Stochastic ranking process with space-time dependent intensities

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November 24, 2011

Abstract

We consider the stochastic ranking process with space-time dependent jump rates for the particles. The process is a simplified model of the time evolution of the rankings such as sales ranks at online bookstores. We prove that the joint empirical distribution of jump rate and scaled position converges almost surely to a deterministic distribution, and also the tagged particle processes converge almost surely, in the infinite particle limit. The limit distribution is characterized by a system of inviscid Burgers-like integral-partial differential equations with evaporation terms, and the limit process of a tagged particle is a motion along a characteristic curve of the differential equations except at its Poisson times of jumps to the origin.

2000 *Mathematics Subject Classification.* Primary 60K35; Secondary 35C05, 82C22.
Key words. Stochastic ranking process, Poisson process, hydrodynamic limit, inviscid Burgers equation, move-to-front rules.

*Partly supported by KAKENHI 21340020 (the Grant-in-Aid for Scientific Research (B)) from the Japan Society for the Promotion of Science.
†Research Fellow of the Japan Society for the Promotion of Science

1 Introduction.

In this paper, we consider stochastic ranking processes whose jump rates depend not only on time but also on their positions. Stochastic ranking processes are a model of a ranking system, such as the sales ranks found at online bookstores. We consider N particles each of which are exclusively located at $1,2,\ldots,N$. Each particle jumps to 1 according to its Poisson clock. When a jump of the particle at position i occurs, the particle moves to position 1 and the locations of the particles at $1,2,\ldots,i-1$ are sifted by $+1$. Particles whose Poisson clocks rang recently are at positions with small numbers, and the others are at positions with large numbers. We regard the number for each particle as the particle's rank. This system enables us to give ranks to N particles, and in this paper we call the time evolution of the particles given by this ranking system the *stochastic ranking process*.

A precise formulation of the stochastic ranking process which we consider in this paper are as follows. Let $(\Omega,\mathcal{F},P)$ be a probability space, and let $\{\nu_i(d\xi ds)\}_{i=1,2,3,\ldots}$ be independent Poisson random measures on $[0,\infty)\times[0,\infty)$ with the intensity measure $d\xi ds$. Let W be a set of non-negative valued C^1 functions

$$w:\ [0,1]\times[0,\infty)\to[0,\infty),$$

such that, for each $T>0$,

$$R_w(T) := \sup_{w\in W}\ \sup_{(y,t)\in[0,1]\times[0,T]} \max\left\{ w(y,t),\ \left|\frac{\partial w}{\partial y}(y,t)\right| \right\} < \infty. \tag{1}$$

Let w_i, $i=1,2,\ldots$ be a sequence in W, and for a positive integer N, put

$$w_i^{(N)}(k,t) := w_i(\frac{k-1}{N},t),\quad k=1,2,\ldots,N,\ t\in[0,\infty),\ i=1,2,\ldots,N.$$

Also, let $x_1^{(N)}$, $x_2^{(N)}$, …, $x_N^{(N)}$ be a rearrangement of $1,2,\ldots,N$. Define a process

$$X^{(N)} = (X_1^{(N)},\ldots,X_N^{(N)})$$

by

$$\begin{aligned} &X_i^{(N)}(t) \\ &= x_i^{(N)} + \sum_{j=1}^{N} \int_{s\in(0,t]} \int_{\xi\in[0,\infty)} \mathbf{1}_{X_i^{(N)}(s-)<X_j^{(N)}(s-)}\, \mathbf{1}_{\xi\in[0,w_j^{(N)}(X_j^{(N)}(s-),s))}\, \nu_j(d\xi ds) \\ &\quad + \int_{s\in(0,t]} \int_{\xi\in[0,\infty)} (1-X_i^{(N)}(s-))\, \mathbf{1}_{\xi\in[0,w_i^{(N)}(X_i^{(N)}(s-),s))}\, \nu_i(d\xi ds),\quad i=1,2,\ldots,N,\ t\geqq 0, \end{aligned} \tag{2}$$

where, $\mathbf{1}_B$ is the indicator function of event B. The integrands in the (2) are predictable, hence the right hand side of (2) is well-defined as the Ito–integrals [20, §IV.9].

$X^{(N)}(t)$ is a rearrangement of $1,2,\ldots,N$ for all $t\geqq 0$, which we regard as ranks or positions of particles $1,2,\ldots,N$ at time t. Moreover, for $i=1,2,\ldots,N$, and $t>t_0\geqq 0$, let

$$J_i^{(N)}(t_0,t) = \left\{ \int_{s\in(t_0,t]} \int_{\xi\in[0,\infty)} \mathbf{1}_{\xi\in[0,w_i^{(N)}(X_j^{(N)}(s-),s))}\, \nu_i(d\xi ds) > 0 \right\}. \tag{3}$$

Then, the last term on the right hand side of (2) implies that $J_i^{(N)}(t_0,t)$ denotes the event that the particle i jumps to the top position ($X_i^{(N)}(s)=1$) in the time interval $(t_0,t]$. Also, the second term on the right hand side of (2) implies that conditioned on the complement $J_i^{(N)}(t_0,t)^c$ of $J_i^{(N)}(t_0,t)$,

$$X_i^{(N)}(s) - X_i^{(N)}(s-) = 0 \ \text{ or } \ 1, \tag{4}$$

for $t_0 < s \leqq t$, where the latter occurs if and only if a particle k at tail side ($X_k^{(N)}(s) > X_i^{(N)}(s)$) jumps to top at time s.

With a great advance in the internet technologies, a new application of the process appeared. The ranking numbers such as those found in the web pages of online retails, e.g., the sales ranks of books at the Amazon online bookstore, are found to follow the predictions of the model [16, 17, 14, 13]. In the ranking of books, each time a book is sold its ranking spontaneously jumps to small numbers (relatively close to 1), regardless of how bad its previous position was (large $X_i^{(N)}(t-)$ in our notation), and regardless of how unpopular (small $w_i^{(N)}$, in our notation) the book is. The stochastic process we consider here corresponds to a mathematical simplification of this observation, that each time a book is sold its ranking jumps to 1 instantaneously. With a view that the process is a model of such on-line, real time, rankings of a large number of items according to their popularity, we will call the model the stochastic ranking processes.

At first thought one might guess that such a naive ranking rules of spontaneous jump to 1 at each sale, as in the definition of the stochastic ranking processes, will not be a good index for the popularity of books. But with a closer look, one notices that the well sold books are dominant near the top position, while books near the tail position are rarely sold. Though the rankings of each book are stochastic with sudden jumps, the spatial distribution of the jump rates is more stable. In the bookstore's view, what matters is not a specific book, but the totality of sales. This motivates an interest on the evolution of the joint empirical distribution of position and jump rates.

In [15, 16, 18, 28], infinite particle (large N) scaling limit for this model is considered, and the explicit formula of the limit distribution of the joint empirical distribution of scaled position and the jump rate is found, which further is characterized as a solution to a system of inviscid Burgers-like equations with a term representing evaporation. The limit formula is successfully applied to the time developments of ranking numbers such as those found in the web pages of online bookstores [16, 17, 14]. Furthermore, convergence of the joint empirical distribution as a process and convergence of tagged particle process are proved in [28].

If the model (2) is independent of spatial position, i.e., if $w^{(N)}$'s are independent of their first variables x, then the law of the process (2) reduces to that of [18, eq. (2)] and [28, eq. (1)], the stochastic ranking process with time dependent (but position independent) intensities. Thus (2) is an extension of [18, 28] to the case where the dynamics is dependent on the value of $X_i^{(N)}(t)$, i.e., to the position dependent case. In the present paper, we mathematically extend the previous results to the case where the jump rates $w^{(N)}$'s are both position and time dependent.

If $w^{(N)}$'s are positive constants, (2) further reduces to the homogeneous case considered in [15, 16]. A discrete time version of the homogeneous case has been known since [32], and has been extensively studied since then and is called move-to-front (MTF) rules [27, 19, 7, 26, 23]. The process and its generalization have, in particular, been extensively studied as a model of least-recently-used (LRU) caching in the field of information theory [29, 9, 4, 8, 5, 30, 10, 12, 11, 21, 31, 22, 1], and also is noted as a time-reversed process of top-to-random shuffling.

A motivation for an on-line web retail store to provide the sales ranks, in their web pages for public access, would be to give information on the popularity of each products which the store provides, to attract consumers' attention on popular products. Extending previous results to the case of position dependent jump rates, which is the main aim of the present paper, corresponds to providing a mathematical framework for considering a possibility of such expected effect of popular products receiving extra attention and effectively increase their jump rates according to their rankings.

We introduce the normalized position for each particle i at time t

$$Y_i^{(N)}(t) = \frac{1}{N}(X_i^{(N)}(t) - 1), \tag{5}$$

and consider the joint empirical distribution of jump rate and normalized position, given by

$$\mu_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{(w_i, Y_i^{(N)}(t))}\,. \tag{6}$$

(We will denote a unit measure on any space by δ_c.) $\mu_\cdot^{(N)}$ is a stochastic process taking values in the set of Borel probability measures. For each $T > 0$, $\mu_t^{(N)}$, $t \in [0, T]$, is regarded as a stochastic process on $C^{1,0}([0,1]\times[0,T])\times[0,1]$, where $C^{1,0}([0,1]\times[0,T])$ is the total set of functions $f \in C([0,1]\times[0,T])$ such that $\dfrac{\partial f}{\partial y}(y,t) \in C([0,1] \times [0,T])$. $C^{1,0}([0,1] \times [0,T])$ is a Polish space (complete separable metric space) with norm

$$\sup_{(y,t)\in[0,1]\times[0,T]} \left\{ |w(y,t)|,\ \left|\frac{\partial w}{\partial y}(y,t)\right| \right\}.$$

Since $C^1([0,1]\times[0,T])$ is a Polish space, so is $C^1([0,1]\times[0,T])\times[0,1]$ [2, Example 26.2]. We assume a standard topology of weak convergence of probability measures on $C^1([0,1] \times [0,T]) \times [0,1]$.

To prove convergence of measures, we work with a distribution function. For each integer N define

$$U^{(N)}(dw, y, t) = \mu_t^{(N)}(dw \times [y, 1)) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{X_i^{(N)}(t-)\geqq Ny+1}\ \delta_{w_i}(dw)\,, 0 \leqq y \leqq 1,\ t \geqq 0. \tag{7}$$

For each (y,t), $U^{(N)}(\cdot, y, t)$ is a Borel measure on W. Note that $U^{(N)}(dw, y, t)$ is non-increasing in y and satisfies

$$\int_W U^{(N)}(dw, y, t) = \frac{[N\,(1-y)]}{N}\,,\quad 0 \leqq y \leqq 1,\ t \geqq 0. \tag{8}$$

where, for real z, $[z]$ is the largest integer not exceding z.

