}t)\in D^o\cap\Xi_d$ for $t>0$, we may use Proposition 16 in a similar way as in the proof of (61) to prove the first claim.

(ii) The second claim is a direct consequence of Proposition 20 for $\vec{t}=\vec{e}$ (defined in (47)), combined with (19).

□

Proof of Theorem 10. To prove the upper bound, define

$$\zeta_n = \sum_{w\in W^{(0)};\ \nu(w)\leqq n} e^{-\beta_c L(w)}, \quad n\in\mathbb{Z}_+. \tag{62}$$

Then a consideration similar to that in the proof of Proposition 4 proves that there exists a polynomial $f_1:\ \mathbb{R}^{\mathcal{I}_d}\to\mathbb{R}$ with positive coefficients, of maximal degree $d+1$, satisfying $f_1(\vec{0})=1$, such that

$$\zeta_{n+1} \leqq f_1(\vec{Z}_n(\beta_c))\zeta_n, \quad n\in\mathbb{Z}_+. \tag{63}$$

The assumption (CS1) implies that $\vec{Z}_n(\beta_c)$ is bounded, hence there exist positive constants A_1 and $A_2\geqq 1$ such that

$$\zeta_n \leqq A_1 A_2^n\,, \quad n\in\mathbb{Z}_+. \tag{64}$$

$L>2^{\nu-1}$ in (55) therefore implies

$$e^{-\beta_c k}N(k) \leqq \sum_{w\in W^{(0)};\ L(w)\leqq k} e^{-\beta_c L(w)} \leqq \zeta_{[\log_2 k]+1} \leqq A_1A_2\, k^{\log_2 A_2},$$

which proves the upper bound.

To prove the lower bound, let $b>0$ be a constant such that the bound in Proposition 23 holds, and let $d_n=\sqrt{2n\log\lambda}c_n=\sqrt{2\log\lambda}\,b\,n\,\lambda^{-n}$, $n\in\mathbb{Z}_+$. Then Proposition 23 implies

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{\xi\in\mathbb{R}}\Big|\int_{[\xi-d_n,\xi+d_n]} h_n(\xi-\eta)\,\bar{p}_{\vec{x}_{can}(\beta_c),n,(1)}(d\eta) - \bar{\rho}^*_{\vec{x}_{can}(\beta_c),(1)}(\xi)\Big|\\
&\leqq \lim_{n\to\infty}\sup_{\xi\in\mathbb{R}}\int_{\mathbb{R}\setminus[\xi-d_n,\xi+d_n]} h_n(\xi-\eta)\,\bar{p}_{\vec{x}_{can}(\beta_c),n,(1)}(d\eta)\\
&\leqq \lim_{n\to\infty} h_n(d_n) = \lim_{n\to\infty}\frac{1}{\sqrt{2\pi b^2 n}} = 0\,.
\end{aligned}$$

With the positivity of $\bar{\rho}^*_{\vec{x}_{can}(\beta_c),(1)}$ (Theorem 9), this implies that there exists an integer n_2 and a positive constant ϵ such that

$$\bar{p}_{\vec{x}_{can}(\beta_c),n,(1)}([\xi - d_n, \xi + d_n]) \geqq c_n\epsilon, \quad n \geqq n_2\,, \ \xi \in [\lambda^{-1}, \lambda].$$

Let k be a positive integer, and let n be a positive integer satisfying $\lambda^{-n}k \in [1, \lambda)$. Assume that k is sufficiently large so that $n \geqq n_2$ and $d_n \leqq 1 - \lambda^{-1}$. Then

$$\bar{p}_{\vec{x}_{can}(\beta_c),n,(1)}([\lambda^{-n}k - 2d_n, \lambda^{-n}k]) \geqq c_n\epsilon.$$

Note that we can construct an injection $\{w \in W^{(n)}_{(1)} \mid L(w) \leqq k\} \to \{w \in W^{(0)} \mid L(w) = k\}$ by extending the path. Hence

$$Z_{n,(1)}(\beta_c)c_n\epsilon \leqq \sum_{w \in W^{(n)}_{(1)};\ k-2d_n\lambda^n \leqq L(w) \leqq k} e^{-\beta_c L(w)} \leqq e^{2\beta_c d_n \lambda^n} e^{-\beta_c k} N(k).$$

Note that $\lambda^{-n}k \geqq 1$ implies $n \leqq \dfrac{\log k}{\log \lambda}$, which further implies

$$d_n\lambda^n \leqq \sqrt{2/\log\lambda}\, b \log k.$$

Since $n \geqq 1$ for sufficiently large k, $\lambda^{-n}k \geqq 1$ also implies $c_n \geqq bk^{-1}$. Hence (for sufficiently large k),

$$N(k) \geqq e^{\beta_c k} k^{-1-2b\beta_c\sqrt{2/\log\lambda}} Z_{n,(1)}(\beta_c) b\epsilon,$$

which implies the lower bound.

□

4.3.2 Large deviation type estimates on long paths and short paths.

Define, for $n, m \in \mathbb{Z}_+$,

$$U_{n,m} = \sum_{\substack{w \in W^{(0)};\\ \nu(w) \leqq n,\\ L(w) \geqq \lambda^{n+(m/2)}}} e^{-\beta_c L(w)} \qquad \text{and} \qquad V_{n,m} = \sum_{\substack{w \in W^{(0)};\\ \nu(w) = n+1,\\ L(w) \leqq \lambda^{n-m}}} e^{-\beta_c L(w)}.$$

Proposition 25 *There exist positive constants C' and C'' such that*

$$U_{n,m} \leqq C' A_2^n e^{-C''\lambda^{m/2}}, \quad \text{and} \quad V_{n,m} \leqq C' A_2^n e^{-C'' 2^m}, \quad n, m \in \mathbb{Z}_+,$$

where A_2 is as in (64).

Proof. Let C'' be a positive constant satisfying $C'' \leqq \dfrac{\lambda - \sqrt{\lambda}}{d+1}$.

To prove the bound for $U_{n,m}$, define

$$S_{n,m,I} = \sum_{w \in W^{(n)}_I,\ L(w) \geqq C''\lambda^{n+(m/2)}} e^{-\beta_c L(w)}, \quad n, m \in \mathbb{Z}_+,\ I \in \mathcal{I}_d\,.$$

Then, in a similar way as deriving (63), we find

$$U_{n+1,m} \leqq f_1(\vec{Z}_n(\beta_c))U_{n,m+1} + \sum_{I \in \mathcal{I}_d} S_{n,m,I} \frac{\partial f_1}{\partial x_I}(\vec{Z}_n(\beta_c))\,\zeta_n\,, \quad n, m \in \mathbb{Z}_+.$$

This with (64) implies that there exists $C_1 > 0$ such that

$$A_2^{-n-1}U_{n+1,m} \leqq A_2^{-n}U_{n,m+1} + C_1 \sum_{I \in \mathcal{I}_d} S_{n,m,I}\,, \quad n, m \in \mathbb{Z}_+.$$

Note that $L(w) \leqq d\,(d+1)^{\nu(w)}$ in (55) implies

$$U_{n,m} = 0, \quad \lambda^{n+(m/2)} > d\,(d+1)^n,$$

which holds if

$$m > \frac{2\log d}{\log\lambda} + 2\alpha n,$$

where we wrote $\alpha = \dfrac{\log(d+1)}{\log\lambda} - 1$. (Note that Proposition 12 implies $\alpha > 0$.) Therefore, if we write

$$k_0(n,m) = \left[\frac{2\alpha n - m}{2\alpha+1} + \frac{2\log d}{(2\alpha+1)\log\lambda}\right],$$

then

$$U_{n,m} \leqq C_1 A_2^n \sum_{k=0}^{k_0(n,m)} \sum_{I\in\mathcal{I}_d} S_{n-k-1,m+k,I}\,, \quad n,m\in\mathbb{Z}_+. \tag{65}$$

On the other hand,

$$\begin{aligned} S_{n,m,I} &\leqq \sum_{w\in W_I^{(n)},\ L(w)\geqq C''\lambda^{n+(m/2)}} e^{-(\beta_c-\lambda^{-n})L(w)} e^{-C''\lambda^{m/2}} \leqq Z_{n,I}(\beta_c-\lambda^{-n}) e^{-C''\lambda^{m/2}} \\ &\leqq C e^{-C''\lambda^{m/2}}, \quad n,m\in\mathbb{Z}_+,\ I\in\mathcal{I}_d\,, \end{aligned}$$

for a positive constant C, where we used Corollary 24. This and (65) imply

$$U_{n,m} \leqq CC_1 \sharp\mathcal{I}_d\, A_2^n \sum_{k=0}^{\infty} e^{-C''\lambda^{(m+k)/2}}.$$

Note that there exists a constant k_0 such that

$$\lambda^{(m+k)/2} \geqq \lambda^{(m-k_0)/2} + \lambda^{(k-k_0)/2}, \quad m,k\in\mathbb{Z}_+.$$