As an extension of the corresponding results in [16, 18], the infinite particle scaling limit U of $U^{(N)}$ turns out to be characterized by a system of inviscid Burgers-like integral–partial differential equations with evaporation terms. Denote the set of 'boundary points' and of 'initial points' by

$$\Gamma_b = \{(0, t_0) \mid t_0 \geqq 0\}, \tag{9}$$

and

$$\Gamma_i = \{(y_0, 0) \mid 0 \leqq y_0 \leqq 1\}, \tag{10}$$

respectively, and put

$$\Gamma = \Gamma_b \cup \Gamma_i\,. \tag{11}$$

Also, for $t \geqq 0$ put

$$\Gamma_t = \{(y_0, t_0) \in \Gamma \mid t_0 \leqq t\} = \Gamma_i \cup \{(0, t_0) \mid 0 \leqq t_0 \leqq t\}. \tag{12}$$

Theorem 1 *Let λ be a Borel probability measure on W, and $\rho:\ W \times [0,1] \to [0,1]$ be a non-negative Borel measurable function such that $\frac{\partial \rho}{\partial y}(w,y)$ exists and continuous, and*

$$\frac{\partial \rho}{\partial y}(w,y) \leqq 0,\ (w,y) \in W \times [0,1], \tag{13}$$

and such that $\rho(w,0) = 1$ and $\rho(w,1) = 0$, $w \in W$. Define a Borel measure on W parametrized by $y \in [0,1]$, by

$$U_0(dw,y) = \rho(w,y)\,\lambda(dw), \quad y \in [0,1],\ w \in W. \tag{14}$$

In particular, $U_0(dw,0) = \lambda(dw)$. Assume also

$$U_0(W,y) = \int_W U_0(dw,y) = 1 - y,\ 0 \leqq y \leqq 1. \tag{15}$$

Then there exists a unique pair of functions

$$y_C:\ \{(\gamma,t) \in \Gamma \times [0,\infty) \mid \gamma \in \Gamma_t\} \to [0,1],$$

and $U = U(dw,y,t)$ on $[0,1] \times [0,\infty)$ taking values in the non-negative Borel measures on W, such that,

- *(i) $y_C(\gamma,t)$ and $\frac{\partial y_C}{\partial t}(\gamma,t)$ is continuous*
- *(ii) for each $t > 0$, $y_C(\cdot,t):\ \Gamma_t \to [0,1]$ is surjective,*
- *(iii) for all bounded continuous $h:\ W \to \mathbb{R}$, $U(h,y,t) := \int_W h(w)U(dw,y,t)$ is Lipschitz continuous in $(y,t) \in [0,1] \times [0,T]$ for any $T > 0$, and non-increasing in y, and*
- *(iv) the following (16), (17), (19), and (20) hold:*

$$y_C(\gamma,t_0) = y_0 \quad \text{and} \quad U(dw,y_0,t_0) = U_0(dw,y_0), \quad \gamma = (y_0,t_0) \in \Gamma, \tag{16}$$

$$U(h,y_C(\gamma,t),t) = U_0(h,y_0) - \int_{t_0}^{t} V(h,y_C(\gamma,s),s)\,ds, \quad t \geqq t_0,\ \gamma = (y_0,t_0) \in \Gamma, \tag{17}$$

for all bounded continuous function $h:\ W \to \mathbb{R}$, where, $U(h,y,t) := \int_W h(w)U(dw,y,t)$, and

$$V(h,y,t) = \int_W h(w)\,w(y,t)\,U(dw,y,t) + \int_y^1 \int_W h(w)\,\frac{\partial w}{\partial z}(z,t)\,U(dw,z,t)\,dz, \tag{18}$$

and

$$\frac{\partial y_C}{\partial t}(\gamma,t) = V(\mathbf{1}_W,y_C(\gamma,t),t), \quad t \geqq t_0,\ \gamma = (y_0,t_0) \in \Gamma, \tag{19}$$

where $\mathbf{1}_W(w) = 1$ for all $w \in W$, and

$$U(\mathbf{1}_W,y,t) = 1 - y, \quad 0 \leqq y \leqq 1,\ t \geqq 0. \tag{20}$$

◇

The claim (20), together with continuity and monotonicity of U, implies that U determines a Borel probability measure μ_t on the direct product $W \times [0,1]$ parametrized by t:

$$U(dw, y, t) = \mu_t(dw \times [y, 1)), \quad 0 \leqq y \leqq 1, \ t \geqq 0. \tag{21}$$

If $U(h, y, t)$ in Theorem 1 is C^1 in a neighborhood of $(y, t) \in (0,1) \times (0, \infty)$, then differentiating (17) by t and using (19), and noting that $y_C(\cdot, t) : \ \Gamma_t \to [0,1]$ is surjective, we have

$$\frac{\partial U}{\partial t}(h, y, t) + V(\mathbf{1}_W, y, t) \, \frac{\partial U}{\partial y}(h, y, t) = -V(h, y, t), \tag{22}$$

where V is as in (18). y_C in (19) determines the characteristic curves for (22). In terms of [6, §3.4], we can therefore say that Theorem 1 claims global existence of the Lipschitz solution (broad solution which is Lipschitz continuous) to the system of quasilinear partial differential equations (22), with components parametrized by (possibly continuous) parameter w. To be more precise, we have extended the definition in [6, §3.4] of Lipschitz solution for (22) to the non-local case (see (18)), and for the case where $V(\mathbf{1}_W, y, t)$ in the left-hand side of Theorem 1 is common for all h. We have also generalized the notion of domain of determinancy defined in [6, §3.4], which in the present case corresponds to

$$\{(y, t) \in [0,1] \times [0, \infty) \mid y \geqq y_C((0,0), t)\},$$

to the domain determined by boundary conditions

$$\{(y, t) \in [0,1] \times [0, \infty) \mid y < y_C((0,0), t)\},$$

with initial data $U(h, \cdot, 0) = U_0(h, \cdot)$ and the boundary condition $U(h, 0, t) = U_0(h, 0)$, $t \geqq 0$, as obtained in (16).

As a simple example, where the jump rates are finitely many space-time constants, there is a natural one to one onto map from W (the space of jump rate functions), to a finite set $\{w_1, w_2, \ldots, w_A\}$ of positive integers for some positive integer A, and the distribution of the jump rates $U_0(dw, 0) = \lambda(dw)$ can be identified with

$$\lambda(dw) = \sum_{a=1}^{A} r_a \, \delta_{w_a},$$

for some positive constants r_a, $a = 1, 2, \ldots, A$, satisfying $\displaystyle\sum_{a=1}^{A} r_a = 1$. In this simple example, (22) reduces to

$$\frac{\partial U_a}{\partial t}(y, t) + \sum_{b=1}^{A} w_b U_b(y, t) \, \frac{\partial U_a}{\partial y}(y, t) = -w_a U_a(y, t), \tag{23}$$

where we wrote $U_a(y, t) = U(h, y, t)$ and $V_a(y, t) = V(h, y, t) = w_a U_a(y, t)$, so that $V(\mathbf{1}_W, y, t) = \displaystyle\sum_{b=1}^{A} w_b U_b(y, t)$. If the right hand side of (23) is 0, the partial differential equation is known (for $A = 1$) as the inviscid Burgers equation in the terminology of fluid dynamics. In terms of fluid dynamics, the right hand side of (23) could be interpreted as the evaporation of the fluid.

For the case (22) which we consider in this paper, non-locality of interaction is inevitable, precisely because of the position dependence of the jump rate functions, hence we need to consider a harder problem of a system of differential–integral equations compared to previous cases [15, 18].

Now we give a norm of measures in order to state the next theorem. Let $||\cdot||_{\rm var}$ be the total variation norm for Borel measures on W, i.e. for a signed measure μ on W define $||\mu||_{\rm var}$ by

$$||\mu||_{\rm var} = \mu^+(W) + \mu^-(W),$$

where μ^+ and μ^- are the positive part and the negative part obtained by Hahn-Jordan decomposition of μ respectively.

We consider a scaling limit of the stochastic ranking process as $N \to \infty$ (the limit for the number of particles to infinity). We are naturally considering a law of large number type of results, and as suggested by the fact, which we state in the following, that the limit distribution satisfies (17), or more intuitively, non-linear equations (22), it is a non-trivial problem of the law of large numbers for dependent variables. This has been the case also for previous results in [15, 18], but in the previous studies, where the jump rate functions are independent of spatial positions, a special combination of quantities we define $(U^{(N)}(B, Y_C^{(N)}(y_0, t_0, t), t)$, in terms of notations in Section 3) turns out to be a sum of independent random variables.

However, the position-dependence of jump rates, as considered in the present paper, implies that the dependence of random variables are built-in in the model, so that the proofs in [15, 18] do not work in the present case. Inspired partly by [28], where the case of finite types of position independent particles are proved [28, Prop. 1.1 and Thm. 1.2], we extend his result to our position-dependent case, and obtain a convergence of empirical distribution and also the limiting dynamics of fixed finite particles (tagged particles) for the case of jump rate functions with space-time dependence as follows.

Theorem 2 *Assume that with probability* 1,

$$\lim_{N\to\infty} \sup_{y\in[0,1)} ||U^{(N)}(\cdot, y, 0) - U_0(\cdot, y)||_{\rm var} = 0, \tag{24}$$

where $U_0(dw, y)$ satisfies all the assumptions in Theorem 1. Then the following hold.

(i) With probability 1*, for all $T > 0$, $\lim_{N\to\infty} U^{(N)}(dw, y, t) = U(dw, y, t)$, uniformly in $y \in [0, 1)$ and $t \in [0, T]$, where U is the solution claimed in Theorem 1.*

(ii) Assume in addition that,

$$\lim_{N\to\infty} \frac{1}{N} x_i^{(N)} = y_i\,, \quad i = 1, 2, \ldots, L, \tag{25}$$

for a positive integer L and $y_i \in [0, 1)$, $i = 1, 2, \ldots, L$. Then, with probability 1*, for all $T > 0$, the tagged particle system*

$$(Y_1^{(N)}(t), Y_2^{(N)}(t), \ldots, Y_L^{(N)}(t))$$

converges as $N \to \infty$, uniformly in $t \in [0, T]$ to a limit $(Y_1(t), Y_2(t), \ldots, Y_L(t))$. Here, for each $i = 1, 2, \ldots, L$, Y_i is the unique solution to

$$\begin{aligned} &Y_i(t) \\ &= y_i + \int_0^t V(\mathbf{1}_W, Y_i(s-), s) ds - \int_{s\in(0,t]} \int_{\xi\in[0,\infty)} Y_i(s-)\, \mathbf{1}_{\xi\in[0, w_i(Y_i(s-),s))}\, \nu_i(d\xi ds), \end{aligned} \tag{26}$$

where, V is as in (18).

$\diamond$

When $\{w_i; i = 1, 2, 3, \ldots\}$ is a finite set of W, because of Proposition 11 in Appendix, we obtain the following corollary easily.

Corollary 3 *When $w_i \in \{\tilde{w}_\alpha \in W; \alpha = 1, 2, \ldots, A\}$ for $i = 1, 2, 3, \ldots$, the assumption (24) of Theorem 2 is relaxed as follows:*

$$\lim_{N\to\infty} U^{(N)}(\{\tilde{w}_\alpha\}, y, 0) = U(\{\tilde{w}_\alpha\}, y, 0), \quad \textit{for each} \ \ y \in [0, 1)$$

with probability 1 *for $\alpha = 1, 2, \ldots, A$.*

A discrete correspondence $Y_C^{(N)}$ of the characteristic curves y_C is defined in (61) in Section 3, both of which have been key quantities since [15]. As pointed in [28], (26) says that a particle moves along a characteristic curve of (22) except at its Poisson times of jumps to $y = 0$.

The plan of the paper is as follows. In Section 2 we prove Theorem 1, and in Section 3 we prove Theorem 2.

Acknowledgment. The authors would like to thank Prof. Y. Nagahata for discussions. T.H. also would like to thank Prof. M. Hino, Prof. I. Shigekawa, Prof. S. Takesue, Prof. K. Yano, Prof. Y. Yano, and Prof. N. Yoshida, for their interest and discussions on the present work, and also for their hospitality at Kyoto University.

2 Proof of Theorem 1.

Consider first the case $(y_0, t_0) \in \Gamma_i$, namely, the case $t_0 = 0$.

Lemma 4 *There exists a unique C^1 function $f: \ [0, 1] \times [0, \infty) \to [0, 1]$ which satisfies*

$$(27)\ f(y,t) = 1 + \int_W \left(\int_y^1 \frac{\partial \rho}{\partial z}(w, z) \ \exp(-\int_0^t w(f(z, s), s)\, ds)\, dz \right) U_0(dw, 0), \ \ y \in [0, 1], \ \ t \geqq 0,$$

where ρ and U_0 are as in the assumptions of Theorem 1. ◇

Proof. For $k \in \mathbb{Z}_+$, define $f_k: \ [0, 1] \times [0, \infty) \to [0, 1]$ inductively by

$$f_0(y, t) = 1 + \int_W \left(\int_y^1 \frac{\partial \rho}{\partial z}(w, z) \ \exp(-\int_0^t w(z, s)\, ds)\, dz \right) U_0(dw, 0),$$

and

$$(28) \quad f_{k+1}(y, t) = 1 + \int_W \left(\int_y^1 \frac{\partial \rho}{\partial z}(w, z) \ \exp(-\int_0^t w(f_k(z, s), s)\, ds)\, dz \right) U_0(dw, 0), \quad k \in \mathbb{Z}_+.$$

Assume that f_k is continuous and takes values in $[0, 1]$. Then (28) is well-defined. Non-increasing assumption of Theorem 1 for ρ implies $\dfrac{\partial \rho}{\partial z}(w, z) \leqq 0$, hence (28) implies $f_{k+1} \leqq 1$. Similarly, using also (14) and (15),

$$\begin{aligned} &f_{k+1}(y, t) \\ &\geqq 1 + \int_W \left(\int_y^1 \frac{\partial \rho}{\partial z}(w, z)\, dz \right) U_0(dw, 0) = 1 + \int_W U_0(dw, 1) - \int_W U_0(dw, y) = y \\ &\geqq 0. \end{aligned}$$

ρ and w are C^1 in z, by assumption of Theorem 1, hence (28) implies that f_{k+1} is continuous. By induction, f_k is continuous and takes values in $[0,1]$, for all k.