(This is because

$$\lambda^{(m+k)/2} - (\lambda^{m/2-k_0} + \lambda^{k/2-k_0}) = (\lambda^{m/2} - \lambda^{-k_0/2})(\lambda^{k/2} - \lambda^{-k_0/2}) - \lambda^{-k_0}$$

is increasing in m and k, hence it is sufficient to choose k_0 such that $1 \geqq 2\lambda^{-k_0/2}$.) Therefore there exists a positive constant C' such that

$$U_{n,m} \leqq CC_1\sharp\mathcal{I}_d \sum_{k=0}^{\infty} e^{-C''\lambda^{(k-k_0)/2}}\, A_2^n e^{-C''\lambda^{(m-k_0)/2}} \leqq C' A_2^n e^{-C''\lambda^{m/2}}, \quad n,m\in\mathbb{Z}_+.$$

To prove the bound for $V_{n,m}$, define

$$T_{n,m,I} = \sum_{w\in W_I^{(n)},\ L(w)\leqq\lambda^{n-m}} e^{-\beta_c L(w)}, \quad n,m\in\mathbb{Z}_+;\ m\leqq n,\ I\in\mathcal{I}_d\,,$$

and write $\vec{T}_{n,m} = (T_{n,m,I}\,,\ I\in\mathcal{I}_d)$. Then, again in a similar way as deriving (63), we find

$$V_{n,m} \leqq (f_1(\vec{T}_{n,m}) - 1)\zeta_n = (f_1(\vec{T}_{n,m}) - f_1(\vec{0}))\zeta_n\,.$$

On the other hand, the condition $L(w) \leqq \lambda^{n-m}$ in the definition of $T_{n,m,I}$ implies

$$T_{n,m,I} \leqq \sum_{w\in W_I^{(n)}} e^{-(\beta_c+\lambda^{m-n})L(w)+1} = eZ_{n,I}(\beta_c+\lambda^{m-n}).$$

With Corollary 24 and (63) we have the statement.

□

Proposition 26 *For sufficiently large positive constant α, it holds that*

$$\varliminf_{k\to\infty} (\log k)^{s\alpha} k^{-s} E_k[\|w\|^{sd_w}] > 0, \qquad \textit{and} \qquad \varlimsup_{k\to\infty} (\log k)^{-s\alpha} k^{-s} E_k[\|w\|^{sd_w}] < \infty,$$

for all $s \geqq 0$, where $E_k[\cdot]$ is the expectation defined in Theorem 11.

Proof. For $k \in \mathbb{N}$, define $\tilde{n}(k) = \left[\dfrac{\log k}{\log \lambda}\right]$. Obviously, $\lambda^{\tilde{n}(k)} \leqq k < \lambda^{\tilde{n}(k)+1}$.

For positive integers m and k satisfying $m \leqq \tilde{n}(k)$, Proposition 25 implies

$$\begin{aligned}
&\sharp\{w \in W^{(0)} \mid L(w) = k,\ \nu(w) \leqq \tilde{n}(k) - m\} \leqq e^{\beta_c k} U_{\tilde{n}(k)-m,2m} \\
&\leqq C' \exp(\beta_c k + (\tilde{n}(k) - m)\log A_2 - C''\lambda^m) \\
&\leqq C' \exp(\beta_c k + \log_{\log\lambda} A_2 \log k - C''\lambda^m).
\end{aligned}$$

This and Theorem 10 imply that for sufficiently large α and for any real ϵ, there exists $C > 0$ such that

$$\begin{aligned}
&\tilde{P}_k[\nu(w) \leqq \tilde{n}(k) - \alpha \log\log k + \epsilon] \leqq C' C_1^{-1} \exp((\log_{\log\lambda} A_2 - C_3)\log k - C''\lambda^{-\epsilon}(\log k)^{\alpha\log\lambda}) \\
&\leqq C e^{-(\log k)^3}, \quad k \in \mathbb{N},
\end{aligned}$$

where $\tilde{P}_k[\cdot]$ is as defined in Theorem 11. This, with (56), implies for sufficiently large α,

$$\lim_{k\to\infty} \tilde{P}_k[\|w\| < (\log k)^{-\alpha/d_w} k^{1/d_w}]\, e^{(\log k)^2} \leqq \lim_{k\to\infty} \tilde{P}_k[2^{\nu(w)} < 2(\log k)^{-\alpha/d_w} k^{1/d_w}]\, e^{(\log k)^2} = 0\,.$$

Chebishev's inequality implies, for $s \geqq 0$,

$$E_k[\|w\|^{sd_w}] \geqq ((\log k)^{-\alpha} k)^s (1 - \tilde{P}_k[\|w\| \leqq (\log k)^{-\alpha/d_w} k^{1/d_w}]).$$

Therefore,

$$\varliminf_{k\to\infty} (\log k)^{s\alpha} k^{-s} E_k[\|w\|^{sd_w}] > 0, \quad s \geqq 0\,.$$

Next, for positive integers m, k, and ℓ, Proposition 25 implies, with an obvious inequality $2^{m+\ell} \geqq 2^{m-1} + 2^{\ell-1}$, $m, \ell \in \mathbb{Z}_+$,

$$\begin{aligned}
&\sharp\{w \in W^{(0)} \mid L(w) = k,\ \nu(w) = \tilde{n}(k) + m + \ell + 2\} \leqq e^{\beta_c k} V_{\tilde{n}(k)+m+\ell+1,m+\ell} \\
&\leqq C' \exp(\beta_c k + (\tilde{n}(k) + m + \ell + 1)\log A_2 - C'' 2^{m+\ell}) \\
&\leqq C' \exp(\beta_c k + \log_{\log\lambda} A_2 \log k + m \log A_2 - C'' 2^{m-1}) A_2^{\ell+1} \exp(-C'' 2^{\ell-1}).
\end{aligned}$$

This and Theorem 10 imply that there exists a positive constant C such that

$$\begin{aligned}
&\tilde{P}_k[\nu(w) \geqq \tilde{n}(k) + \frac{\alpha}{\log 2}\log\log k] = \sum_{\ell=0}^{\infty} \tilde{P}_k[\nu(w) = \tilde{n}(k) + \ell + \frac{\alpha}{\log 2}\log\log k] \\
&\leqq \frac{C'}{C_1 A_2} \sum_{\ell=0}^{\infty} A_2^{\ell} e^{-C'' 2^{\ell-1}} \exp((\log_{\log\lambda} A_2 - C_3)\log k + \alpha \log_2 A_2 \log\log k - \frac{C''}{8}(\log k)^{\alpha}) \\
&\leqq C \exp((\log_{\log\lambda} A_2 - C_3)\log k + \alpha \log_2 A_2 \log\log k - \frac{C''}{8}(\log k)^{\alpha}), \quad k \in \mathbb{N}.
\end{aligned}$$

This, with (56), implies for sufficiently large α,

$$\lim_{k\to\infty} \tilde{P}_k[\|w\| \geqq (\log k)^{\alpha/d_w} k^{1/d_w}]\, e^{(\log k)^2} \leqq \lim_{k\to\infty} \tilde{P}_k[2^{\nu(w)} \geqq (\log k)^{\alpha/d_w} k^{1/d_w}]\, e^{(\log k)^2} = 0\,.$$

Let $s \geqq 0$. Note that $\|w\| \leqq L(w)$, which implies

$$E_k[\|w\|^{sd_w}] \leqq ((\log k)^{\alpha} k)^s + k^{sd_w} \tilde{P}_k[\|w\|^{d_w} \geqq (\log k)^{\alpha} k].$$

Therefore,

$$\varlimsup_{k\to\infty} (\log k)^{-s\alpha} k^{-s} E_k[\|w\|^{sd_w}] < \infty,$$

□

4.3.3 Reflection principle.

The definition of pre-dSG in (1) and (2) induces a natural coordinate system on G which is an onto map $\pi:\ \{0,1,2,\cdots,d\}^{\mathbb{Z}_+}\to G$ defined as follows. For each $v_{0,i}=v_i\in G_0$ $(i=0,1,2,\cdots,d)$ we assign a coordinate $(i,0,0,0,\cdots)$;

$$\pi(i,0,0,0,\cdots)=v_{0,i},\ \ i=0,1,2,3,\cdots,d.$$

For $n=0,1,2,\cdots$ and $i=0,1,2,3,\cdots,d$, put $G_{n,i}=G_n+2^n v_{0,i}$, and define a $1:1$ onto map $\iota_{n,i}:\ G_{n,i}\to G_n$ by $\iota_{n,i}(v)=v-2^n v_{0,i}$. $\iota_{n,i}$ naturally induces a $1:1$ onto map $B_{n,i}\to B_n$, which we also denote by $\iota_{n,i}$.

We proceed with by induction in n and assume that a coordinate system π on G_{n-1} $(=G_{n-1,0})$ has been defined for an $n\geqq 1$, in such a way that $\pi(v)\in G_{n-1}$ holds for any $v=(i_0,i_1,i_2,\cdots,i_{n-1},0,0,0,\cdots)$ with $i_k\in\{0,1,2,\cdots,d\}$, $k=0,1,2,\cdots,n-1$. For $v\in G_{n-1,j}$, with $j\in\{0,1,2,\cdots,d\}$, define

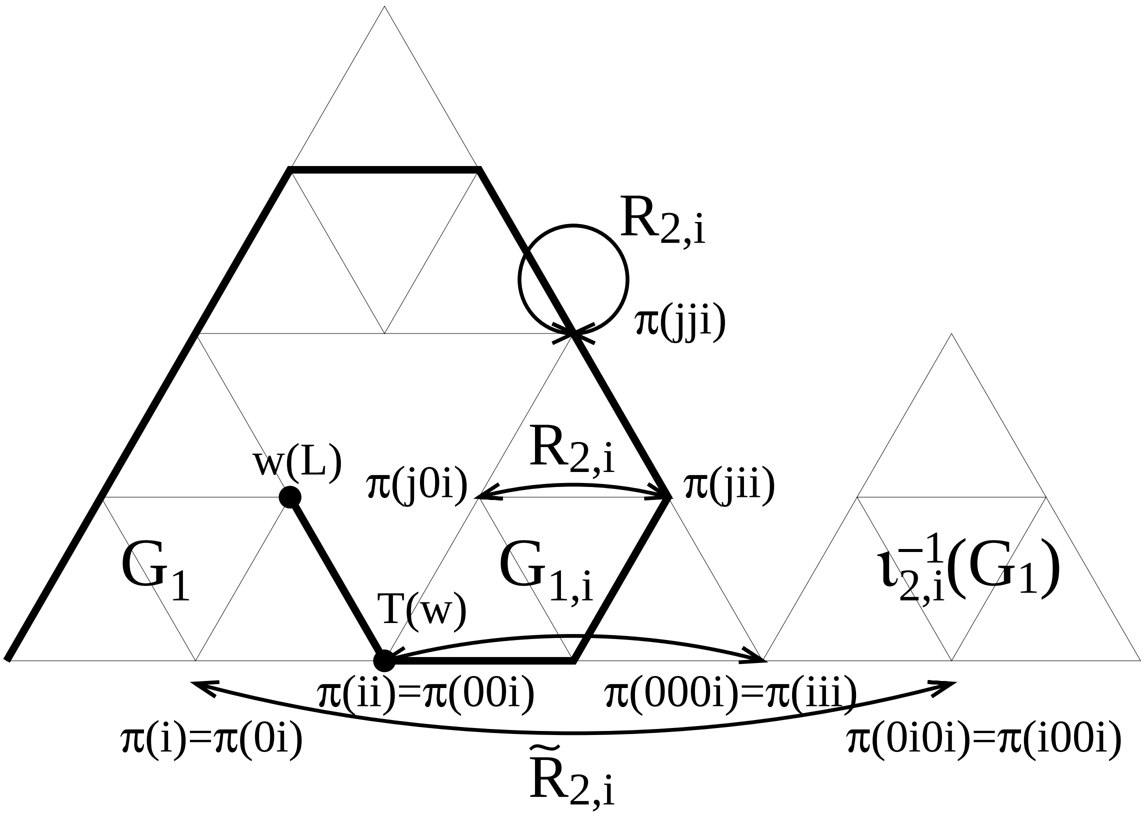
$$\pi(i_0,i_1,i_2,\cdots,i_{n-1},j,0,0,\cdots)=v,\quad\text{if}\ \ \pi(i_0,i_1,i_2,\cdots,i_{n-1},0,0,0,\cdots)=\iota_{n-1,j}(v).$$

Note that this definition is compatible with $G_{n-1,0}=G_{n-1}\subset G_n$.

Each point in $G\setminus\{O\}$ has exactly two coordinate representations, because

$$\pi(j,j,\cdots,j,i,0,\cdots)=\pi(i,i,\cdots,i,j,0,\cdots)\in G_{m,i}\cap G_{m,j}\ ,\quad 0\leqq i<j\leqq d,\ \ m\in\mathbb{Z}_+.$$

Note also that if $\pi(i_0,i_1,i_2,\cdots)\in G_n$ then $i_k=0$, $k=n+1,n+2,\cdots$.



Reflection in the i-j hyperplane

We now define a reflection map (see the Figure) with which we formulate a reflection principle in Theorem 28. For each $i = 1, 2, 3, \cdots, d$ define $R_{0,i} : \ \{0, 1, 2, \cdots, d\} \to \{0, 1, 2, \cdots, d\}$ by

$$\begin{cases} R_{0,i}(0) = i, \\ R_{0,i}(i) = 0, \\ R_{0,i}(j) = j, \ \ j \neq 0, i. \end{cases}$$

For $n = 1, 2, 3, \cdots$ and $i = 1, 2, \cdots, d$, we define $1:1$ maps $R_{n,i} : \ G \to G$ and $\tilde{R}_{n,i} : \ G \to G$ ('partial reflections' with respect to a hyperplane containing $\pi(j, j, \cdots, j, i, 0, 0, \cdots) \in G_{n-1,i}$, $j \neq 0, i$, and 'perpendicular to i-th axis'), by:

$$\begin{aligned} & R_{n,i}(\pi(x_0, x_1, x_2, \cdots)) \\ & = \begin{cases} \pi(x_0, x_1, x_2, \cdots), \ \text{if } \pi(x_0, x_1, x_2, \cdots, x_n, \cdots) \notin G_{n-1,i}, \\ \pi(R_{0,i}(x_0), R_{0,i}(x_1), \cdots, R_{0,i}(x_{n-1}), x_n, 0, 0, 0, \cdots), \\ \qquad\qquad\qquad \text{if } \pi(x_0, x_1, x_2, \cdots, x_n, \cdots) \in G_{n-1,i} \ \text{ and } \ x_n = i, \end{cases} \end{aligned} \tag{66}$$

and

$$\begin{aligned} & \tilde{R}_{n,i}(\pi(x_0, x_1, x_2, \cdots)) \\ & = \begin{cases} \pi(x_0, x_1, x_2, \cdots), \ \text{if } \pi(x_0, x_1, x_2, \cdots, x_n, \cdots) \notin G_{n-1} \cup \iota_{n,i}^{-1}(G_{n-1}), \\ \pi(R_{0,i}(x_0), R_{0,i}(x_1), \cdots, R_{0,i}(x_{n-1}), x_n, R_{0,i}(x_{n+1}), 0, 0, \cdots), \\ \qquad\qquad\qquad \text{if } \pi(x_0, x_1, x_2, \cdots, x_n, \cdots) \in G_{n-1} \cup \iota_{n,i}^{-1}(G_{n-1}) \ \text{ and } x_n = 0\,. \end{cases} \end{aligned} \tag{67}$$

Note that G_{n-1} , $G_{n-1,i}$, and $\iota_{n,i}^{-1}(G_{n-1})$ are three copies of G_{n-1} aligned in 'i-th axis' direction, such that

$$G_{n-1} \cap G_{n-1,i} = \{\pi(\overset{0,1,2}{0,0,0}, \cdots, 0, \overset{n}{i}, 0, 0, \cdots)\} = \{\pi(\overset{0,1,2}{i,i,i}, \cdots, \overset{n-1}{i}, 0, 0, 0, \cdots)\}, \tag{68}$$

and

$$G_{n-1,i} \cap \iota_{n,i}^{-1}(G_{n-1}) = \{\pi(\overset{0,1,2}{0,0,0}, \cdots, 0, \overset{n+1}{i}, 0, 0, \cdots)\} = \{\pi(\overset{0,1,2}{i,i,i}, \cdots, i, \overset{n}{i}, 0, 0, \cdots)\}. \tag{69}$$

Note also that, by construction all the points in $G_{n-1,i}$ can be written as $\pi(x_0, x_1, \cdots, x_n, 0, 0, \cdots)$ with $x_n = i$, those in G_{n-1} as $\pi(x_0, \cdots, x_n, 0, 0, \cdots)$ with $x_n = 0$, and those in $\iota_{n,i}^{-1}(G_{n-1})$ as $\pi(x_0, \cdots, x_n, x_{n+1}, 0, \cdots)$ with $x_n = 0$ and $x_{n+1} = i$.

Proposition 27 *$R_{n,i}$ and $\tilde{R}_{n,i}$ are $1:1$ maps. Moreover, the following hold.*

(i) If $x \in G_{n-1}$ then $\tilde{R}_{n,i}(x) \in \iota_{n,i}^{-1}(G_{n-1})$.

(ii) If $(x, y) \in B_{n-1}$, then $(\tilde{R}_{n,i}(x), \tilde{R}_{n,i}(y)) \in B$.

(iii) If $x \in G_{n-1} \cap G_{n-1,i}$ then $R_{n,i}(x) = \tilde{R}_{n,i}(x)$.

(iv) If $x \in G_n$ then $R_{n,i}(x) \in G_n$.

(v) If $x \in G_{n-1,i} \cap \bigcup_{j \neq 0, i} G_{n-1,j}$ then $R_{n,i}(x) = x$.

(vi) If $(x, y) \in B_{n-1,i}$ then $(R_{n,i}(x), R_{n,i}(y)) \in B$.

Proof. By definition, $R_{n,i}^2(x) = \tilde{R}_{n,i}^2(x) = x$, $x \in G$, hence, in particular, $R_{n,i}$ and $\tilde{R}_{n,i}$ are $1:1$ maps.

(i) If $x \in G_{n-1}$ then x has a coordinate of the form $x = \pi(x_0, x_1, \cdots, x_{n-1}, 0, 0, 0, \cdots)$, which, with (67), implies $\tilde{R}_{n,i}(x) = \pi(x_0, x_1, \cdots, x_{n-1}, 0, i, 0, \cdots)$, which is in $\iota_{n,i}^{-1}(G_{n-1})$.