For $k \in \mathbb{Z}_+$, put $F_k(y,t) = |f_{k+1}(y,t) - f_k(y,t)|$. Then, using (1) and the assumptions of Theorem 1 as above, we have

$$F_{k+1}(y,t) \leqq R_w(T) \int_y^1 \int_0^t F_k(z,s)\, ds\, dz, \quad y \in [0,1],\ t \in [0,T],\ k \in \mathbb{Z}_+, \tag{29}$$

for any $T > 0$. Since all f_k's are continuous and take values in $[0,1]$, F_k, $k = 1,2,\ldots$, are also continuous and take values in $[0,1]$. Then it holds by the argument of [6, §3.8, Lemma 3.4], that

$$0 \leqq F_k(y,t) \leqq e^{2R_w(T)t} 2^{-k}, \quad y \in [0,1],\ t \in [0,T],\ k \in \mathbb{Z}_+. \tag{30}$$

In fact, since F_0 takes values in $[0,1]$, (30) holds for $k = 0$. Assume (30) holds for some k. Then (29) implies

$$F_{k+1}(y,t) \leqq 2^{-k} R_w(T) \int_y^1 \int_0^t e^{2R_w(T)s}\, ds \leqq e^{2R_w(T)t} 2^{-k-1}, \quad 0 \leqq y < 1,\ 0 \in [0,T].$$

By induction, (30) holds for all $k \in \mathbb{Z}_+$. In particular, $f_0(y,t) + \sum_{k=0}^{\infty} F_k(y,t)$ converges uniformly in (y,t) for any bounded range of t. Hence, $f_k(y,t) = f_0(y,t) + \sum_{j=0}^{k-1} (f_{j+1}(y,t) - f_j(y,t))$ converges as $k \to \infty$ to a function, continuous in y and t. Let

$$f(y,t) = \lim_{k\to\infty} f_k(y,t), \quad y \in [0,1],\ t \geqq 0.$$

Then (28) implies that f satisfies (27). Also, $0 \leqq f_k \leqq 1$ implies

$$0 \leqq f(y,t) \leqq 1,\ 0 \leqq y \leqq 1,\ t \geqq 0. \tag{31}$$

The right hand side of (27), with the assumptions in Theorem 1 implies that $f(y,t)$ is C^1.

Next, we prove the uniqueness. Suppose for $i = 1,2$, $f^{(i)} : [0,1] \times [0,\infty) \to [0,1]$ are continuous functions which satisfy (27). Then $|f^{(1)}(y,0) - f^{(2)}(y,0)| = 0$ and, as above, for each $T > 0$,

$$|f^{(1)}(y,t) - f^{(2)}(y,t)| \leqq R_w(T) \int_y^1 \int_0^t |f^{(1)}(z,s) - f^{(2)}(z,s)|\, ds\, dz \quad y \in [0,1],\ t \in [0,T],$$

which implies $f^{(1)} = f^{(2)}$. □

Next, consider the case $(y_0,t_0) \in \Gamma_b$, namely, the case $y_0 = 0$.

Lemma 5 *For each continuous function* $\tilde g : \{(s,t) \in [0,\infty)^2 \mid 0 \leqq s \leqq t\} \to [0,1]$, *there exists a unique non-negative function* $\eta : W \times [0,\infty) \to [0,\infty)$, *integrable with respect to* $U_0(dw,0)$, *continuous in the second variable, which satisfy, for each* $w \in W$,

$$\begin{aligned} \eta(w,t) = &\int_0^t \eta(w,u)\, w(\tilde g(u,t),t)\, \exp(-\int_u^t w(\tilde g(u,v),v)\, dv)\, du \\ &- \int_0^1 \frac{\partial \rho}{\partial z}(w,z)\, w(f(z,t),t)\, \exp(-\int_0^t w(f(z,v),v)\, dv)\, dz, \quad t \geqq 0, \end{aligned} \tag{32}$$

where ρ is as in the assumption of Theorem 1, and f is the function given by Lemma 4.

Moreover, it holds that

$$\begin{aligned}
&\int_0^t \eta(w,u)\,\exp(-\int_u^t w(\tilde g(u,v),v)\,dv)\,du \\
&= 1+\int_0^1 \frac{\partial\rho}{\partial z}(w,z)\exp(-\int_0^t w(f(z,s),s)\,ds)\,dz.
\end{aligned} \tag{33}$$

In particular, for any $T>0$, there exists $C(T)>0$, which is independent of $\tilde g$, such that

$$0 \leqq \int_W \eta(w,t)\,U_0(dw,0) \leqq C(T),\ \ 0\leqq t\leqq T. \tag{34}$$

◇

Proof. Define a sequence of continuous functions $\eta_k:\ W\times[0,\infty)\to[0,\infty)$, $k=0,1,2,\ldots$, inductively, by

$$\eta_0(w,t)=0,\quad w\in W,\ t\geqq 0,$$

and

$$\begin{aligned}
\eta_{k+1}(w,t) = &\int_0^t \eta_k(w,u)\,w(\tilde g(u,t),t)\,\exp(-\int_u^t w(\tilde g(u,v),v)\,dv)\,du \\
&-\int_0^1 \frac{\partial\rho}{\partial z}(w,z)\,w(f(z,t),t)\,\exp(-\int_0^t w(f(z,v),v)\,dv)\,dz.
\end{aligned} \tag{35}$$

For $k\in\mathbb{Z}_+$ put $H_k(t)=\displaystyle\int_W |\eta_{k+1}(w,t)-\eta_k(w,t)|\,U_0(dw,0)$. Non-negativity of $w\in W$ and (1) imply

$$H_{k+1}(t)\leqq R_w(T)\int_0^t H_k(u)\,du,\ \ 0\leqq t\leqq T.$$

[6, §3.8, Lemma 3.4] implies that there exists a positive constant $C(T)$ such that

$$H_k(t)\leqq C(T)\,2^{-k},\ \ t\in[0,T],\ \ k\in\mathbb{Z}_+,$$

hence, as in the proof of Lemma 4, $\eta(w,t)=\lim_{k\to\infty}\eta_k(w,t)$ exists, is continuous, non-negative, and satisfies (32). Integrability inductively follows from (35) by

$$\begin{aligned}
&\sup_{t\in[0,T]}\int_W \eta_{k+1}(w,t)\,U_0(dw,0) \\
&\leqq R_w(T)\sup_{t\in[0,T]}\int_W\int_0^t \eta_k(w,u)\,du\,U_0(dw,0) + R_w(T)\int_W\int_0^1\left(-\frac{\partial\rho}{\partial z}(w,z)\right)dz\,U_0(dw,0) \\
&= R_w(T)\sup_{t\in[0,T]}\int_W\int_0^t \eta_k(w,u)\,du\,U_0(dw,0) + R_w(T),
\end{aligned}$$

where we also used (13), (14) and (15).

Next, we prove the uniqueness. Suppose for $i=1,2$, $\eta^{(i)}:\ W\times[0,\infty)\to[0,\infty)$ are functions, continuous in the second variable and satisfy (32). Then $|\eta^{(1)}(w,0)-\eta^{(2)}(w,0)|=0$ and, as above, for each $T>0$,

$$|\eta^{(1)}(w,t)-\eta^{(2)}(w,t)|\leqq R_w(T)\int_0^t |\eta^{(1)}(w,s)-\eta^{(2)}(w,s)|\,ds\ \ t\in[0,T],$$

which implies $\eta^{(1)} = \eta^{(2)}$.

Changing the variable t in (32) to s, and then integrating from 0 to t, and changing the order of integration in the first term on the right hand side, we have

$$\begin{aligned}
&\int_0^t \eta(w,s)\,ds \\
&\quad = -\int_0^t \eta(w,u)\left(\int_u^t \frac{\partial}{\partial s}\exp(-\int_u^s w(\tilde{g}(u,v),v)\,dv)ds\right)du \\
&\qquad + \int_0^1 \frac{\partial \rho}{\partial z}(w,z)\left(\int_0^t \frac{\partial}{\partial s}\exp(-\int_0^s w(f(z,v),v)\,dv)ds\right)dz \\
&\quad = \int_0^t \eta(w,u)\left(1-\exp(-\int_u^t w(\tilde{g}(u,v),v)\,dv)\right)du \\
&\qquad - \int_0^1 \frac{\partial \rho}{\partial z}(w,z)\left(1-\exp(-\int_0^t w(f(z,v),v)\,dv)\right)dz,
\end{aligned}$$

which, with $\rho(w,0) = 1$ and $\rho(w,1) = 0$, proves (33).

Combining (32) and (1), together with $\dfrac{\partial \rho}{\partial z}(w,z) \leqq 0$, $\rho(w,0) = 1$ and $\rho(w,1) = 0$, we see that

$$\int_W \eta(w,t)\,U_0(dw,0) \leqq R_w(T)\int_0^t \int_W \eta(w,u)\,U_0(dw,0)\,du + R_w(T).$$

[6, §3.8, Lemma 3.4] again implies that there exists $C(T) > 0$, independent of $\tilde{g}$, such that $\int_W \eta(w,t)\,U_0(dw,0) \leqq C(T)$, $0 \leqq t \leqq T$. □

Corollary 6 *For $i = 1,2$, let η_i be η in Lemma 5 with g_i in place of $\tilde{g}$, respectively. Then, for each $T > 0$ there exists a positive constant $C(T)$ such that*

$$\int_W |\eta_1(w,t) - \eta_2(w,t)|\,U_0(dw,0) \leqq C(T)\int_0^t \sup_{v\in[u,T]} |g_1(u,v) - g_2(u,v)|\,du. \tag{36}$$

◇

Proof. Put

$$\Delta\eta(t) = \int_W |\eta_1(w,t) - \eta_2(w,t)|\,U_0(dw,0)$$

and

$$\Delta g(u) = \sup_{v\in[u,T]} |g_1(u,v) - g_2(u,v)|.$$

Lemma 5, in particular, (32), (34), and (1), implies that

$$\Delta\eta(t) \leqq C_1(T)\int_0^t \Delta\eta(u)\,du + C_2(T)\int_0^t \Delta g(u)\,du,\ \ t\in[0,T],$$

for each T and for positive constants $C_i(T)$, $i = 1,2$. Hence

$$\Delta\eta(t) \leqq C_2(T)\int_0^t e^{C_1(T)(t-s)}\Delta g(s)\,ds \leqq C_2(T)e^{TC_1(T)}\int_0^t \Delta g(s)\,ds,$$

which implies (36). □

Lemma 7 *There exists a unique* C^1 *function* $g:\ \{(s,t)\in[0,\infty)^2 \mid 0 \leqq s \leqq t\} \to [0,1]$ *such that*

$$(37)\qquad \begin{aligned} g(s,t) = 1 &+ \int_W \int_0^1 \frac{\partial \rho}{\partial z}(w,z)\exp(-\int_0^t w(f(z,u),u)\,du)\,dz\,U_0(dw,0) \\ &- \int_W \int_0^s \eta(w,u)\exp(-\int_u^t w(g(u,v),v)\,dv)\,du\,U_0(dw,0), \quad 0 \leqq s \leqq t. \end{aligned}$$

Here, $f(s,t)$ *is defined in (27) and* η *is the function given by Lemma 5 with* g *in place of* $\tilde g$. ◇

Proof. For $k\in\mathbb{Z}_+$, define a sequence of functions, g_k and η_k, inductively by $g_0(s,t)=1$, $0\leqq s\leqq t$, and, for $k\in\mathbb{Z}_+$, η_k the function η in Lemma 5 with g_k in place of $\tilde g$, and

$$(38)\qquad \begin{aligned} g_{k+1}(s,t) = 1 &+ \int_W \int_0^1 \frac{\partial \rho}{\partial z}(w,z)\exp(-\int_0^t (f(z,u),u)\,du)\,dz\,U_0(dw,0) \\ &- \int_W \int_0^s \eta_k(u)\exp(-\int_u^t w(g_k(u,v),v)\,dv)\,du\,U_0(dw,0), \quad 0 \leqq s \leqq t. \end{aligned}$$

Note that (13) and $\eta_k(w,z)\geqq 0$ implies $g_k(s,t)\leqq 1$, and that (33) and (15), with $\eta_k(w,z)\geqq 0$ imply

$$1-g_k(s,t) \leqq \int_W \rho(w,0)\,U_0(dw,0) = 1,$$

hence, $0\leqq g_k(s,t)\leqq 1$, implying that η_k is well-defined.

Put $\Delta g_k = |g_{k+1}-g_k|$ and $\Delta\eta_k = |\eta_{k+1}-\eta_k|$. Repeating the arguments of Lemma 4 or Lemma 5, we see that (38) implies, with (34),

$$\Delta g_{k+1}(s,t) \leqq \int_W \int_0^s \Delta\eta_k(w,u)\,du\,U_0(dw,0) + C_1(T)\int_0^s \left(\int_u^t \Delta g_k(u,v)\,dv\right)du,$$

for $0\leqq s\leqq t\leqq T$, where $C_1(T)$ is a positive constant. Putting $G_k(s) = \sup_{t\in[s,T]} \Delta g_k(s,t)$, we have, with Corollary 6,

$$\begin{aligned} G_{k+1}(s) &\leqq C_2(T)\int_0^s \left(\int_0^u G_k(v)\,dv\right)du + T\,C_1(T)\int_0^s G_k(u)\,du \\ &\leqq (C_2(T)+C_1(T))\,T\int_0^s G_k(u)\,du, \end{aligned}$$

where $C_2(T)$ is a positive constant. As in the proof of Lemma 4 or Lemma 5, this implies that the limit $g = \lim_{k\to\infty} g_k$ exists and is continuous. Also, $0\leqq g_k(s,t)\leqq 1$ implies

$$(39)\qquad 0\leqq g(s,t)\leqq 1,\ t\geqq s\geqq 0.$$

Then $\eta = \lim_{k\to\infty}\eta_k$ also exist and are continuous, and these functions satisfy (32) with g in place of $\tilde g$, and (37). C^1 properties follow from the right hand side of (37), and uniqueness also follows as in the proof of Lemma 5. □

Corollary 8 *The following hold.*

$$(40)\qquad f(y,0) = y, \quad y\in[0,1].$$

$$\frac{\partial f}{\partial y}(y,t) > 0,\ \frac{\partial f}{\partial t}(y,t) \geqq 0, \quad (y,t) \in [0,1]\times[0,\infty). \tag{41}$$

$$\frac{\partial g}{\partial s}(s,t) \leqq 0,\ \frac{\partial g}{\partial t}(s,t) \geqq 0, \quad 0 \leqq s \leqq t. \tag{42}$$

$$g(t,t) = 0, \quad t \geqq 0. \tag{43}$$

$$g(0,t) = f(0,t),\ t \geqq 0. \tag{44}$$

◇

Proof. The claims on f, (40) and (41), are consequences of (27) and the assumptions inTheorem 1. The only perhaps less obvious claim is that the derivative of f in y cannot be 0 in (41), which follows from (13) and (1), with

$$\frac{\partial f}{\partial y}(y,t) \geqq -e^{-T\,R_w(T)} \int_W \frac{\partial \rho}{\partial y}(w,y)\, U_0(dw,0) = -e^{-T\,R_w(T)} \frac{\partial}{\partial y} U_0(\mathbf{1}_W, y) = e^{-T\,R_w(T)} > 0.$$

Differentiating (33) with $\tilde{g}$ replaced by g,

$$\begin{aligned} &\eta(w,t) - \int_0^t \eta(w,u)\, w(g(u,t),t)\ \exp(-\int_u^t w(g(u,v),v)\, dv)\, du \\ &= -\int_0^1 \frac{\partial \rho}{\partial z}(w,z)\, w(f(z,t),t)\ \exp(-\int_0^t w(f(z,s),s)\, ds)\, dz. \end{aligned} \tag{45}$$

Integrating (45) over W with measure $U_0(dw,0)$, and recalling that η and $w \in W$ are non-negative, and using (27) and (41),

$$\begin{aligned} &\int_W \eta(w,t)\, U_0(dw,0) \\ &\geqq -\int_W \int_0^1 \frac{\partial \rho}{\partial z}(w,z)\, w(f(z,t),t)\ \exp(-\int_0^t w(f(z,s),s)\, ds)\, dz\, U_0(dw,0) \\ &= \frac{\partial f}{\partial t}(0,t) \geqq 0. \end{aligned} \tag{46}$$

Differentiating (37) by s, and using (46) we then have

$$\frac{\partial g}{\partial s}(s,t) = -\int_W \eta(w,s) \exp(-\int_s^t w(g(s,v),v)\, dv)\, U_0(dw,0) \leqq 0.$$

Similarly, differentiating $g(s,t)$ by t and using (37) and (41),

$$\frac{\partial g}{\partial t}(s,t) \geqq \frac{\partial f}{\partial t}(0,t) \geqq 0.$$

The rest of the claims are obtained easily. Indeed, (43) follows from (15), (37) and (33), and (44) from (37) and (27). □

We are ready to define the characteristic curves $y = y_C(\gamma,t)$ for (22). For $\gamma = (y_0,t_0) \in \Gamma$ and $t \geqq t_0$, put

$$y_C(\gamma,t) := \begin{cases} f(y_0,t) & \text{if } \gamma \in \Gamma_i\,, \ \text{ i.e., } t_0 = 0, \\ g(t_0,t) & \text{if } \gamma \in \Gamma_b\,, \ \text{ i.e., } y_0 = 0. \end{cases} \tag{47}$$

Note that (44) implies that (47) is well-defined on $(y_0, t_0) = (0, 0) \in \Gamma_i \cap \Gamma_b$. Lemma 4 and Lemma 7 imply continuity of $y_C(\gamma, t)$, and C^1 property in t. (In fact, it is also C^1 in (γ, t) except on $y = y_C((0, 0), t)$.) Also, (40) and (43) imply the first equality in (16).

Note also that, for each $t \geqq 0$, $y_C(\cdot, t) : \Gamma_t \to [0, 1]$ is surjective. In fact, f and g are continuous, (40) and (41) imply $f(1, t) \geqq f(1, 0) = 1$. These and (43) and (44) imply that y_C is surjective:

$$\{y_C(\gamma, t) \mid \gamma \in \Gamma_t\} = [0, 1].$$

Note that (41) implies that there exists a unique C^1, increasing, one-to-one onto inverse function $\hat{f} : [f(0, t), 1] \to [0, 1]$ of $f(y, t)$ with respect to y. For $y < y_C(0, 0, t) = f(0, t) = g(0, t)$ we define $\hat{g} : [0, g(0, t)] \to [0, t]$ by

$$\hat{g}(y, t) = \inf\{s \geqq 0 \mid g(s, t) = y\}. \tag{48}$$

Since, as noted above, $g(\cdot, t) : [0, t] \to [0, g(0, t)]$ is surjective, $\hat{g}$ is well-defined, and Since g is continuous, $g(\hat{g}(y, t), t) = y$. Also (42) implies that $\hat{g}(y, t)$ is non-increasing with respect to y. Put

$$\hat{\gamma}(y, t) = \begin{cases} (\hat{f}(y, t), 0) \in \Gamma_i & \text{if } f(0, t) \leqq y \leqq 1, \\ (0, \hat{g}(y, t)) \in \Gamma_b \cap \Gamma_t & \text{if } 0 \leqq y \leqq g(0, t). \end{cases} \tag{49}$$

The definition implies

$$y_C(\hat{\gamma}(y, t), t) = y, \quad y \in [0, 1], \quad \text{and} \quad \hat{\gamma}(y_C(\gamma, t), t) = \gamma, \quad \gamma \in \Gamma_i. \tag{50}$$

Note that the second equality may fail on $\gamma \in \Gamma_b$.

For $t \geqq 0$, define a measure valued function

$$\varphi(dw, \cdot, t) : \Gamma_t \to [0, \infty)$$

as follows: If $\gamma = (y_0, 0) \in \Gamma_i$,

$$\varphi(dw, \gamma, t) := -\int_{y_0}^{1} \frac{\partial \rho}{\partial z}(w, z) \exp\Big(-\int_0^t w(f(z, s), s)\, ds\Big)\, dz\, U_0(dw, 0), \tag{51}$$

where f is as in Lemma 4, and if $\gamma = (0, t_0) \in \Gamma_b \cap \Gamma_t$,

$$\begin{aligned} &\varphi(dw, \gamma, t) \\ &:= -\int_0^1 \frac{\partial \rho}{\partial z}(w, z) \exp\Big(-\int_0^t w(f(z, s), s)\, ds\Big)\, dz\, U_0(dw, 0) \\ &\quad + \int_0^{t_0} \eta(w, u) \exp\Big(-\int_u^t w(g(u, v), v) dv\Big)\, du\, U_0(dw, 0), \end{aligned} \tag{52}$$

where, f is as in Lemma 4, and η and g are as in Lemma 7. Let

$$\varphi(h, \gamma, t) := \int_W h(s)\, \varphi(dw, \gamma, t)$$

for a continuous bounded function h, $\gamma \in \Gamma$ and $t \in [0, \infty)$.

Proposition 9 *The following hold.*

$$y_C(\gamma, t) = 1 - \varphi(\mathbf{1}_W, \gamma, t) := 1 - \int_W \varphi(dw, \gamma, t), \quad \gamma \in \Gamma_t,\ t \geqq 0. \tag{53}$$

$$\varphi(dw,\gamma,t_0)=U_0(dw,y_0),\quad \gamma=(y_0,t_0)\in\Gamma. \tag{54}$$

For bounded continuous $h:\ W\to\mathbb{R}$ and $t>0$,

$$\frac{\partial\varphi}{\partial t}(h,(y_0,0),t)=\int_W\int_{y_0}^1 w(y_C((z,0),t),t)\frac{\partial\varphi}{\partial z}(h,(z,0),t)\,dz\,U_0(dw,0),\quad 0\leqq y_0\leqq 1, \tag{55}$$

and

$$\frac{\partial\varphi}{\partial t}(h,(0,t_0),t)=\frac{\partial\varphi}{\partial t}(h,(0,0),t)-\int_W\int_0^{t_0} w(y_C((0,u),t),t)\frac{\partial\varphi}{\partial u}(h,(0,u),t)\,du\,U_0(dw,0),\quad 0\leqq t_0\leqq t. \tag{56}$$

◇

Proof. The definitions (47), (51) and (52), with Lemma 4 and Lemma 7 imply (53), and (54) follows from (33), (51) and (52). The definitions (47) and (51) imply that both hand sides of (55) are equal to

$$\int_W h(w)\int_{y_0}^1\frac{\partial\rho}{\partial z}(w,z)\,w(f(z,t),t)\,\exp(-\int_0^t w(f(z,s),s)\,ds)\,dz\,U_0(dw,0).$$

Similarly, (47) and (52) imply that both hand sides of (56) are equal to

$$\frac{\partial\varphi}{\partial t}(h,(0,0),t)-\int_W h(w)\int_0^{t_0}\eta(w,u)\,w(g(u,t),t)\,\exp(-\int_u^t w(g(u,v),v)\,dv)\,du\,U_0(dw,0).$$

□

For $(y,t)\in[0,1]\times[0,\infty)$ put

$$U(dw,y,t):=\varphi(dw,\hat\gamma(y,t),t)=\begin{cases}\varphi(dw,(\hat f(y,t),0),t) & f(0,t)\leqq y\leqq 1,\\ \varphi(dw,(0,\hat g(y,t)),t) & 0\leqq y\leqq g(0,t),\end{cases} \tag{57}$$

where $\hat\gamma$ is defined in (49).

Theorem 10 *It holds that*

$$\varphi(dw,\gamma,t)=U(dw,y_C(\gamma,t),t),\quad \gamma\in\Gamma_t,\ t\geqq 0. \tag{58}$$

Furthermore, for bounded continuous function $h:\ W\to\mathbb{R}$ $U(h,\cdot,\cdot):\ [0,1]\times[0,\infty)\to[0,\infty)$ is Lipschitz continuous in $(y,t)\in[0,1]\times[0,T]$ for any $T>0$, and satisfies the second equality in (16), (17), (19), and (20). ◇

Proof. For $\gamma\in\Gamma_i$, (58) follows from (50). The point is the case $\gamma\in\Gamma_b$, where $y_C((0,s),t)=g(s,t)$, as a function of s, may fail to be one-to-one. Suppose $g(s,t)=g(s',t)$ for some s and s' satisfying $0\leqq s<s'\leqq t$. Then (37) and non-negativity of $\eta(w,u)$ implies

$$\int_s^{s'}\eta(w,u)\exp(-\int_u^t w(g(u,v),v)dv)\,du=0,\ U_0(dw,0)\text{–almost surely.}$$

Hence (52) implies $\varphi(dw,(0,s'),t)=\varphi(dw,(0,s),t)$. On the other hand, the first equality of (50) implies

$$y_C(\hat\gamma(y_C(\gamma,t),t),t)=y_C(\gamma,t),\ \gamma\in\Gamma_b.$$

Therefore, $\varphi(dw, \hat{\gamma}(y_C(\gamma,t),t),t) = \varphi(dw,\gamma,t)$, with which (57) implies

$$U(dw, y_C(\gamma,t),t) = \varphi(dw, \hat{\gamma}(y_C(\gamma,t),t),t) = \varphi(dw,\gamma,t),$$

so that (58) holds.