(ii) let $(x, y) \in B_{n-1}$. Then $x, y \in G_{n-1}$, hence their coordinates can be written as

$$x = \pi(x_0, x_1, \cdots, x_{n-1}, 0, 0, 0, \cdots), \quad y = \pi(y_0, y_1, \cdots, y_{n-1}, 0, 0, 0, \cdots).$$

With (67), we have

$$\tilde{R}_{n,i}(x) = \pi(R_{0,i}(x_0), R_{0,i}(x_1), \cdots, R_{0,i}(x_{n-1}), 0, i, 0, 0, \cdots) \in \iota_{n,i}^{-1}(G_{n-1}),$$

and a similar expression holds for y. Noting that $\iota_{n,i}^{-1}(G_{n-1})$ is a copy of G_{n-1}, it then suffices to prove

$$(\pi(R_{0,i}(x_0),\cdots,R_{0,i}(x_{n-1}),0,0,\cdots),\pi(R_{0,i}(y_0),\cdots,R_{0,i}(y_{n-1}),0,0,\cdots))\in B_{n-1}\,.$$

Noting the definition of B_{n-1}, we may assume, from $(x,y)\in B_{n-1}$, that $x_{n-1}=y_{n-1}$ and

$$(\pi(x_0,x_1,\cdots,x_{n-2},0,0,0,\cdots),\pi(y_0,y_1,\cdots,y_{n-2},0,0,0,\cdots))\in B_{n-2}\,.$$

Inductively, it eventually turns out that it suffices to prove

$$(\pi(R_{0,i}(x),0,0,\cdots),\pi(R_{0,i}(y),0,0,\cdots))\in B_0 \quad\text{if}\quad (\pi(x,0,0,\cdots),\pi(y,0,0,\cdots))\in B_0\,,$$

which is obvious from the definition of $F_0=(G_0,B_0)$.

(iii) Applying (68) in (66) and (67), we have

$$R_{n,i}(x)=R_{n,i}(\pi(\overset{0,1,2}{0,0,0},\cdots,0,\overset{n}{i},0,0,\cdots))=\pi(\overset{0,1,2}{i,i,i},\cdots,i,\overset{n}{i},0,0,\cdots),$$

and

$$\begin{aligned}\tilde{R}_{n,i}(x)&=\tilde{R}_{n,i}(\pi(\overset{0,1,2}{i,i,i},\cdots,\overset{n-1}{i},0,0,0,\cdots))=\pi(\overset{0,1,2}{0,0,0},\cdots,0,0,\overset{n+1}{i},0,\cdots)\\&=\pi(\overset{0,1,2}{i,i,i},\cdots,i,\overset{n}{i},0,0,\cdots).\end{aligned}$$

Hence $R_{n,i}(x)=\tilde{R}_{n,i}(x)$.

(iv) If $x\in G_n$ then x has a coordinate $x=\pi(x_0,x_1,\cdots,x_n,0,0,\cdots)$. Then (66) implies that $R_{n,i}(x)$ has a similar coordinate, hence $R_{n,i}(x)\in G_n$.

(v) Noting (68) and (69), we see that x has a coordinate $x=\pi(j,j,\cdots,j,\overset{n}{i},0,0,\cdots)$, $j\neq 0,i$. Then (66) implies $R_{n,i}(x)=x$.

(vi) This can be proved in a similar way as (ii).

□

For each self-avoiding path $w\in W^{(0)}$ satisfying $|w(L(w))|<2^{\nu(w)-1}$, we want to assign a self-avoiding path $R(w)\in W^{(0)}$ ('reflected path') such that $|R(w)(L(R(w)))|>2^{\nu(R(w))-1}$ (the right hand side stands for the Euclidean distance of the endpoints of $R(w)$). This is possible using (66) and (67), as follows.

Given $w=(w(0),w(1),\cdots,w(L(w)))\in W^{(0)}$ with the property

$$w(L(w))\in G_{n-1}\setminus\bigcup_{i=1}^{d}G_{n-1,i}\,,\quad\text{where}\quad n=\nu(w), \tag{70}$$

we shall define a path $R(w)$ as follows.

Let $T(w)$ be a positive integer satisfying

$$w(k)\in G_{n-1},\ T(w)\leqq k\leqq L(w),\quad\text{and}\quad w(T(w)-1)\notin G_{n-1}\,.$$

The condition (70) implies that such an integer (uniquely) exists. Since $w(T(w))\in G_{n-1}$, $w(T(w)-1)\in\bigcup_{i=1}^{d}G_{n-1,i}$. Let $i(w)\in\{1,2,\cdots,d\}$ be such that $w(T(w)-1)\in G_{n-1,i(w)}$. Clearly, such an integer is also unique. Also the definitions of $T(w)$ and $i(w)$ imply

$$w(T(w))\in G_{n-1}\cap G_{n-1,i(w)}\,. \tag{71}$$

Define $R(w)$ by

$$R(w)(k)=\begin{cases}R_{n,i(w)}(w(k)), & 0\leqq k<T(w),\\ \tilde{R}_{n,i(w)}(w(k)), & T(w)\leqq k\leqq L(w).\end{cases}$$

Theorem 28 *For each* $w \in W^{(0)}$ *satisfying* $|w(L(w))| < 2^{\nu(w)-1}$, $R(w) \in W^{(0)}$ *(i.e., is a self-avoiding path starting from O) which satisfies* $L(R(w)) = L(w)$, $\nu(R(w)) = \nu(w)+1$, *and* $2^{\nu(w)} < |R(w)(L(w))|$ *.*

Proof. The non-trivial part of the claim is that $R(w)$ is a self-avoiding path. All the other properties are obvious from the definition of the reflections.

We first prove that $R(w)$ is self-avoiding, when w is self-avoiding. Let $n = \nu(w)$. Note that (71) and Proposition 27 (iii) imply

$$R_{n,i(w)}(w(T(w))) = \tilde{R}_{n,i(w)}(w(T(w))). \tag{72}$$

Since $R_{n,i}$ and $\tilde{R}_{n,i}$ are 1 : 1 maps (Proposition 27), the path segments $w_1 = \{R(w)(k),\ 0 \leqq k \leqq T(w)\}$ and $w_2 = \{R(w)(k),\ T(w) \leqq k \leqq L(w)\}$ are self-avoiding.

The definition of $T(w)$ implies $w(k) \in G_{n-1}$, $T(w) \leqq k \leqq L(w)$, which, with Proposition 27 (i), implies $\tilde{R}_{n,i}(w(k)) \in \iota_{n,i}^{-1}(G_{n-1})$, hence, in particular, $w_2 \cap G_n = \{R_{n,i}(w(T(w)))\}$. On the other hand, Proposition 27 (iv) implies $w_1 \subset G_n$. Therefore w_1 and w_2 are mutually avoiding. This proves that $R(w)$ is self-avoiding.

We are left with proving that $R(w)$ is a path, i.e.,

$$(R(w)(k), R(w)(k+1)) \in B,\ \ k = 0, 1, 2, \cdots, L(w)-1\,. \tag{73}$$

Definition of $T(w)$ and Proposition 27 (ii) imply (73) for $T(w) \leqq k < L(w)$. Hence we may assume $0 \leqq k < T(w)$. Definition of $T(w)$ and (72) imply $R(w)(k) = R_{n,i(w)}(w(k))$, $0 \leqq k \leqq T(w)$. If $w(k) \notin G_{n-1,i(w)}$, then $(w(k), w(k+1)) \in B$ and $n = \nu(w)$ and (71) with self-avoiding property of w imply $w(k+1) \notin G_{n-1,i(w)}$ or $w(k+1) \in G_{n-1,i(w)} \cap \displaystyle\bigcup_{j\neq 0,i(w)} G_{n-1,j}$. The definition of $R_{n,i(w)}$ and Proposition 27(v) imply that in either case $R_{n,i(w)}(w(k+1)) = w(k+1)$. Also $w(k) \notin G_{n-1,i(w)}$ implies $R_{n,i(w)}(w(k)) = w(k)$. Hence in this case (73) holds.

The remaining case is when $0 \leqq k < T(w)$ and $w(k) \in G_{n-1,i(w)}$ hold. If $w(k+1) \notin G_{n-1,i(w)}$, then a similar reasoning as above applies. Therefore we may also assume $w(k+1) \in G_{n-1,i(w)}$. Hence $(w(k), w(k+1)) \in B_{n-1,i(w)}$, in which case Proposition 27 (vi) applies and we have (73).

□

Corollary 29 *If* $s \geqq 0$ *and* $k \in \mathbb{N}$,

$$E_k[2^{s(\nu(w)-1)};\ |w(k)| \leqq 2^{\nu(w)-1}] \leqq E_k[2^{s(\nu(w)-1)};\ |w(k)| \geqq 2^{\nu(w)-1}],$$

where $E_k[\cdot]$ *is the expectation defined in Theorem 11.*

Proof. Recalling the definition of the measure $\tilde{P}_k$, we find from Theorem 28,

$$\begin{aligned}
&E_k[2^{s(\nu(w)-1)};\ |w(k)| < 2^{\nu(w)-1}] = \sum_{n=0}^{\infty} 2^{s(n-1)}\tilde{P}_k[\nu(w) = n,\ |w(k)| < 2^{n-1}] \\
&\leqq \sum_{n=0}^{\infty} 2^{s(n-1)}\tilde{P}_k[\nu(w) = n+1,\ |w(k)| > 2^{n}] = 2^{-s}E_k[2^{s(\nu(w)-1)};\ |w(k)| > 2^{\nu(w)-1}] \\
&\leqq E_k[2^{s(\nu(w)-1)};\ |w(k)| > 2^{\nu(w)-1}].
\end{aligned}$$

Hence

$$\begin{aligned}
&E_k[2^{s(\nu(w)-1)};\ |w(k)| \leqq 2^{\nu(w)-1}] = E_k[2^{s(\nu(w)-1)};\ |w(k)| = 2^{\nu(w)-1}] + E_k[2^{s(\nu(w)-1)};\ |w(k)| < 2^{\nu(w)-1}] \\
&\leqq E_k[2^{s(\nu(w)-1)};\ |w(k)| \geqq 2^{\nu(w)-1}].
\end{aligned}$$

□

Proof of Theorem 11. Note an obvious bound

$$|w(L(w))| \leqq \|w\| \leqq 2^{\nu(w)},\ \ w \in W^{(0)}.$$

This and Corollary 29 imply

$$\begin{aligned}
&2^{-s-1}E_k[\|w\|^s] \\
&\leqq 2^{-s-1}E_k[2^{s\nu(w)}] = \frac{1}{2}(E_k[2^{s(\nu(w)-1)};\ |w(k)| \leqq 2^{\nu(w)-1}] + E_k[2^{s(\nu(w)-1)};\ |w(k)| > 2^{\nu(w)-1}]) \\
&\leqq E_k[2^{s(\nu(w)-1)};\ |w(k)| \geqq 2^{\nu(w)-1}] \leqq E_k[|w(k)|^s;\ |w(k)| \geqq 2^{\nu(w)-1}] \\
&\leqq E_k[|w(k)|^s] \leqq E_k[\|w\|^s], \quad s \geqq 0,\ k \in \mathbb{Z}.
\end{aligned}$$

This and Proposition 26 further imply, for sufficiently large α,

$$\varliminf_{k\to\infty} (\log k)^{s\alpha} k^{-s} E_k[|w(k)|^{s\, d_w}] > 0, \qquad \text{and} \qquad \varlimsup_{k\to\infty} (\log k)^{-s\alpha} k^{-s} E_k[|w(k)|^{s\, d_w}] < \infty.$$

Namely, there exist constants k_0, C, and C' such that

$$s\log k - s\alpha \log\log k + C \leqq \log E_k[|w(k)|^{s\, d_w}] \leqq s\log k + s\alpha\log\log k + C', \quad k \geqq k_0,\ s \geqq 0,$$

which completes a proof of Theorem 11. □

5 Restricted model on the 4-dimensional Sierpiński gasket.

In this section, we consider the restricted model for $d = 4$. The RG map $\vec{\Phi}$ (Proposition 4) is a map on 6 dimensional space $\mathbb{C}^{\mathcal{I}_4}$ where, as in Section 2, $\mathcal{I}_4 = \{(1), (1,1), (2), (3), (4), (1,2)\}$. (For convenience, we assign the second coordinate to $(1,1)$ in this section.)

We will consider the restricted self-avoiding walks, the self-avoding paths w starting from O with the property $s_J(w) = 0,\ J \notin \mathcal{K}_{res}$, where $\mathcal{K}_{res} = \{(1), (11)\}$ (see (13) and (15)). We regard $\mathbb{R}_+^{\mathcal{K}_{res}} = \{(x_{(1)}, x_{(11)}, 0, \cdots, 0) \mid x_{(1)}, x_{(11)} \in \mathbb{R}_+\} \subset \mathbb{R}_+^{\mathcal{I}_4}$.

To apply the results of previous sections, we use the following explicit properties of $\vec{\Phi}$.

Proposition 30 *The map $\vec{\Phi}$ satisfies the following.*

(i) $\vec{\Phi}$ *is a* 6 *dimensional vector valued function whose components are polynomials in* 6 *variables* (1), (11), (2), (3), (4), (12) *with positive integer coefficients. The degree of each term in the polynomials are no less than* 2 *and no greater than* 5.

(ii)

$$\begin{aligned}
\Phi_{(1)}(x,y,0,0,0,0) &= x^2 + 3x^3 + 6x^4 + 6x^5 + 12x^3y + 30x^4y + 18x^2y^2 \\
&\quad + 78x^3y^2 + 96x^2y^3 + 132xy^4 + 132y^5, \\
\Phi_{(1,1)}(x,y,0,0,0,0) &= x^4 + 2x^5 + 4x^3y + 13x^4y + 32x^3y^2 + 88x^2y^3 \\
&\quad + 22y^4 + 220xy^4 + 186y^5.
\end{aligned} \tag{74}$$

(iii) There exist polynomials $\Phi_{I,i}$, $I = (1), (11)$, $i = 0,1,2,3,4$, of positive coefficients, such that $\Phi_{(1),0}$ contains a term $x_{(1)}^2$, $\Phi_{(11),0}$ contains a term $x_{(1)}^4$, and

$$\begin{aligned}
\Phi_{(1)}(\vec{x}) &= \Phi_{(1),1}(\vec{x})\,x_{(1)} + \tfrac{1}{2}\Phi_{(1),2}(\vec{x})\,x_{(2)} + \tfrac{1}{3}\Phi_{(1),3}(\vec{x})\,x_{(3)} + \Phi_{(1),4}(\vec{x})\,x_{(11)} + \Phi_{(1),0}(\vec{x}), \\
\Phi_{(2)}(\vec{x}) &= \Phi_{(1),1}(\vec{x})\,x_{(2)} + \Phi_{(1),2}(\vec{x})\,x_{(3)} + \Phi_{(1),3}(\vec{x})\,x_{(4)} + \Phi_{(1),4}(\vec{x})\,x_{(12)}\,, \\
\Phi_{(11)}(\vec{x}) &= \Phi_{(11),1}(\vec{x})\,x_{(1)} + \tfrac{1}{2}\Phi_{(11),2}(\vec{x})\,x_{(2)} + \tfrac{1}{3}\Phi_{(11),3}(\vec{x})\,x_{(3)} + \Phi_{(11),4}(\vec{x})\,x_{(11)} + \Phi_{(11),0}(\vec{x}), \\
\Phi_{(12)}(\vec{x}) &= \Phi_{(11),1}(\vec{x})\,x_{(2)} + \Phi_{(11),2}(\vec{x})\,x_{(3)} + \Phi_{(11),3}(\vec{x})\,x_{(4)} + \Phi_{(11),4}(\vec{x})\,x_{(12)}\,.
\end{aligned} \tag{75}$$

(iv) For each $I \notin \mathcal{K}_{res}$ there exist positive integers $m = m_I$ and $m' = m'_I$ such that $\Phi_{(1)}$ and $\Phi_{(11)}$ contain terms $x_{(1)}^m x_I$ and $x_{(1)}^{m'} x_I$, respectively.

(v) If $I \notin \mathcal{K}_{res}$, then each term in Φ_I contains a positive power of x_J for some $J \notin \mathcal{K}_{res}$. Furthermore, each term in $\Phi_{(3)}$ and $\Phi_{(4)}$ has total degree 2 or more of x_J's with $J \notin \mathcal{K}_{res}$. $\Phi_{(2)}$ contains a term $x_{(1)}^3 x_{(11)} x_{(12)}$ and $\Phi_{(12)}$ contains a term $x_{(1)}^4 x_{(2)}$.

Proof. (i) This is already proved in Proposition 4.

(ii) The explicit forms (74) of $\Phi_I(x,y,0,0,0,0)$, $I=(1),(1,1)$, are obtained by explicit counting of paths in the right hand side of the definition (9) for $\vec{\Phi}=\vec{X}_1$, with aid of computer calculations. The explicit forms (74) are given in [9, eqs. (A1) (A2)].

(iii) The expression for $\Phi_{(2)}$ in (75) follows from an observation (from the definition (5)) that each path w in $W^{(1)}_{(2)}$, which starts from O and ends at $v_{1,2}$, must pass through $v_{1,1}$. w therefore has a path segment of length more than 1 in the unit simplex $F_0+v_{0,1}$ (which contains $v_{1,1}$). The contribution x_I to $\Phi_{(2)}(\vec{x})$ from this simplex is therefore in $\{x_I \mid I \notin \mathcal{K}_{res}\}$. Classifying the terms in$\Phi_{(2)}(\vec{x})$ by this x_I, we obtain the claimed expression.

The expression for $\Phi_{(12)}$ follows in a similar way.

To relate $\Phi_{(2)}$ to $\Phi_{(1)}$, classify the terms again by the contribution from $F_0+v_{0,1}$. Let $w\in W^{(1)}_{(2)}$. According to what is just proved, $\hat{w}\cap(B_0+v_{0,1})$ is congruent to one of $\{\Delta_I \mid I\notin\mathcal{K}_{res}\}$, in the notation of Proposition 3.

If it is congruent to $\Delta_{(2)}$, then w has a form $\overline{v_a v_{1,1} v_b}$ in the simplex. To this w assign a path w' obtained by replacing $\overline{v_a v_{1,1} v_b}$ by $\overline{v_a v_b}$. Then $w'\in W^{(1)}_{(1)}$, and contributes a term to $\Phi_{(1)}$ similar to the contribution of w to $\Phi_{(2)}$ but with $x_{(2)}$ replaced by $x_{(1)}$.