The Lipschitz continuity of $U(h,y,t)$ for $f(0,t) \leqq y \leqq 1$, $0 \leqq t \leqq T$ is obvious, since the definitions (57), (51), and the definition of $\hat{f}$ stated just before (48) imply that $U(h,y,t)$ is C^1. To prove the Lipschitz continuity of $U(h,y,t)$ for $0 \leqq g(0,t) = f(0,t) \leqq y \leqq 1$, $0 \leqq t \leqq T$, let (y,t) and (y',t') be 2 points in this domain. Use (57) to decompose

$$|U(h,y',t') - U(h,y,t)| \leqq |\varphi(h,\hat{\gamma}(y',t'),t') - \varphi(h,\hat{\gamma}(y',t'),t)| + |\varphi(h,\hat{\gamma}(y',t'),t) - \varphi(h,\hat{\gamma}(y,t),t)|.$$

Since by definition (52) $\varphi(h,\gamma,t)$ is C^1 in t, the first term on the right hand side is bounded by a global constant times $|t'-t|$. To evaluate the second term, let M be such that $|h(w)| \leqq M$, $w \in W$, and denote by h_+ and h_- the positive and negative part of h, respectively, so that $h = h_+ - h_-$, $0 \leqq h_\pm \leqq M$. Definitions (49) and (52), and the non-negativity of η imply

$$\begin{aligned}
&|\varphi(h,\hat{\gamma}(y',t'),t) - \varphi(h,\hat{\gamma}(y,t),t)| = \left| \int_{\hat{g}(y',t')}^{\hat{g}(y,t)} \int_W h(w)\, \eta(w,u) \exp(-\int_u^t w(g(u,v),v)dv)\, U_0(dw,0)\, du \right| \\
&\leqq 2M \left| \int_{\hat{g}(y',t')}^{\hat{g}(y,t)} \int_W \mathbf{1}_W(w)\, \eta(w,u) \exp(-\int_u^t w(g(u,v),v)dv)\, U_0(dw,0)\, du \right|,
\end{aligned}$$

which, with with (49), (52), and (53), is equal to

$$2M\,|y_C(\hat{\gamma}(y',t'),t) - y_C(\hat{\gamma}(y,t),t)|.$$

This with (50) implies

$$\begin{aligned}
&|\varphi(h,\hat{\gamma}(y',t'),t) - \varphi(h,\hat{\gamma}(y,t),t)| \\
&\leqq 2M\,|y_C(\hat{\gamma}(y',t'),t') - y_C(\hat{\gamma}(y',t'),t)| + 2M\,|y_C(\hat{\gamma}(y',t'),t') - y_C(\hat{\gamma}(y,t),t)| \\
&\quad = 2M\,|y_C(\hat{\gamma}(y',t'),t') - y_C(\hat{\gamma}(y',t'),t)| + 2M\,|y'-y|.
\end{aligned}$$

Since $y_C(\gamma,t)$ is C^1 in t, we have the global Lipschitz continuity.

The property (20) follows from (53) and (50). The second equality in (16) then follows from (54), (58), (58), and the first equality in (16). (Note that the first equality in (16) and other claims in Theorem 1 for y_C is proved below (47).)

To prove (17) for $(y_0,t_0) \in \Gamma_i$, namely, for $t_0 = 0$, use (18), (57), (58), and (51), and change the order of integration, to find

$$\begin{aligned}
&-V(h, y_C((y_0,0),t),t) \\
&= -\int_W h(w)\, w(y_C((y_0,0),t),t)\, \varphi(dw,(y_0,0),t) \\
&\quad + \int_W h(w) \int_{y_C((y_0,0),t)}^1 \frac{\partial w}{\partial z}(z,t) \left(\int_{\hat{f}(z,t)}^1 \frac{\partial \rho}{\partial z'}(w,z') \exp(-\int_0^t w(f(z',s),s)\, ds)\, dz' \right) dz\, U_0(dw,0) \\
&= -\int_W h(w)\, w(y_C((y_0,0),t),t)\, \varphi(dw,(y_0,0),t) \\
&\quad + \int_W h(w) \int_{y_0}^1 \left(\int_{y_C((y_0,0),t)}^{y_C((z',0),t)} \frac{\partial w}{\partial z}(z,t) dz \right) \frac{\partial \rho}{\partial z'}(w,z') \exp(-\int_0^t w(f(z',s),s)\, ds)\, dz'\, U_0(dw,0),
\end{aligned}$$

which, with the definition (51), is equal to $\dfrac{\partial \varphi}{\partial t}(h,\gamma,t)$. Integrating from t_0 to t and using (58) and (54), we have (17).

To prove (17) for $(y_0, t_0) \in \Gamma_b$, namely, for $y_0 = 0$, first decompose the integration range in (18) with $y = y_C((0, t_0), t)$ as

$$[y_C((0, t_0), t), 1] = [g(t_0, t), g(0, t)] \cup [f(0, t), 1],$$

then use the definitions (57) and (51) or (52), and change the order of integration, to find

$$\begin{aligned}
&-V(h, y_C((0, t_0), t), t)\\
&= -\int_W h(w)\, w(y_C((0, t_0), t), t)\varphi(dw, (0, t_0), t)\\
&\quad - \int_W h(w) \int_{g(t_0,t)}^{g(0,t)} \frac{\partial w}{\partial z}(z, t) \Big(-\int_0^1 \frac{\partial \rho}{\partial z'}(w, z') \exp(-\int_0^t w(f(z', s), s)\, ds)\, dz'\\
&\qquad\qquad + \int_0^{\hat g(z,t)} \eta(w, u) \exp(-\int_u^t w(g(u, v), v) dv)\, du \Big)\, dz\, U_0(dw, 0)\\
&\quad + \int_W h(w) \int_{f(0,t)}^1 \frac{\partial w}{\partial z}(z, t) \Big(\int_{\hat f(z,t)}^1 \frac{\partial \rho}{\partial z'}(w, z') \exp(-\int_0^t w(f(z', s), s)\, ds)\, dz' \Big)\, dz\, U_0(dw, 0)\\
&= -\int_W h(w)\, w(y_C((0, t_0), t), t)\varphi(dw, (0, t_0), t)\\
&\quad + \int_W h(w)\, (w(g(0, t), t) - w(g(t_0, t), t)) \int_0^1 \frac{\partial \rho}{\partial z'}(w, z') \exp(-\int_0^t w(f(z', s), s)\, ds)\, dz'\, U_0(dw, 0)\\
&\quad - \int_W h(w) \int_0^{t_0} \eta(w, u) \exp(-\int_u^t w(g(u, v), v) dv) \Big(\int_{g(t_0,t)}^{g(u,t)} \frac{\partial w}{\partial z}(z, t)\, dz \Big)\, du\, U_0(dw, 0)\\
&\quad + \int_W h(w) \int_0^1 \frac{\partial \rho}{\partial z'}(w, z') \exp(-\int_0^t w(f(z', s), s)\, ds) \Big(\int_{f(0,t)}^{f(z',t)} \frac{\partial w}{\partial z}(z, t)\, dz \Big)\, dz'\, U_0(dw, 0).
\end{aligned}$$

Using (52), this further is simplified as

$$\begin{aligned}
\int_W h(w) \Big(&-\int_0^{t_0} \eta(u) w(g(u, t), t) \exp(-\int_u^t w(g(u, v), v) dv)\, du\\
&+ \int_0^1 \frac{\partial \rho}{\partial z'}(w, z') w(f(z', t), t) \exp(-\int_0^t w(f(z', s), s)\, ds)\, dz' \Big)\, U_0(dw, 0),
\end{aligned}$$

which, by using (52), is seen to be equal to $\dfrac{\partial \varphi}{\partial t}(h, \gamma, t)$. Integrating from t_0 to t and using (58) and (54), we have (17).

Substituting $h = \mathbf{1}_W$ in (17), and using (20), (53) and (58), we have (19). $\square$

To complete a proof of Theorem 1, it only remains to prove uniqueness. Besides the pair y_C and U which we constructed and proved so far to satisfy the properties stated in Theorem 1, assume that there are another such pair $\tilde y_C$ and $\tilde U$. For $T > 0$, let $L(T) > 0$ be such that

$$\begin{aligned}
&\max\{|U(h, y, t) - U(h, y', t')|,\ |\tilde U(h, y, t) - \tilde U(h, y', t')|\}\\
&\leqq L(T)\ \|(y, t) - (y', t')\|,\quad (y, t), (y', t') \in [0, 1] \times [0, T],\ h:\ W \to [-1, 1];\ \text{conti.}.
\end{aligned}$$

Put

$$I(t) = \sup_{h:\ W \to [-1,1];\ \text{conti.}} \sup_{y \in [0,1]} |U(h, y, t) - \tilde U(h, y, t)|$$

and

$$J(t) = \sup_{\gamma \in \Gamma_t} |\tilde y_C(\gamma, t) - y_C(\gamma, t)|.$$

Then (16) and its correspondence for $\tilde U$ imply $I(0)=0$. Since $y_C(\cdot,t):\ \Gamma_t\to[0,1]$ is onto,

$$I(t)=\sup_{h:\ W\to[-1,1];\ \text{conti.}}\ \sup_{\gamma\in\Gamma_t}|U(h,y_C(\gamma,t),t)-\tilde U(h,y_C(\gamma,t),t)|. \tag{59}$$

Note also that since $\tilde U(dw,y,t)$ is, by assumption, a non-negative measure, for h with $|h(w)|\leqq 1$, $w\in W$, we have

$$\tilde U(h,y,t)\leqq \tilde U(\mathbf{1}_W,y,t)=1-y\leqq 1,$$

where we also used (20).

It holds that

$$I(t)\leqq L(T)J(t)$$

Subtracting

$$\tilde U(h,\tilde y_C(\gamma,t),t)=U_0(h,y_0)-\int_{t_0}^t \tilde V(h,\tilde y_C(\gamma,s),s)\,ds, \tag{60}$$

from (17), and using (59), (18) and (1), we have

$$\begin{aligned}&|\tilde y_C(\gamma,t)-y_C(\gamma,t)|=|\tilde U(\mathbf{1}_W,\tilde y_C(\gamma,t),t)-U(\mathbf{1}_W,y(\gamma,t),t)|\\&\leqq 2R_w(T)\int_{t_0}^t J(s)\,ds+R_w(T)\int_{t_0}^t I(s)\,ds+R_w(T)L(T)\int_{t_0}^t J(s)\,ds.\end{aligned}$$

Therefore,

$$J(t)\leqq 2R_w(T)\int_{t_0}^t J(s)\,ds+R_w(T)\int_{t_0}^t I(s)ds+R_w(T)L(T)\int_{t_0}^t J(s)\,ds.$$

Then,

$$I(t)\leqq L(T)\left(2R_w(T)\int_{t_0}^t J(s)\,ds+R_w(T)\int_{t_0}^t I(s)ds+R_w(T)L(T)\int_{t_0}^t J(s)\,ds\right),$$

so that if we put $K(t)=\max\{I(t),J(t)\}$ then there exists $C(T)$ such that

$$K(t)\leqq C(T)\int_{t_0}^t K(s)\,ds,\ K(t_0)=0,$$

implying $K(t)=0$. Hence $\tilde U=U$ and $\tilde y_C=y_C$.

This completes a proof of Theorem 1.

3 Proof of Theorem 2.

Let Γ be as in (11). To simplify the notation, for $\gamma=(y_0,t_0)\in\Gamma$, we will write $y_C((y_0,t_0),t)$ defined in (47) as $y_C(y_0,t_0,t)$.

We first prepare a random variable which converges as $N\to\infty$ to $y_C(t)=y_C(y_0,t_0,t)$, for $(y_0,t_0)\in\Gamma$, $t\geqq t_0$;

$$Y_C^{(N)}(y_0,t_0,t)=y_0+\frac{1}{N}\sum_{i;\ X_i^{(N)}(t_0)\geqq Ny_0+1}\mathbf{1}_{J_i^{(N)}(t_0,t)}\,,\quad (y_0,t_0)\in[0,1)\times[0,\infty),\ t>t_0\,, \tag{61}$$

where $J_i^{(N)}$ is defined in (3). In particular, if we put, as an analogue to (5),

(62) $$y_i^{(N)} = \frac{1}{N}(x_i^{(N)} - 1), \quad i = 1, 2, \dots, N,$$

then (3) and (4) imply

(63) $$Y_i^{(N)}(t) \geqq Y_C^{(N)}(y_0, t_0, t) \quad \Leftrightarrow \quad Y_i^{(N)}(t_0) \geqq y_0 \;\text{ and }\; J_i^{(N)}(t_0, t) \;\text{ does not hold.}$$

Hence, we have

(64) $$\begin{aligned} &Y_C^{(N)}(y_0, t_0, t) \\ &= y_0 + \frac{1}{N}\sum_i \int_{s\in(t_0,t]} \int_{\xi\in[0,\infty)} \mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(y_0,t_0,s-)} \mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))} \nu_i(d\xi ds). \end{aligned}$$

For the spatially homogeneous case, $Y_A^{(N)}(t_0, t)$ in [18] is equal to $Y_C^{(N)}(0, t - t_0, t)$, $Y_B^{(N)}(y_0, t)$ to $Y_C^{(N)}(y_0, 0, t)$, and $Y_C^{(N)}(t)$ in [18] is equal to $Y_C^{(N)}(0, 0, t)$ of (61).

Let Γ be as in (11). Let $(y_0, t_0) \in \Gamma$, $t \geqq t_0$. The definition (7) and the properties (3), (4), and (63) imply that for $B \in \mathscr{B}(W)$, $U^{(N)}(B, Y_C^{(N)}(y_0, t_0, t), t)$ as a function of t changes its value if and only if $J_i^{(N)}(t_0, t)$ occurs for some i satisfying $y_i^{(N)} \geqq y_0$ and $w_i \in B$. Therefore, for $B \in \mathscr{B}(W)$

$$\begin{aligned} &U^{(N)}(B, Y_C^{(N)}(y_0, t_0, t), t) - U^{(N)}(B, y_0, t_0) \\ &= -\frac{1}{N}\sum_{i;\ w_i\in B} \int_{s\in(t_0,t]} \int_{\xi\in[0,\infty)} \mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(y_0,t_0,s-)} \mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))} \nu_i(d\xi ds). \end{aligned}$$

In analogy to (18) define for $B \in \mathscr{B}(W)$

(65) $$V^{(N)}(B, y, t) = \int_B w(y, t)\, U^{(N)}(dw, y, t) + \int_y^1 \int_B \frac{\partial w}{\partial z}(z, t)\, U^{(N)}(dw, z, t)\, dz.$$

By definition (7), for $B \in \mathscr{B}(W)$

(66) $$\begin{aligned} V^{(N)}(B, y, t) &= \frac{1}{N}\sum_{m\geqq Ny+1}\sum_{i;\ w_i\in B} w_i(\frac{m-1}{N}, t)\ \mathbf{1}_{X_i^{(N)}(t-)=m} \\ &= \frac{1}{N}\sum_{i;\ w_i\in B,\, Y_i^{(N)}(t-)\geqq y} w_i(Y_i^{(N)}(t-), t). \end{aligned}$$

Denote the compensated Poisson process by

(67) $$\tilde{\nu}_i(d\xi ds) = \nu_i(d\xi ds) - d\xi ds,$$

and put for $B \in \mathscr{B}(W)$

(68) $$\begin{aligned} &M_U^{(N)}(B, y_0, t_0, t) \\ &= -\frac{1}{N}\sum_{i;\ w_i\in B} \int_{s\in(t_0,t]} \int_{\xi\in[0,\infty)} \mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(y_0,t_0,s-)} \mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))} \tilde{\nu}_i(d\xi ds). \end{aligned}$$

Then, we have for $B \in \mathscr{B}(W)$

$$
\begin{aligned}
&U^{(N)}(B, Y_C^{(N)}(y_0,t_0,t),t) \\
&= U^{(N)}(B,y_0,t_0) + M_U^{(N)}(B,y_0,t_0,t) \\
&\quad -\frac{1}{N}\sum_{i;\ w_i\in B}\int_{s\in(t_0,t]}\int_{\xi\in[0,\infty)} \mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(y_0,t_0,s-)}\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,d\xi ds \\
&= U^{(N)}(B,y_0,t_0) + M_U^{(N)}(B,y_0,t_0,t) \\
&\quad -\frac{1}{N}\sum_{i;\ w_i\in B}\int_{t_0}^{t} w_i\left(Y_i^{(N)}(s-),s\right)\mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(y_0,t_0,s-)}\,ds \\
&= U^{(N)}(B,y_0,t_0) + M_U^{(N)}(B,y_0,t_0,t) - \int_0^t V^{(N)}(B, Y_C^{(N)}(y_0,t_0,s),s)ds.
\end{aligned}
$$

Combining this equality with (17), we have for $B \in \mathscr{B}(W)$

$$
\begin{aligned}
&U^{(N)}(B, Y_C^{(N)}(y_0,t_0,t),t) - U(B, y_C(y_0,t_0,t),t) \\
&= U^{(N)}(B,y_0,t_0) - U(B,y_0,t_0) + M_U^{(N)}(B,y_0,t_0,t) \\
&\quad -\int_{t_0}^{t}\left(V^{(N)}(B, Y_C^{(N)}(y_0,t_0,s),s) - V(B, y_C(y_0,t_0,s),s)\right)ds.
\end{aligned} \tag{69}
$$

Put

$$
\begin{aligned}
W^{(N)}(t) = &\sup_{(y_0,t_0)\in\Gamma; t_0\leqq t}\ \sup_{s\in[t_0,t]} |Y_C^{(N)}(y_0,t_0,s) - y_C(y_0,t_0,s)| \\
&\vee \sup_{(y_0,t_0)\in\Gamma; t_0\leqq t}\ \sup_{s\in[t_0,t]} ||U^{(N)}(\cdot, Y_C^{(N)}(y_0,t_0,s),s) \\
&\qquad\qquad - U(\cdot, y_C(y_0,t_0,s),s)||_{\mathrm{var}} \\
&\vee \sup_{B\in\mathscr{B}(W)}\ \sup_{(y_0,t_0)\in\Gamma; t_0\leqq t}\ \sup_{s\in[t_0,t]} |V^{(N)}(B, Y_C^{(N)}(y_0,t_0,s),s) \\
&\qquad\qquad - V(B, y_C(y_0,t_0,s),s)|.
\end{aligned} \tag{70}
$$

Since for all $z \in [0,1)$ there exist $(y_0,t_0) \in \Gamma$ such that $y_C(y_0,t_0,t) = z$, for $B \in \mathscr{B}(W)$

$$
\begin{aligned}
&\sup_{z\in[0,1]} |U^{(N)}(B,z,t) - U(B,z,t)| \\
&\leqq \sup_{(y_0,t_0)\in\Gamma; t_0\leqq t}\ \sup_{s\in[t_0,t]} |U^{(N)}(B, y_C(y_0,t_0,s),s) - U(B, y_C(y_0,t_0,s),s)| \\
&\leqq \sup_{(y_0,t_0)\in\Gamma; t_0\leqq t}\ \sup_{s\in[t_0,t]} |U^{(N)}(B, Y_C^{(N)}(y_0,t_0,s),s) - U(B, y_C(y_0,t_0,s),s)| \\
&\quad + \sup_{(y_0,t_0)\in\Gamma; t_0\leqq t}\ \sup_{s\in[t_0,t]} |U^{(N)}(B, Y_C^{(N)}(y_0,t_0,s),s) - U^{(N)}(B, y_C(y_0,t_0,s),s)|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sup_{z\in[0,1]} |U^{(N)}(B,z,t) - U(B,z,t)| \\
&\leqq W^{(N)}(t) + \sup_{(y_0,t_0)\in\Gamma; t_0\leqq t}\ \sup_{s\in[t_0,t]} |U^{(N)}(B, Y_C^{(N)}(y_0,t_0,s),s) \\
&\qquad\qquad - U^{(N)}(B, y_C(y_0,t_0,s),s)|.
\end{aligned} \tag{71}
$$

By (7) it holds that

$$\sup_{(y_0,t_0)\in\Gamma;t_0\leqq t}\ \sup_{s\in[t_0,t]} |U^{(N)}(B,Y_C^{(N)}(y_0,t_0,s),s)-U^{(N)}(B,y_C(y_0,t_0,s),s)| \\ \leqq \sup_{(y_0,t_0)\in\Gamma;t_0\leqq t}\ \sup_{s\in[t_0,t]} |Y_C^{(N)}(y_0,t_0,s)-y_C(y_0,t_0,s)|+\frac{1}{N}. \tag{72}$$

By (71) and (72) we obtain

$$\sup_{z\in[0,1]} |U^{(N)}(B,z,t)-U(B,z,t)| \leqq 2W^{(N)}(t)+\frac{1}{N}. \tag{73}$$

This implies

$$\sup_{z\in[0,1]} ||U^{(N)}(\cdot,z,t)-U(\cdot,z,t)||_{\mathrm{var}} \leqq 4W^{(N)}(t)+\frac{2}{N}. \tag{74}$$

By (69) and (70), we have for $B\in\mathscr{B}(W)$

$$\left|U^{(N)}(B,Y_C^{(N)}(y_0,t_0,t),t)-U(B,y_C(y_0,t_0,t),t)\right| \\ \leqq \left|U^{(N)}(B,y_0,t_0)-U(B,y_0,t_0)\right|+\left|M_U^{(N)}(B,y_0,t_0,t)\right|+\int_{t_0}^t W^{(N)}(s)ds. \tag{75}$$

Similarly, combining (65) with (18), we have

$$\begin{aligned} &V^{(N)}(B,Y_C^{(N)}(y_0,t_0,t),t)-V(B,y_C(y_0,t_0,t),t)\\ &=\int_B w(Y_C^{(N)}(y_0,t_0,t),t)\,(U^{(N)}(dw,Y_C^{(N)}(y_0,t_0,t),t)-U(dw,y_C(y_0,t_0,t),t))\\ &\quad+\int_B (w(Y_C^{(N)}(y_0,t_0,t),t)-w(y_C(y_0,t_0,t),t))\,U(dw,y_C(y_0,t_0,t),t)\\ &\quad+\int_{Y_C^{(N)}(y_0,t_0,t)}^1\int_B \frac{\partial w}{\partial z}(z,t)\,(U^{(N)}(dw,z,t)-U(dw,z,t))\,dz\\ &\quad-\int_{y_C(y_0,t_0,t)}^{Y_C^{(N)}(y_0,t_0,t)}\int_B \frac{\partial w}{\partial z}(z,t)\,U(dw,z,t)\,dz. \end{aligned}$$

Hence, using this estimate, (1), (74) and the fact that $0\leqq U^{(N)}\leqq 1$, we have for $B\in\mathscr{B}(W)$

$$\left|V^{(N)}(B,Y_C^{(N)}(y_0,t_0,t),t)-V(B,y_C(y_0,t_0,t),t)\right| \\ \leqq R_w(T)||U^{(N)}(\cdot,Y_C^{(N)}(y_0,t_0,t),t)-U(\cdot,y_C(y_0,t_0,t),t)||_{\mathrm{var}} \\ +2R_w(T)\left|Y_C^{(N)}(y_0,t_0,t)-y_C(y_0,t_0,t)\right|+R_w(T)\left(4\int_{t_0+}^t W^{(N)}(s)ds+\frac{2}{N}\right). \tag{76}$$

To estimate $Y_C^{(N)}(y_0,t_0,t)-y_C(y_0,t_0,t)$, by using (64), (66) and (68) calculate

$$Y_C^{(N)}(y_0,t_0,t)=y_0+M_U^{(N)}(W,y_0,t_0,t)+\int_{t_0}^t V^{(N)}(W,Y_C^{(N)}(y_0,t_0,s),s)\,ds. \tag{77}$$

Combining with (19),

$$Y_C^{(N)}(y_0,t_0,t)-y_C(y_0,t_0,t) \\ =M_U^{(N)}(W,y_0,t_0,t)+\int_{t_0}^t\left[V^{(N)}(W,Y_C^{(N)}(y_0,t_0,s),s)-V(W,y_C(y_0,t_0,s),s)\right]\,ds. \tag{78}$$

Hence, by (70) we have

$$\left|Y_C^{(N)}(y_0,t_0,t)-y_C(y_0,t_0,t)\right| \leqq |M_U^{(N)}(W,y_0,t_0,t)|+\int_{t_0}^t W^{(N)}(s)ds. \tag{79}$$