If $w\cap(B_0+v_{0,1})$ is congruent to $\Delta_{(3)}$, then w has a form $\overline{v_a v_{1,1} v_c v_b}$ or $\overline{v_a v_c v_{1,1} v_b}$ in the simplex. To this w assign a path w' obtained, as before, by shortcutting $v_{1,1}$, to obtain $\overline{v_a v_c v_b}$. Then $w'\in W^{(1)}_{(1)}$, and contributes a term to $\Phi_{(1)}$ similar to the contribution of w to $\Phi_{(2)}$ but with $x_{(3)}$ replaced by $x_{(2)}$. This correspondence is 2 to 1, which explains the factor $\dfrac{1}{2}$ in the expression of $\Phi_{(1)}$ in (75).

The other possibilities (4) and (12) are similar.

That $x^2_{(1)}$ is contained in $\Phi_{(1)}(\vec{x})$ is proved in Proposition 4. This term can not be obtained by an above mentioned 'shortcutting' procedure on paths, so it must be contained in $\Phi_{(1),0}(\vec{x})$.

The expression for $\Phi_{(11)}$ follows in a similar way.

(iv) $W^{(1)}_{(1)}$ contains paths $\overline{Ov_1(v_1+v_2)v_{1,1}}$, $\overline{Ov_1(v_1+v_2)(v_1+v_3)v_{1,1}}$, $\overline{Ov_1(v_1+v_2)(v_1+v_3)(v_1+v_4)v_{1,1}}$, and $\overline{Ov_2(v_2+v_3)v_3v_4v_1v_{1,1}}$, which respectively contribute terms $x_{(1)}x_{(i)}$, $i=2,3,4$, and $x^3_{(1)}x_{(12)}$ in $\Phi_{(1)}$.

Similarly, paths sets

$$\begin{array}{l}(\overline{Ov_1v_{1,1}}\,,\overline{v_{1,2}(v_3+v_2)(v_3+v_1)v_{1,3}}),\\ (\overline{Ov_1v_{1,1}}\,,\overline{v_{1,2}(v_3+v_2)(v_3+v_1)(v_3+v_4)v_{1,3}}),\\ (\overline{Ov_1v_{1,1}}\,,\overline{v_{1,2}(v_3+v_2)(v_3+v_1)(v_3+v_4)v_3v_{1,3}}),\\ (\overline{Ov_1v_{1,1}}\,,\overline{v_{1,2}v_2v_4v_3v_{1,3}}),\end{array}$$

respectively contribute terms $x^3_{(1)}x_{(i)}$, $i=2,3,4$, and $x^3_{(1)}x_{(12)}$ in $\Phi_{(11)}$.

(v) The first part follows from (75) and similar expressions for $\Phi_{(3)}$, $\Phi_{(4)}$. The second part follows from an observation similar to that used to prove (75). That $\Phi_{(2)}$ contains a term $x^3_{(1)}x_{(11)}x_{(12)}$ follows from an existence of a path

$$\overline{Ov_3(v_1+v_3)(v_1+v_4)v_4v_1v_{1,1}(v_1+v_2)v_{1,2}}\,,$$

and $\Phi_{(12)}$ contains a term $x^4_{(1)}x_{(2)}$ because of a path

$$(\overline{Ov_1v_{1,1}}\,,\overline{v_{1,2}(v_2+v_3)v_{1,3}(v_3+v_4)v_{1,4}}).$$

□

Theorem 31 *(i) $\Phi_{(1)}(x,y,0,0,0,0)=x$, and $\Phi_{(11)}(x,y,0,0,0,0)=y$ has a unique solution $\vec{x}_c=(x_c,y_c,0,0,0,0)$ in $\{(x,y,0,0,0,0)\in\mathbb{R}_+^{\mathcal{K}_{res}} \mid (x,y,0,0,0,0)\in\Xi_4\setminus\{\vec{0}\}\}$.*

$x_c=0.326490898\cdots$ and $y_c=0.027929572\cdots$ are positive. (In particular, $\mathcal{K}_{res}=\mathcal{K}_{\vec{x}_c}$.)

(ii) $\vec{x}_c$ is a self-avoiding fixed point, i.e., satisfies (FP1) – (FP4).

(iii) There exists a critical point of the restricted model $\beta_{c,res}$.

Proof. (i) Since (74) implies $x > x^2 + 3x^3 + 6x^4 + 6x^5$, it follows that $0 < x < \dfrac{3}{8}$.

Eliminating y from $\Phi_{(1)}(x, y, 0, 0, 0, 0) = x$ and $\Phi_{(11)}(x, y, 0, 0, 0, 0) = y$ (in a similar spirit with that of the proof in [6, Proposition 3.1], but with lengthy calculations), we obtain an algebraic equation $g(x) = 0$, where

$$\begin{aligned} g(x) \;=\; & -3162456 + 3162456x + 27935028x^2 + 82351390x^3 + 534340195x^4 - 22712313853x^5 \\ & - 22749190609x^6 + 173488539516x^7 + 520491536505x^8 + 159919155293x^9 \\ & - 1067593750255x^{10} - 3355567112768x^{11} - 7117707818273x^{12} - 8049744033921x^{13} \\ & + 3218074725393x^{14} + 29132597332920x^{15} + 58986824992938x^{16} + 74778447132144x^{17} \\ & + 70897214418552x^{18} + 55063893147408x^{19} + 36096140965140x^{20} \\ & + 19669482325692x^{21} + 7841354208804x^{22} + 1771680351168x^{23} + 149567809608x^{24}, \end{aligned}$$

and y is expressed by a rational function of x, say $h(x)$.

The signs of the coefficients imply (by an argument from elementary calculus similar to those in the proof of [6, Proposition 3.1]) that $g'(x)$ changes sign no more than 4 times on $x \geqq 0$ (i.e., $g'(x)$ has no more than 4 positive roots). Since, by explicit calculation, $g'(0) > 0$, $g'(1/5) < 0$, $g'(3/8) > 0$, $g'(3/7) < 0$, and $g'(+\infty) > 0$, $g'(x)$ changes sign 4 times on $x \geqq 0$, and 2 times on $0 \leqq x \leqq 3/8$. Also we have $g(0) < 0$, $g(1/5) < 0$, and $g(3/8) > 0$. In particular, we have one and only one solution $x = x_c$ of $g(x) = 0$ in $x \in (1/5, 3/8)$.

Again by explicit calculation, we have $g'(21/200) > 0$ and $g'(11/100) < 0$, therefore, the other point that $g'(x)$ changes sign in $0 \leqq x \leqq 3/8$ is in the interval $(21/200, 11/100)$. By explicit calculation $g'(x) \leqq g'(21/200) \leqq 9 \times 10^5$, $x \in [21/200, 11/100]$, and $g(21/200) \leqq -2 \times 10^6$. Therefore, if $x \in [21/200, 11/100]$, then $g(x) \leqq -2\times 10^6 + 4.5\times 10^3 < 0$. This implies $g(x) < 0$, $0 \leqq x \leqq 1/5$, hence x_c obtained above is the unique solution to $g(x) = 0$ on $x \geqq 0$. With $y_c = h(x_c)$, (x_c, y_c) is therefore the unique solution to $\Phi_{(1)}(x, y, 0, 0, 0, 0) = x$, and $\Phi_{(11)}(x, y, 0, 0, 0, 0) = y$ on $\mathbb{R}^2_+$. It is easy to see that $x_c = 0.326490898\cdots$ and $y_c = 0.027929572\cdots$, which prove that $(x_c, y_c, 0, 0, 0, 0) \in \Xi_4 \setminus \{\vec{0}\}$.

(ii) We prove each of (FP1) – (FP4) in turn.

(a) Proposition 30 (v) implies that if $I \notin \mathcal{K}_{res} = \{(1), (11)\}$ then $\Phi_I(\vec{x}_c) = 0$. Therefore $\vec{\Phi}(\vec{x}_c) = \vec{x}_c$.

(b) Proposition 30 (v) implies

$$\mathcal{J}_{(3)J}(\vec{x}_c) = \mathcal{J}_{(4)J}(\vec{x}_c) = 0, \quad J \in \mathcal{I}_d\,,$$

and

$$\mathcal{J}_{I(1)}(\vec{x}_c) = \mathcal{J}_{I(11)}(\vec{x}_c) = 0, \quad I \notin \mathcal{K}_{res}\,.$$

Therefore, $B = \mathcal{J}(\vec{x}_c)$ has a form

$$B = \begin{pmatrix} p & 3q & B_{13} & B_{14} & B_{15} & B_{16} \\ q & r & B_{23} & B_{24} & B_{25} & B_{26} \\ 0 & 0 & B_{33} & B_{34} & B_{35} & B_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{63} & B_{64} & B_{65} & B_{66} \end{pmatrix},$$

where, by (74),

$$\begin{aligned} p &= 2x_c + 9x_c^2 + 24x_c^3 + 30x_c^4 + 36x_c^2y_c + 120x_c^3y_c + 36x_cy_c^2 + 234x_c^2y_c^2 + 192x_cy_c^3 + 132y_c^4\,, \\ q &= 4x_c^3 + 10x_c^4 + 12x_c^2y_c + 52x_c^3y_c + 96x_c^2y_c^2 + 176x_cy_c^3 + 220y_c^4\,, \\ r &= 4x_c^3 + 13x_c^4 + 64x_c^3y_c + 264x_c^2y_c^2 + 88y_c^3 + 880x_cy_c^3 + 930y_c^4\,, \end{aligned}$$

and B_{ij}'s are non-negative, and (75) implies

$$\begin{aligned} B_{33} &= \mathcal{J}_{(2)(2)}(\vec{x}_c) = \Phi_{(1),1}(\vec{x}_c), \\ B_{36} &= \mathcal{J}_{(2)(12)}(\vec{x}_c) = \Phi_{(1),4}(\vec{x}_c), \\ B_{63} &= \mathcal{J}_{(12)(2)}(\vec{x}_c) = \Phi_{(11),1}(\vec{x}_c), \\ B_{66} &= \mathcal{J}_{(12)(12)}(\vec{x}_c) = \Phi_{(11),4}(\vec{x}_c). \end{aligned}$$

A 2×2 matrix $\begin{pmatrix} p & 3q \\ q & r \end{pmatrix}$ has eigenvalues $\lambda = 3.17282866849\cdots > 1$ and $\lambda_2 = 0.249393708\cdots$. In fact, they are the solutions of $x^2-(p+r)x+pr-3q^2=0$.