By (1), (75), (76), and (79),we obtain for $B\in\mathscr{B}(W)$

$$\begin{aligned}&\left|V^{(N)}(B,Y_C^{(N)}(y_0,t_0,t),t)-V(B,y_C(y_0,t_0,t),t)\right|\\ &\leqq R_w(T)||U^{(N)}(\cdot,y_0,t_0)-U(\cdot,y_0,t_0)||_{\text{var}}+7R_w(T)\int_{t_0}^t W^{(N)}(s)ds\\ &\quad+3R_w(T)\left|M_U^{(N)}(B,y_0,t_0,t)\right|+\frac{2R_w(T)}{N}.\end{aligned} \tag{80}$$

Because of (70), (75), (79), and (80), we have

$$\begin{aligned}W^{(N)}(t)\leqq{}& C_1\sup_{(y_0,t_0)\in\Gamma}||U^{(N)}(\cdot,y_0,t_0)-U(\cdot,y_0,t_0)||_{\text{var}}\\ &+(1+R_w(T))\sup_{B\in\mathscr{B}(W)}\sup_{(y_0,t_0)\in\Gamma;t_0\leqq t}\left|M_U^{(N)}(B,y_0,t_0,t)\right|+C_2\int_0^t W^{(N)}(s)ds+\frac{2R_w(T)}{N}\end{aligned}$$

where C_1 and C_2 are constants depending on $A,T,R_w(T)$. Hence, Gronwall's inequality implies

$$\begin{aligned}&\sup_{t\in[0,T]}W^{(N)}(t)\\ &\leqq e^{C_2T}\left[C_1\sup_{(y_0,t_0)\in\Gamma}||U^{(N)}(\cdot,y_0,t_0)-U(\cdot,y_0,t_0)||_{\text{var}}\right.\\ &\qquad\left.+(1+R_w(T))\sup_{B\in\mathscr{B}(W)}\sup_{(y_0,t_0)\in\Gamma;t_0\leqq t}\left|M_U^{(N)}(B,y_0,t_0,t)\right|+\frac{2R_w(T)}{N}\right].\end{aligned} \tag{81}$$

By the definition of Γ, we have

$$\begin{aligned}&E\left[\sup_{B\in\mathscr{B}(W)}\sup_{(y_0,t_0)\in\Gamma,t_0\leqq T}\sup_{t\in[t_0,T]}\left|M_U^{(N)}(B,y_0,t_0,t)\right|^2\right]\\ &\leqq E\left[\sup_{B\in\mathscr{B}(W)}\sup_{y_0\in[0,1)}\sup_{t\in[0,T]}\left|M_U^{(N)}(B,y_0,0,t)\right|^2\right]\\ &\quad+E\left[\sup_{B\in\mathscr{B}(W)}\sup_{0\leqq t_0\leqq t\leqq T}\left|M_U^{(N)}(B,0,t_0,t)\right|^2\right].\end{aligned} \tag{82}$$

Note that $Y_i^{(N)}(t)\in\{0,1/N,\dots(N-1)/N\}$ for $t\in[0,T]$ and $i=1,2,\dots,N$, $Y_C^{(N)}(y_0,t_0,\cdot)$ is a process of pure jumps by $1/N$ and for each $k=1,2,\dots,N$, $Y_C^{(N)}(y_0,t_0,\cdot)-y_0$ is independent of y_0 as long as $y_0\in(k-1/N,k/N]$. Also, note that $\{M_U^{(N)}(B,\cdot,0,\cdot);B\in\mathscr{B}(W)\}=\{M_U^{(N)}(B,\cdot,0,\cdot);B\in$

$2^{\{w_i;i=1,2,\dots,N\}}\}$. Hence, by (68)

$$
\begin{aligned}
&E\left[\sup_{B\in\mathscr{B}(W)}\sup_{y_0\in[0,1)}\sup_{t\in[0,T]}\left|M_U^{(N)}(B,y_0,0,t)\right|^2\right]\\
&=\frac{1}{N^2}E\left[\sup_{B\in\mathscr{B}(W)}\sup_{y_0\in[0,1)}\sup_{t\in[0,T]}\left|\sum_{i;\,w_i\in B}\int_{s\in(0,t]}\int_{\xi\in[0,\infty)}\right.\right.\\
&\qquad\left.\left.\mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(y_0,0,s-)}\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\tilde{\nu}_i(d\xi ds)\right|^2\right]\\
&=\frac{1}{N^2}\sup_{B\in 2^{\{w_i;i=1,2,\dots,N\}}}\max_{k=0,1,\dots,N-1}E\left[\sup_{t\in[0,T]}\left|\sum_{i;\,w_i\in B}\int_{s\in(0,t]}\int_{\xi\in[0,\infty)}\right.\right.\\
&\qquad\left.\left.\mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(k/N,0,s-)}\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\tilde{\nu}_i(d\xi ds)\right|^2\right].
\end{aligned}
$$

Hence, by Doob's martingale inequality (see (6.16) of Chapter I in [20]) and (3.9) of Chapter II in [20] we have

$$
\begin{aligned}
&E\left[\sup_{B\in\mathscr{B}(W)}\sup_{y_0\in[0,1)}\sup_{t\in[0,T]}\left|M_U^{(N)}(B,y_0,0,t)\right|^2\right]\\
&\leqq\frac{C_3}{N^2}\max_{k=0,1,\dots,N-1}E\left[\sum_i\int_{s\in(0,T]}\int_{\xi\in[0,\infty)}\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}d\xi ds\right]\\
&\leqq\frac{C_3R_w(T)T}{N}
\end{aligned}
$$

where C_3 is a positive constant. Thus, it holds that

$$
\lim_{N\to\infty}E\left[\sup_{B\in\mathscr{B}(W)}\sup_{y_0\in[0,1)}\sup_{t\in[0,T]}\left|M_U^{(N)}(B,y_0,0,t)\right|^2\right]=0. \tag{83}
$$

By (68) again, similarly to the case of $t_0 = 0$

$$
\begin{aligned}
&E\left[\sup_{B\in\mathscr{B}(W)}\sup_{0\leqq t_0\leqq t\leqq T}\left|M_U^{(N)}(B,0,t_0,t)\right|^2\right]\\
&=\frac{1}{N^2}\sup_{B\in 2^{\{w_i;i=1,2,\dots,N\}}}E\left[\sup_{0\leqq t_0\leqq t\leqq T}\left|\sum_{i;\,w_i\in B}\int_{s\in(t_0,t]}\int_{\xi\in[0,\infty)}\right.\right.\\
&\qquad\left.\left.\mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(0,t_0,s-)}\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,\tilde{\nu}_i(d\xi ds)\right|^2\right]\\
&=\frac{1}{N^2}\sup_{B\in 2^{\{w_i;i=1,2,\dots,N\}}}E\left[\sup_{0\leqq t_0\leqq t\leqq T}\left|\sum_{i;\,w_i\in B}\int_{s\in(0,t]}\int_{\xi\in[0,\infty)}\right.\right.\\
&\qquad\mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(0,t_0,s-)}\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,\tilde{\nu}_i(d\xi ds)\\
&\qquad\left.\left.-\sum_{i;\,w_i\in B}\int_{s\in(0,t_0]}\int_{\xi\in[0,\infty)}\mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(0,t_0,s-)}\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,\tilde{\nu}_i(d\xi ds)\right|^2\right]\\
&\leqq\frac{1}{N^2}\sup_{B\in 2^{\{w_i;i=1,2,\dots,N\}}}E\left[\left(2\sup_{0\leqq t\leqq T}\left|\sum_{i;\,w_i\in B}\int_{s\in(0,t]}\int_{\xi\in[0,\infty)}\right.\right.\right.\\
&\qquad\left.\left.\left.\mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(0,t_0,s-)}\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,\tilde{\nu}_i(d\xi ds)\right|\right)^2\right].
\end{aligned}
$$

Hence, by Doob's martingale inequality and (3.9) of Chapter II in [20] imply

$$
\begin{aligned}
&E\left[\sup_{B\in\mathscr{B}(W)}\sup_{0\leqq t_0\leqq t\leqq T}\left|M_U^{(N)}(B,0,t_0,t)\right|^2\right]\\
&\leqq\frac{4C_3}{N^2}E\left[\sum_i\int_{s\in(0,T]}\int_{\xi\in[0,\infty)}\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,d\xi ds\right]\\
&\leqq\frac{4C_3R_w(T)T}{N}.
\end{aligned}
$$

Thus, we obtain

$$
\lim_{N\to\infty}E\left[\sup_{B\in\mathscr{B}(W)}\sup_{0\leqq t_0\leqq t\leqq T}\left|M_U^{(N)}(B,0,t_0,t)\right|^2\right]=0. \tag{84}
$$

Combining (82), (83) and (84), we have

$$
\lim_{N\to\infty}E\left[\sup_{B\in\mathscr{B}(W)}\sup_{(y_0,t_0)\in\Gamma;\,t_0\leqq T}\sup_{t\in[t_0,T]}\left|M_U^{(N)}(B,y_0,t_0,t)\right|^2\right]=0. \tag{85}
$$

On the other hand, by (24) we have

$$
\lim_{N\to\infty}||U^{(N)}(\cdot,0,t_0)-U(\cdot,0,t_0)||_{\mathrm{var}}=\lim_{N\to\infty}||U^{(N)}(\cdot,0,0)-U_0(\cdot,0)||_{\mathrm{var}}=0. \tag{86}
$$

Thus, (86) and (24) implies

$$\lim_{N\to\infty} \sup_{(y_0,t_0)\in\Gamma} ||U^{(N)}(\cdot, y_0, t_0) - U(\cdot, y_0, t_0)||_{\mathrm{var}} = 0. \tag{87}$$

(81), (85) and (87) yields

$$\lim_{N\to\infty} E\left[\sup_{t\in[0,T]} W^{(N)}(t)^2\right] = 0.$$

Hence, there exists a subsequence $\{N(k)\}$ such that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} W^{(N(k))}(t) = 0$$

almost surely. However, the argument above is also available even if we replace N by any subsequence $N(k)$. Therefore, we have

$$\lim_{N\to\infty} \sup_{t\in[0,T]} W^{(N)}(t) = 0$$

almost surely. This proves the first assertion of Theorem 2.

We turn to a proof of the second assertion of Theorem 2. First, we show the uniqueness of the stochastic differential equation (26). Note that

$$E\left[\int_{s\in(0,t]}\int_{\xi\in[0,R_w(T)]} \nu_i(d\xi ds)\right] = tR_w(T)$$

and that for all i

$$\begin{aligned} &\int_{s\in(0,t]}\int_{\xi\in[0,\infty)} Y_i(s-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(s-),s))}\,\nu_i(d\xi ds) \\ &= \int_{s\in(0,t]}\int_{\xi\in[0,R_w(T)]} Y_i(s-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(s-),s))}\,\nu_i(d\xi ds). \end{aligned}$$

Moreover, there exists a constant C_T such that

$$\sup_{s\in[0,T]} |V(W,x,s) - V(W,y,s)| \leqq C_T|x-y|,\ \ x,y\in[0,1).$$

The proof of Theorem 9.1 in Chapter IV of [20] is available by taking $U := [0, R_w(T)]$ and $U_0 := \emptyset$. Thus, we obtain the uniqueness.

Next, we show that $(Y_1^{(N)}(t), Y_2^{(N)}(t), \dots, Y_L^{(N)}(t))$ converges to $(Y_1(t), Y_2(t), \dots, Y_L(t))$ uniformly in $t\in[0,T]$ almost surely and also converges in the sense of L^2. Let $i\in\{1,2,\dots,L\}$ be fixed. By (66) it is easy to see

$$\begin{aligned} Y_i^{(N)}(t) &= y_i^{(N)} + M_i^{(N)}(t) + \int_0^t V^{(N)}(W, Y_i^{(N)}(s-), s)ds \\ &\quad - \int_{s\in(0,t]}\int_{\xi\in[0,\infty)} Y_i^{(N)}(s-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,\nu_i(d\xi ds) \end{aligned} \tag{88}$$

where

$$M_i^{(N)}(t) := \frac{1}{N}\sum_{j=1}^N \int_{s\in(0,t]}\int_{\xi\in[0,\infty)} \mathbf{1}_{Y_i^{(N)}(s-)<Y_j^{(N)}(s-)}\,\mathbf{1}_{\xi\in[0,w_j(Y_j^{(N)}(s-),s))}\,\tilde{\nu}_j(d\xi ds). \tag{89}$$

Hence, (26) and (88) imply

$$
\begin{aligned}
&E\left[\sup_{s\in[0,t]}|Y_i^{(N)}(s)-Y_i(s)|^2\right]\\
&\leqq 4|y_i^{(N)}-y_i|^2+4E\left[\sup_{s\in[0,t]}|M_i^{(N)}(s)|^2\right]\\
&\quad +4\int_0^t E\left[|V^{(N)}(W,Y_i^{(N)}(s-),s)-V(W,Y_i(s-),s)|^2\right]ds\\
&\quad +4E\left[\sup_{s\in[0,t]}\left|\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}[Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))}\right.\right.\\
&\qquad\qquad\left.\left.-\,Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}]\nu_i(d\xi du)\right|^2\right]
\end{aligned}
\tag{90}
$$

By (18) and (65), we have

$$
\begin{aligned}
&V^{(N)}(W,Y_i^{(N)}(t-),t)-V(W,Y_i(t-),t)\\
&=\int_W w(Y_i^{(N)}(t-),t)\,(U^{(N)}(dw,Y_i^{(N)}(t-),t)-U(dw,Y_i(t-),t))\\
&\quad+\int_W (w(Y_i^{(N)}(t-),t)-w_i(Y_i(t-),t))\;U(dw,Y_i(t-),t)\\
&\quad+\int_{Y_i^{(N)}(t-)}^1\int_W\frac{\partial\,w}{\partial z}(z,t)\,(U^{(N)}(dw,z,t)-U(dw,z,t))\,dz\\
&\quad-\int_{Y_i(t-)}^{Y_i^{(N)}(t-)}\int_W\frac{\partial\,w}{\partial z}(z,t)\,U(dw,z,t)\,dz.
\end{aligned}
$$

Hence, noting that $0\leqq U^{(N)}\leqq 1$ and $0\leqq U\leqq 1$, there exist positive constants C_5 and C_6 such that

$$
\begin{aligned}
&|V^{(N)}(W,Y_i^{(N)}(t-),t)-V(W,Y_i(t-),t)|\\
&\leqq C_4|Y_i^{(N)}(t-)-Y_i(t-)|+C_5\sup_{z\in[0,1)}||U^{(N)}(\cdot,z,t)-U(\cdot,z,t)||_{\mathrm{var}}.
\end{aligned}
\tag{91}
$$

Now we show that

$$
\int_{\xi\in[0,\infty)}|x\,\mathbf{1}_{\xi\in[0,w_i(x,t))}-y\,\mathbf{1}_{\xi\in[0,w_i(y,t))}|^2d\xi\leqq C_4|x-y|,\quad x,y\in[0,1)
\tag{92}
$$

where C_4 is a positive constant. Let $x,y\in[0,1)$ and consider the case that $w_i(y,t)\leqq w_i(x,t)$. Then,

$$
\begin{aligned}
&\int_{\xi\in[0,\infty)}|x\,\mathbf{1}_{\xi\in[0,w_i(x,t))}-y\,\mathbf{1}_{\xi\in[0,w_i(y,t))}|^2d\xi\\
&=\int_{\xi\in[0,\infty)}|(x-y)\,\mathbf{1}_{\xi\in[0,w_i(x,t))}+y\,\mathbf{1}_{\xi\in[w_i(y,t),w_i(x,t))}|^2d\xi\\
&\leqq\int_{\xi\in[0,\infty)}\left(2(x-y)^2\,\mathbf{1}_{\xi\in[0,w_i(x,t))}+2y^2\,\mathbf{1}_{\xi\in[w_i(y,t),w_i(x,t))}\right)d\xi\\
&=2(x-y)^2w_i(x,t)+y^2(w_i(x,t)-w_i(y,t)).
\end{aligned}
$$

Since w_i and the spatial derivative of w_i are bounded, we have

$$\int_{\xi\in[0,\infty)} |x\,\mathbf{1}_{\xi\in[0,w_i(x,t))} - y\,\mathbf{1}_{\xi\in[0,w_i(y,t))}|^2 d\xi \leqq C_4|x-y|,\ \ x,y\in[0,1),$$

where C_4 is a positive constant. Therefore, (92) holds. The case that $w_i(y,t) \geqq w_i(x,t)$ is shown similarly. By (67), Doob's martingale inequality and (3.9) of Chapter II in [20], there exists a positive constant C_6 and we have

$$\begin{aligned}
&E\left[\sup_{s\in[0,t]}\left|\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}[Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))} - Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}]\nu_i(d\xi du)\right|^2\right]\\
&\leqq 2E\left[\sup_{s\in[0,t]}\left|\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}[Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))} - Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}]\tilde{\nu}_i(d\xi du)\right|^2\right]\\
&\quad +2E\left[\sup_{s\in[0,t]}\left|\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}[Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))} - Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}]d\xi du\right|^2\right]\\
&\leqq 2C_6E\left[\sup_{s\in[0,t]}\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}\left|Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))} - Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}\right|^2 d\xi du\right]\\
&\quad +2E\left[\sup_{s\in[0,t]}\left|\int_0^s[Y_i^{(N)}(u-)w_i(Y_i^{(N)}(u-),u) - Y_i(u-)w_i(Y_i(u-),u)]du\right|^2\right].
\end{aligned}$$

By (92) and boundedness of the spatial derivative of w_i, there exists a positive constant C_7 such that

$$\begin{aligned}
&E\left[\sup_{s\in[0,t]}\left|\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}[Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))} - Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}]\nu_i(d\xi du)\right|^2\right]\\
&\leqq 2C_4C_6E\left[\int_0^t\left|Y_i^{(N)}(u-)-Y_i(u-)\right|du\right] + 2C_7E\left[\sup_{s\in[0,t]}\left(\int_0^s\left|Y_i^{(N)}(u-)-Y_i(u-)\right|du\right)^2\right]\\
&\leqq 2C_4C_6\int_0^t E\left[\left|Y_i^{(N)}(u-)-Y_i(u-)\right|^2\right]du + 2C_7tE\left[\int_0^t\left|Y_i^{(N)}(u-)-Y_i(u-)\right|^2du\right].
\end{aligned}$$

Thus, we obtain for $t\in[0,T]$

$$\begin{aligned}
&E\left[\sup_{s\in[0,t]}\left|\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}[Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))}\right.\right.\\
&\qquad\left.\left. - Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}]\nu_i(d\xi du)\right|^2\right]\\
&\leqq (2C_4C_6+4TC_7)\int_0^t E\left[\left|Y_i^{(N)}(u-)-Y_i(u-)\right|^2\right]du.
\end{aligned}\tag{93}$$

Hence, (90), (91) and (93) imply that for $t \in [0,T]$

$$
\begin{aligned}
&E\left[\sup_{s\in[0,t]} |Y_i^{(N)}(s) - Y_i(s)|^2\right] \\
&\leqq 4|y_i^{(N)} - y_i|^2 + 4E\left[\sup_{s\in[0,t]} |M_i^{(N)}(s)|^2\right] \\
&\quad + 4\int_0^t E\left[(C_4|Y_i^{(N)}(s-) - Y_i(s-)| + C_5 \sup_{z\in[0,1),s\in[0,T]} ||U^{(N)}(\cdot,z,s) - U(\cdot,z,s)||_{\mathrm{var}})^2\right] ds \\
&\quad + 4(2C_4C_6 + 4TC_7)\int_0^t E\left[\left|Y_i^{(N)}(u-) - Y_i(u-)\right|^2\right] du \\
&\leqq 4|y_i^{(N)} - y_i|^2 + 4E\left[\sup_{s\in[0,t]} |M_i^{(N)}(s)|^2\right] + 8C_5^2 \sup_{z\in[0,1),s\in[0,T]} ||U^{(N)}(\cdot,z,s) - U(\cdot,z,s)||_{\mathrm{var}}^2 \\
&\quad + [8C_4^2 + 4(2C_4C_6 + 4TC_7)]\int_0^t E\left[\sup_{u\in[0,s]} |Y_i^{(N)}(u) - Y_i(u)|^2\right] ds.
\end{aligned}
$$

By Gronwall's inequality, we obtain

$$
\begin{aligned}
&E\left[\sup_{t\in[0,T]} |Y_i^{(N)}(t) - Y_i(t)|^2\right] \\
&\leqq 4e^{C_8T}\left(|y_i^{(N)} - y_i|^2 + E\left[\sup_{t\in[0,T]} |M_i^{(N)}(t)|^2\right]\right. \\
&\qquad\qquad \left. + 2C_5^2 \sup_{z\in[0,1),s\in[0,T]} ||U^{(N)}(\cdot,z,s) - U(\cdot,z,s)||_{\mathrm{var}})^2\right)
\end{aligned} \tag{94}
$$

where C_8 is a positive constant. Doob's martingale inequality and (3.9) of Chapter II in [20] and (89) imply there exists a positive constant C_9 such that

$$
\begin{aligned}
&E\left[\sup_{s\in[0,t]} |M_i^{(N)}(s)|^2\right] \\
&\leqq \frac{C_9}{N^2}E\left[\sum_{j=1}^N \int_{s\in(0,t]}\int_{\xi\in[0,\infty)} \mathbf{1}_{Y_i^{(N)}(s-)<Y_j^{(N)}(s-)}\,\mathbf{1}_{\xi\in[0,w_j(Y_j^{(N)}(s-),s))}\,d\xi ds\right] \\
&\leqq \frac{C_9R_w(T)t}{N}.
\end{aligned}
$$

Hence,

$$
\lim_{N\to\infty} E\left[\sup_{s\in[0,t]} |M_i^{(N)}(s)|^2\right] = 0. \tag{95}
$$

Therefore, by the first assertion of Theorem 2, (25), (94) and (95) we obtain

$$
\lim_{N\to\infty} E\left[\sup_{t\in[0,T]} |Y_i^{(N)}(t) - Y_i(t)|^2\right] = 0
$$

for $i = 1, 2, \ldots, L$. This implies that $(Y_1^{(N)}(t), Y_2^{(N)}(t), \ldots, Y_L^{(N)}(t))$ converges to $(Y_1(t), Y_2(t), \ldots, Y_L(t))$ uniformly in $t \in [0,T]$ in the sense of L^2.

To show the almost sure convergence, see that there exists a subsequence $\{N(k)\}$ such that $(Y_1^{(N(k))}(t), Y_2^{(N(k))}(t), \ldots, Y_L^{(N(k))}(t))$ converges to $(Y_1(t), Y_2(t), \ldots, Y_L(t))$ uniformly in $t \in [0,T]$ almost surely. However, the argument above is also available even if we replace N by any subsequence $N(k)$. Therefore, we have $(Y_1^{(N)}(t), Y_2^{(N)}(t), \ldots, Y_L^{(N)}(t))$ converges to $(Y_1(t), Y_2(t), \ldots, Y_L(t))$ uniformly in $t \in [0,T]$ almost surely.

4 Appendix

Proposition 11 *Let $\{\phi_n\}$ be nondecreasing functions on $[0,1]$ and ϕ be a continuous function on $[0,1]$. Assume that $\phi_n(x)$ converges to $\phi(x)$ for all $x \in [0,1]$. Then, $\phi_n(x)$ converges to $\phi(x)$ uniformly in $x \in [0,1]$.* ◇

Proof. Let $\varepsilon > 0$. Since ϕ is uniformly continuous on $[0,1]$, we can choose a positive integer N such that

$$|\phi(x) - \phi(y)| < \varepsilon, \quad |x-y| \leqq \frac{1}{N}.$$

By the assumption, there exists a integer n_0 such that

$$\left|\phi_n\left(\frac{k}{N}\right) - \phi\left(\frac{k}{N}\right)\right| < \varepsilon, \quad n \geqq n_0 \text{ and } k = 1, 2, \ldots, N.$$

For all $x \in [0,1]$ we can choose $k_x \in \{1, 2, \ldots, N\}$ such that $0 \leqq x - k_x/N \leqq 1/N$. Hence, we have for all $x \in [0,1]$ and $n \geqq n_0$

$$\begin{aligned}
&|\phi_n(x) - \phi(x)| \\
&\leqq \left|\phi_n(x) - \phi_n\left(\frac{k_x}{N}\right)\right| + \left|\phi_n\left(\frac{k_x}{N}\right) - \phi\left(\frac{k_x}{N}\right)\right| + \left|\phi\left(\frac{k_x}{N}\right) - \phi(x)\right| \\
&< \phi_n\left(\frac{k_x+1}{N}\right) - \phi_n\left(\frac{k_x}{N}\right) + 2\varepsilon \\
&\leqq \left|\phi_n\left(\frac{k_x+1}{N}\right) - \phi\left(\frac{k_x+1}{N}\right)\right| + \left|\phi\left(\frac{k_x+1}{N}\right) - \phi\left(\frac{k_x}{N}\right)\right| + \left|\phi\left(\frac{k_x}{N}\right) - \phi_n\left(\frac{k_x}{N}\right)\right| + 2\varepsilon \\
&\leqq 5\varepsilon.
\end{aligned}$$

This completes the proof. □

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