Consider a 2×2 matrix $B' = \begin{pmatrix} B_{33} & B_{36} \\ B_{63} & B_{66} \end{pmatrix}$. Its characteristic polynomial $f(x) = \det(x\,I - B)$ is

$$f(x) = x^2-(B_{33}+B_{66})x+(B_{33}B_{66}-B_{36}B_{63}) = (x-\frac{1}{2}(B_{33}+B_{66}))^2-\frac{1}{4}(B_{33}-B_{66})^2-B_{36}B_{63}\,. \tag{76}$$

Note that (75) implies

$$\begin{aligned} x_c &= \Phi_{(1)}(\vec{x}_c) = B_{33}\,x_c + B_{36}\,y_c + \Phi_{(1),0}(\vec{x}_c) \geqq B_{33}\,x_c + B_{36}\,y_c + x_c^2, \\ y_c &= \Phi_{(11)}(\vec{x}_c) = B_{63}\,x_c + B_{66}\,y_c + \Phi_{(11),0}(\vec{x}_c) \geqq B_{63}\,x_c + B_{66}\,y_c + x_c^4, \end{aligned}$$

which further imply

$$\begin{aligned} &0 \leqq x_a := \frac{1}{2}(B_{33}+B_{66}) \leqq \frac{1}{2}((1-x_c)+(1-\frac{x_c^4}{y_c})) < 1, \\ &1-B_{33}-B_{66}+B_{33}B_{66} = (1-B_{33})(1-B_{66}) \geqq (\frac{y_c}{x_c}B_{36}+x_c)(\frac{x_c}{y_c}B_{63}+\frac{x_c^4}{y_c}) > B_{36}B_{63}\,. \end{aligned}$$

Note also that Proposition 30 (v) implies that

$$B_{36} \geqq x_c^3 y_c \quad \text{and} \quad B_{63} \geqq x_c^4.$$

Using these estimates in (76) we find that $f(x)$ is increasing on $x \geqq x_a$, symmetric with respect to $x = x_a$ (hence in particular, $f(-1) > f(-1+2x_a) = f(1)$), $f(x_a) \leqq -B_{36}B_{63} \leqq -x_c^7 y_c < 0$, and $f(1) > 0$. Therefore the two eigenvalues λ_3 and λ_4 of B' are real, distinct, and have absolute values less than 1.

The other two eigenvalues are 0, for which we have 2 independent left eigen vectors $(0,0,0,1,0,0)$ and $(0,0,0,0,1,0)$.

Therefore, $B = \mathcal{J}(\vec{x}_c)$ is diagonalizable by an invertible matrix, whose eigenvalues are λ, λ_2, λ_3, λ_4, 0, 0, and the eigenvalue λ which is largest in absolute value satisfies $\lambda > 1$ and all the other eigenvalues have absolute values strictly less than 1.

Denote the left eigenvector of B for λ by $\vec{v}_L = (v_{L,(1)}, v_{L,(11)}, v_{L,(2)}, v_{L,(3)}, v_{L,(4)}, v_{L,(12)})$. It is defined by

$$\begin{aligned} \frac{v_{L,(11)}}{v_{L,(1)}} &= \frac{\lambda - p}{q} = \frac{3q}{\lambda - r}\,, \\ v_{L,(2)} &= \frac{1}{\lambda - B_{33}}(B_{13}v_{L,(1)} + B_{23}v_{L,(11)} + B_{63}v_{L,(12)}) \\ v_{L,(3)} &= \frac{1}{\lambda}(B_{14}v_{L,(1)} + B_{24}v_{L,(11)} + B_{34}v_{L,(2)} + B_{64}v_{L,(12)}), \\ v_{L,(4)} &= \frac{1}{\lambda}(B_{15}v_{L,(1)} + B_{25}v_{L,(11)} + B_{35}v_{L,(2)} + B_{65}v_{L,(12)}), \\ v_{L,(12)} &= \frac{1}{\lambda - B_{66}}(B_{16}v_{L,(1)} + B_{26}v_{L,(11)} + B_{36}v_{L,(2)}). \end{aligned}$$

Obviously, we can take $v_{L,(1)} = 1$. λ is the larger solution of $x^2-(p+r)x+pr-3q^2=0$, hence $\lambda > p$. Therefore, $v_{L,(11)} > 0$. Note that Proposition 30 (iv) implies

$$B_{i,j} > 0, \quad i = 1,2, \; j = 3,4,5,6\,.$$

Note also that $\lambda > 1 > \max\{B_{33}, B_{66}\}$. Therefore $v_{L,I} > 0$, $I = (2),(3),(4),(12)$.

Finally, note that the right eigenvector of B for λ can be chosen to be $\begin{pmatrix} 3q \\ \lambda - p \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, hence has positive (non-zero) (1)-component.

(c) The explicit form (74) shows that $\Phi_{(1)}$ and $\Phi_{(11)}$ contain terms $x_{(1)}^2$ and $x_{(1)}^4$, respectively.

(d) We have already shown $\vec{x}_c \in \Xi_4 \cap \mathbb{R}_+^{\mathcal{K}_{res}}$ and since $x_c > 0$, $\vec{x}_c \neq \vec{0}$.

(iii) Theorem 15 implies that there exists one and only one $\beta_0 \in \mathbb{R}$ such that $\vec{x}_{can,\mathcal{K}_{res}}(\beta_0) \in \partial D$. To prove that β_0 is the critical point of the restricted model, it is sufficient to prove, from (CS2) and (DA1) – (DA2), that

$$\lim_{n\to\infty} \vec{X}_n(\vec{x}_{can,\mathcal{K}_{res}}(\beta_0)) = \vec{x}_c\,, \tag{77}$$

and

$$x_{can,\mathcal{K}_{res},I}(\beta_0) \neq 0,\ I \in \mathcal{K}_{res}\,.$$

The latter obviously holds, because

$$\vec{x}_{can,\mathcal{K}_{res}}(\beta_0) = (e^{-\beta_0}, e^{-2\beta_0}, 0, 0, 0, 0).$$

Hence it suffices to prove (77).

Since $\mathbb{R}_+^{\mathcal{K}_{res}} \subset \mathbb{R}_+^{\mathcal{I}_4}$ is an invariant subset (Proposition 6 or Proposition 8), we may restrict (77) to $\mathbb{R}_+^{\mathcal{K}_{res}}$. Define

$$\begin{aligned} &x_n(x,y) = X_{n,(1)}(x,y,0,0,0,0), \quad y_n(x,y) = X_{n,(11)}(x,y,0,0,0,0),\\ &\vec{\phi} = (\phi_1,\phi_2); \quad \phi_1(x,y) = \Phi_{(1)}(x,y,0,0,0,0), \quad \phi_2(x,y) = \Phi_{(11)}(x,y,0,0,0,0). \end{aligned}$$

Then $(x_0(x,y), y_0(x,y)) = (x,y)$ and $(x_{n+1}, y_{n+1}) = \vec{\phi}\circ(x_n,y_n)$, $n \in \mathbb{Z}_+$.

Next define

$$\Xi_4^{(2)} = \{(x,y) \in \mathbb{R}_+^2 \mid (x,y,0,0,0,0) \in \Xi_4\} = \{(x,y) \in \mathbb{R}_+^2 \mid x^2 \geqq y\},$$

and

$$D^{(2)} = \{(x,y) \in \Xi_4^{(2)} \mid (x,y,0,0,0,0) \in D\},$$

where $D \subset \mathbb{R}_+^6$ is as in (31). Denote by $D^{(2)o}$, $D^{(2)c}$, $\partial D^{(2)}$, the interior, exterior, and boundary of $D^{(2)}$ in $\Xi_4^{(2)}$.

Note that

$$(e^{-\beta_0}, e^{-2\beta_0}) \in \partial D^{(2)}.$$

To prove (77), it then suffices to prove

$$\lim_{n\to\infty}(x_n(x,y), y_n(x,y)) = (x_c, y_c), \quad (x,y) \in \partial D^{(2)}. \tag{78}$$

The definition and Theorem 15 implies that these sets are invariant sets of (ϕ_1,ϕ_2) and that

$$\begin{aligned} D^{(2)} &= \{(x,y) \in \Xi_4^{(2)} \mid \sup_{n\in\mathbb{Z}_+} \max\{x_n(x,y), y_n(x,y)\} < \infty\}\\ &= \{(x,y) \in \Xi_4^{(2)} \mid \sup_{n\in\mathbb{Z}_+} \max\{x_n(x,y), y_n(x,y)\} \leqq 1\}, \end{aligned}$$

and

$$D^{(2)o} = \{(x,y) \in \Xi_4^{(2)} \mid \lim_{n\to\infty} \max\{x_n(x,y), y_n(x,y)\} = 0\}.$$

The explicit form (74) and Theorem 15 also imply, by an elementary argument (as in [6, Proposition 3.2 (2)]), that there exist a constant c, satisfying $0 < c < \dfrac{1}{2}$, and a continuous sitrictly decreasing function $p:\ [0,c] \to \mathbb{R}_+$, satisfying $p(0) < \dfrac{1}{2}$ and $p(c) = 0$, such that

$$\partial D^{(2)} = \{(x, p(x)) \mid x \in [0,c]\}. \tag{79}$$

The Jacobian of the map $(x,y) \to \vec{\phi}(x,y)$ is, by (74),

$$\begin{aligned} \det \mathcal{J}^{(2)}(x,y) &= \det\left(\frac{\partial(\phi_1,\phi_2)}{\partial(x,y)}(x,y)\right)\\ &= 8x^4 + 62x^5 + 165x^6 + 192x^7 + 90x^8 + 128x^4y + 432x^5y + 516x^6y + 360x^7y\\ &\quad + 528x^3y^2 + 2088x^4y^2 + 3996x^5y^2 + 4770x^6y^2 + 176xy^3 + 2552x^2y^3\\ &\quad + 10032x^3y^3 + 25200x^4y^3 + 35040x^5y^3 + 1860xy^4 + 11538x^2y^4 + 56640x^3y^4\\ &\quad + 113520x^4y^4 + 3168xy^5 + 69912x^2y^5 + 206640x^3y^5 + 50376xy^6\\ &\quad + 201780x^2y^6 + 11616y^7 + 62400xy^7 - 22440y^8. \end{aligned}$$

We note the following fact proved by an elementary argument in [6, Proposition 3.3].

Lemma 32 *If there exists $\epsilon > 0$ such that*

$$\mathcal{J}^{(2)}(x,y) > \epsilon,\ (x,y) \in \partial D^{(2)}, \tag{80}$$

then, $\phi_1(x_1,y_1) < \phi_1(x_2,y_2)$ holds for any $(x_1,y_1) \in \partial D^{(2)}$ and $(x_2,y_2) \in \partial D^{(2)}$ satisfying $x_1 < x_2$.

Remark. The proof of [6, Proposition 3.3] applies to the present case. The proof there (after [6, (3.7)]) uses only (79) and the fact that ϕ_1 and ϕ_2 are polynomials of positive coefficients. $\diamond$

To prove (80), note that the only negative term in $\mathcal{J}^{(2)}(x,y)$ is $-22440y^8$. If $(x,y) \in \partial D^{(2)}$, then $y = p(x) < p(0) < 1/2$, hence $22440y^8 < 11220y^7 < 11616y^7$. With other positive terms in $\mathcal{J}^{(2)}(x,y)$, (80) follows easily.

We are ready to prove (78). Let $(x,y) \in \partial D^{(2)}$, and assume that $\{(x_n(x,y), y_n(x,y)) \mid n \in \mathbb{Z}_+\}$ accumulates at $(x_a,y_a) \neq (x_c,y_c)$. Let $(x_b,y_b) = \vec{\phi}(x_a,y_a)$. Since Theorem 31 implies that (x_c,y_c) is the only fixed point in $\Xi_4^{(2)} \setminus \{\vec{0}\}$, it follows that $x_b \neq x_a$.

Assume that $x_b > x_a$. For $\epsilon > 0$ denote the ϵ neighborhood of (x,y) by $U_\epsilon(x,y)$. Since $\vec{\phi}$ is continuous, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\vec{\phi}(U_\delta(x_a,y_a)) \subset U_\epsilon(x_b,y_b)$. Taking ϵ small enough so that $(x',y') \in U_\epsilon(x_b,y_b)$ implies $x' > x_m := (x_a + x_b)/2$, we therefore see that there exists $\delta > 0$ such that $(x,y) \in U_\delta(x_a,y_a) \cap \partial D^{(2)}$ implies $\phi_1(x,y) > x_m$. Therefore if $\phi_1(x,y) > x$, then Lemma 32 implies $\phi_1(\vec{\phi}(x,y)) > \phi_1(x,y)$ $(> x_m)$, hence by induction,

$$\phi_1(\vec{\phi}^n(x,y)) > x_m\,,\quad n \in \mathbb{N},\ (x,y) \in U_\epsilon(x_a,y_a) \cap \partial D^{(2)}.$$

By assumption that $\{(x_n(x,y), y_n(x,y)) \mid n \in \mathbb{Z}_+\}$ accumulates at (x_a,y_a), there exists a positive integer N such that $(x_N(x,y), y_N(x,y)) \in U_\epsilon(x_a,y_a) \cap \partial D^{(2)}$. Therefore $x_n(x,y) > x_m$, $n = N+1, N+2, \cdots$, which contradicts the assumption that $\{x_n(x,y) \mid n \in \mathbb{Z}_+\}$ accumulates at $x_a < x_m$.

Similar argument also holds for if we assume $x_b < x_a$. Therefore (x_c,y_c) is the only point that $\{(x_n(x,y), y_n(x,y)) \mid n \in \mathbb{Z}_+\}$ accumulates, which proves (78).

$\square$

Theorem 31 and the results in Section 3.2 imply the following results on the asymptotic behaviors of restricted self-avoiding paths on the 4 dimensional pre-Sierpiński gasket.

Theorem 33 *Let $\vec{x}_c = (x_c, y_c, 0, 0, 0, 0) \in \mathbb{R}_+^6$ and $\beta_{c,res} = \beta_{c,\mathcal{K}_{res}} \in \mathbb{R}$ be the constants defined in Theorem 31. Then the following holds for the restricted self-avoiding paths on the 4 dimensional pre-Sierpiński gasket.*

(i) If $\vec{x} \in \mathcal{D}om(\vec{x}_c)$, then the following hold.

For $I \in \mathcal{K}_{res} = \{(1),(11)\}$, the joint distribution of scaled generalized path length $(\lambda^{-n}s_{(1)}\,,\ \lambda^{-n}s_{(11)})$ under $\mu_{\vec{x},n,I}$ converges weakly to a Borel probability measure $p^*_{\vec{x},I}$ on $\mathbb{R}^6$ as $n \to \infty$.

The generating function $\varphi^*_{\vec{x},I}$, defined by

$$\varphi^*_{\vec{x},I}(\vec{t}) = \int_0^\infty e^{\vec{t}\cdot\vec{\xi}} p^*_{\vec{x},I}[d\vec{\xi}], \quad \vec{t} \in \mathbb{C}^6\,,$$

is an entire function in $\vec{t}$, and the set of functions $(\varphi^*_{\vec{x},(1)}\,,\ \varphi^*_{\vec{x},(11)})$ are uniquely determined by (25) for $d = 4$.

If $\vec{x} \in \mathcal{D}om(\vec{x}_c) \cap \Xi_4$ and $I \in \mathcal{K}_{res}$, then the distribution of $\lambda^{-n}L(w)$, the scaled length of $w \in W_I^{(n)}$, under $\mu_{\vec{x},n,I}$ converges weakly to a Borel probability measure $\bar{p}^*_{\vec{x},I}$, which has a C^∞ density $\bar{\rho}^*_{\vec{x},I}$.

In particular, $\bar{\rho}^*_{\vec{x},(1)}(\xi) > 0$, $\xi > 0$.

(ii) For $I \in \mathcal{I}_4 = \{(1),(11),(2),(3),(4),(12)\}$ *,*

$$\lim_{n\to\infty} Z_{\mathcal{K}_{res},n,I}(\beta) = \begin{cases} 0, & \beta > \beta_{c,res}\,, \\ x_{c,I}\,, & \beta = \beta_{c,res}\,, \end{cases}$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{4} Z_{\mathcal{K}_{res},n,(i)}(\beta) = \infty, \quad \beta < \beta_{c,res}\,.$$

(iii) The number $N_{res}(k) = N_{\mathcal{K}_{res}}(k)$ *of restricted self-avoiding paths of length* k *starting from* 0 *satisfies*

$$\lim_{k\to\infty} \frac{1}{k} \log N_{res}(k) = \beta_{c,res}\,.$$

(iv) The exponent for mean square displacement for the restricted model is $d_w = \dfrac{\log\lambda}{\log 2} = 1.6657696\cdots$*, in the sense that*

$$\lim_{k\to\infty} \frac{1}{\log k} \log E_{res,k}[|w(k)|^{s\,d_w}] = s, \quad s \geqq 0,$$

where $E_{res,k}$ *is the expectation with respect to the probability measure with equal weight on length* k *restricted self-avoiding paths starting at* O*.*

Remark. The convergence of $\mu_{\vec{x},n,I}$ and the properties of the limit measure holds both for the full model and the restricted model (if $\vec{x} \in \mathcal{D}om(\vec{x}_c)$), because they hold independently of (CS2).

◇

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